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REGULARIZED FUNCTIONS ON THE PLANE AND NEMYTSKII OPERATORS

FUNCIONES REGULARES EN EL PLANO Y OPERADORES DE NEMYTSKII

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Abstract

In this paper we show that the space of the so-called regularized functions defined on some rectangle in the plane is a Banach space. Moreover, under suitable hypotheses we give a necessary and sufficient condition for the Nemytskii operator to map the space of regularized functions into itself.

Keywords: regularized functions of two variables, Banach spaces, Nemytskii operator.

Resumen

En este artículo mostramos que el espacio de las llamadas funciones regularizadas definidas sobre algún rectángulo del plano es un espacio de Banach. Más aún, bajo las hipótesis adecuadas damos condición necesaria y suficiente para el operador de Nemytskii aplique el espacio de funciones regularizadas sobre el mismo.

Palabras clave: funciones regulares de dos variables, espacios de Banach, operador de Nemytskii.

Mathematics Subject Classification: 47B33, 26B30.

1 Introduction

The regularized functions (also called regulated functions) were introduced by Georg Aumann in 1954 [2], that is, functions of a real variable which at each point of their domain of definition admit both finite one-sided limits. In [13] the space of regularized functions on $[a, b]$ is denoted by $G(a, b)$.

Regularized functions play an important role, for instance, in applications to differential equations with singular right-hand sides or with distributional coefficients, see [7], or to the Skorokhod problem, see [4]. In the study of the controllability of systems governed by evolution equations, as well as in existence and expansion of solutions of differential or functional equations the Nemytskii operator appears in a natural way. To the best of our knowledge, this operator has not been studied up to now in spaces of regularized functions. Given a rectangle $I := [a, b] \times [c, d]$ in the plane $\mathbb{R}^2$, in this paper we show that the space $G^{-}(I)$ of so called left-left regularized functions $h : I \to \mathbf{R}$ is a Banach space, and we characterize the Nemytskii operator acting in this space.

2 Regularized functions

Let $I = [a, b] \times [c, d]$ as above, and let $h : I \longrightarrow \mathbb{R}$ be some function. Following [5] we call the function h_- defined by

$$h_-(t,s) = \begin{cases} \lim\limits_{(x,y)\to(t^-,s^-)} h(x,y), & (t,s) \in (a,b] \times (c,d], \\ \lim\limits_{(x,y)\to(t^-,c^+)} h(x,y), & (t,s) \in (a,b] \times \{c\}, \\ \lim\limits_{(x,y)\to(a^+,s^-)} h(x,y), & (t,s) \in \{a\} \times (c,d], \\ \lim\limits_{(x,y)\to(a^+,c^+)} h(x,y), & (t,s) = (a,c) \end{cases}$$

the *left-left regularization* of h. In the sequel the class of functions h for which the left-left regularization exists will be denoted by $G^-(I;\mathbb{R})$. Finding functions in this class is trivial: for example, any continuous function $h : I \to \mathbb{R}$ satisfies $h_-(t,s) \equiv h(t,s)$, and so belongs to $G^-(I;\mathbb{R})$.

The *right-right regularization* of a function $h : I \to \mathbb{R}$ is defined in a similar way by

$$h_+(t,s) = \begin{cases} \lim\limits_{(x,y)\to(t^+,s^+)} h(x,y), & (t,s) \in [a,b) \times [c,d), \\ \lim\limits_{(x,y)\to(t^+,d^-)} h(x,y), & (t,s) \in [a,b) \times \{d\}, \\ \lim\limits_{(x,y)\to(b^-,s^+)} h(x,y), & (t,s) \in \{b\} \times [c,d), \\ \lim\limits_{(x,y)\to(b^-,d^-)} h(x,y), & (t,s) = (b,d). \end{cases}$$

Similarly to the previous case we denote the class of functions which admit a right-right regularization by $G^+(I;\mathbb{R})$. Finally, the class $G^{-+}(I;\mathbb{R})$ of left-right regularized and the class $G^{+-}(I;\mathbb{R})$ of right-left regularized functions are defined analogously.

It is easy to see that the classes $G^-(I,\mathbb{R})$ and $G^+(I,\mathbb{R})$ are different. For example, the function h defined on $I = [-1,1] \times [-1,1]$ by

$$h(x,y) = \begin{cases} \dfrac{1}{x+y}, & x > 0 \text{ and } y > 0, \\ 1 & \text{otherwise} \end{cases}$$

satisfies

$$h_-(0,0) = h(0,0) = 1, \qquad h_+(0,0) = \infty$$

and therefore belongs to $G^+(I;\mathbb{R})$, but not to $G^-(I;\mathbb{R})$.

3 Properties of regularized functions

In this section we show that the class $G^-(I;\mathbb{R})$ is a Banach space.

Proposition 3.1 *The class* $(G^-(I;\mathbb{R}),+,\cdot)$ *is a linear space.*

Proof. Given $f,g\in G^-(I;\mathbb{R})$ and $\alpha\in\mathbb{R}$, we have to show that $f+g\in G^-(I;\mathbb{R})$ and $\alpha f\in G^-(I;\mathbb{R})$. Denoting by f_- the left-left regularization of f and by g_- the left-left regularization of g we get for $(t,s)\in(a,b]\times(c,d]$

$$\begin{aligned}(f+g)_-(t,s) &= \lim_{(x,y)\to(t^-,s^-)} f(x,y)+\lim_{(x,y)\to(t^-,s^-)} g(x,y)\\ &= \lim_{(x,y)\to(t^-,s^-)}\big[f(x,y)+g(x,y)\big]=f_-(t,s)+g_-(t,s).\end{aligned}$$

The other three cases for $(t,s)\in I$ are treated similarly, and so we have shown that $f+g\in G^-(I;\mathbb{R})$. For $\alpha\in\mathbb{R}$ and $(t,s)\in(a,b]\times(c,d]$ we obtain

$$(\alpha f)_-(t,s)=\lim_{(x,y)\to(t^-,s^-)}(\alpha f)(x,y)=\alpha\lim_{(x,y)\to(t^-,s^-)} f(x,y)=\alpha f_-(t,s),$$

and analogously for the other choices of $(t,s)\in I$. We conclude that $\alpha f\in G^-(I;\mathbb{R})$ which proves the assertion. ■

In the sequel we consider the linear space $G^-(I;\mathbb{R})$ equipped with the supremum norm

$$\|f\|_\infty := \sup\Big\{|f(x,y)| : (x,y)\in I\Big\}.$$

We have then the following result.

Theorem 3.1 $(G^-(I;\mathbb{R}),\ \|\cdot\|_\infty)$ *is a Banach space.*

Proof. Let $\{f_n\}_{n\geq 1}\in G^-(I;\mathbb{R})$ be a Cauchy sequence, which means that for each $\varepsilon>0$ there exists $N=N_\varepsilon>0$ such that $n,m\geq N$ implies $\|f_n-f_m\|_\infty<\varepsilon$. Since

$$\big|f_n(t,s)-f_m(t,s)\big|\leq\|f_n-f_m\|_\infty<\varepsilon$$

for all $(t,s)\in I$, we conclude that $\big\{f_m(t,s)\big\}_{m\geq 1}$ is a Cauchy sequence in $\mathbb{R}$, and so we know that the pointwise limit

$$f(t,s):=\lim_{m\to\infty} f_m(t,s)\qquad((t,s)\in I)$$

exists. We claim that $f\in G^-(I;\mathbb{R})$ and $\lim\limits_{m\to\infty}\|f_n-f\|_\infty=0$. To prove this assertion fix $\varepsilon>0$, and choose N such that $n,m\geq N$ implies that $\|f_n-f_m\|_\infty<\varepsilon$. We distinguish now four possibilities for (t,s):

(i) Let $(t,s) \in (a,b] \times (c,d]$. In this case we have

$$
\begin{aligned}
&|f_n(t,s) - f(t,s)| \\
&\quad = \quad |f_n(t,s) - \lim_{m\to\infty} f_m(t,s)| \\
&\quad = \quad |\lim_{(x,y)\to(t^-,s^-)} f_n(x,y) - \lim_{m\to\infty} \lim_{(x,y)\to(t^-,s^-)} f_m(x,y)| \\
&\quad = \quad \lim_{m\to\infty} \lim_{(x,y)\to(t^-,s^-)} |f_n(x,y) - f_m(x,y)| \\
&\quad \leq \quad \|f_n - f_m\|_\infty < \varepsilon.
\end{aligned}
$$

(ii) In the case $(t,s) \in (a,b] \times \{c\}$ we obtain

$$
\begin{aligned}
&|f_n(t,c) - f(t,c)| \\
&\quad = \quad |f_n(t,c) - \lim_{m\to\infty} f_m(t,c)| \\
&\quad = \quad |\lim_{(x,y)\to(t^-,c^+)} f_n(x,y) - \lim_{m\to\infty} \lim_{(x,y)\to(t^-,c^+)} f_m(x,y)| \\
&\quad = \quad \lim_{m\to\infty} \lim_{(x,y)\to(t^-,c^+)} |f_n(x,y) - f_m(x,y)| \\
&\quad \leq \quad \|f_n - f_m\|_\infty < \varepsilon.
\end{aligned}
$$

(iii) The case when $(t,s) \in \{a\} \times (c,d]$ is similar to those considered above.

(iv) If $(t,s) = (a,c)$ we get

$$
\begin{aligned}
&|f_n(a,c) - f(a,c)| \\
&\quad = \quad |f_n(a,c) - \lim_{m\to\infty} f_m(a,c)| \\
&\quad = \quad |\lim_{(x,y)\to(a^+,c^+)} f_n(x,y) - \lim_{m\to\infty} \lim_{(x,y)\to(a^+,c^+)} f_m(x,y)| \\
&\quad = \quad \lim_{m\to\infty} \lim_{(x,y)\to(a^+,c^+)} |f_n(x,y) - f_m(x,y)| \\
&\quad \leq \quad \|f_n - f_m\|_\infty < \varepsilon.
\end{aligned}
$$

In all cases we have shown that $\|f_n - f\|_\infty < \varepsilon$. Moreover, since $G^-(I;\mathbb{R})$ is a linear space we have $f_n, f_n - f \in G^-(I;\mathbb{R})$. Therefore $f = (f - f_n) + f_n \in G^-(I;\mathbb{R})$, and the proof is complete. ■

Clearly, in a similar way we can prove that $G^{-+}(I;\mathbb{R})$, $G^{+-}(I;\mathbb{R})$ and $G^{+}(I;\mathbb{R})$ are Banach spaces with the supremum norm. We remark that we have not only $C(I;\mathbb{R}) \subset G^{-}(I;\mathbb{R})$, but also $BV(I;\mathbb{R}) \subset G^{-}(I;\mathbb{R})$, see [3].

A function $f : I \to \mathbb{R}$ is said to be *left-left continuous* if

$$\lim_{(x,y)\to(t,s)} f(x,y) = f(t,s) \text{ for all } x \in (a,b] \text{ and } y \in (c,d].$$

We denote by $BV^{*}(I;\mathbb{R})$ the subspace of $BV(I;\mathbb{R})$ of those functions which are left–left continuous on $(a,b] \times (c,d]$.

Lemma 3.2 *(cf. [3]) If $h \in BV(I;\mathbb{R})$, then $h_{-} \in BV^{*}(I;\mathbb{R})$.*

4 The Nemytskii operator

For $I = [a,b] \times [c,d]$ as before, consider the linear space $\mathcal{F}$ of all functions $f : I \to \mathbb{R}$ and the nonlinear operator $H : \mathcal{F} \to \mathcal{F}$ defined by the formula

$$(H_f)(t,s) = h(t,s,f(t,s)),$$

where $h : I \times \mathbb{R} \to \mathbb{R}$ is a mapping. In this case we say that H is the *Nemytskii operator* generated by h. When h is independent on $(t,s) \in I$, the Nemyskii operator generated by h is called *autonomous*.

In [6] Josephy proved that the autonomous Nemytskii operator generated by $h : \mathbb{R} \to \mathbb{R}$ is a self mapping of $BV([a,b];\mathbb{R})$ if and only if h is locally Lipschitz on $\mathbb{R}$. Subsequently, several authors proved this result for many other functions spaces (see [1, 8, 9, 10, 11, 12], for example).

Here we will give conditions to assure that the Nemytskii operator defined on the space $G^{-}(I;\mathbb{R})$ maps this space into itself. To this end we consider the space $G^{-} \cdot Lip(I \times \mathbb{R};\mathbb{R})$ of all left-left regularized functions in the first two variables and Lipschitzian in the third variable, i.e. all functions h which satisfy the following two conditions:

(i) The map $(t,s) \mapsto h(t,s,u)$ is a left-left regularized function for all $u \in \mathbb{R}$.

(ii) There exists $M > 0$ such that

$$|h(t,s,u) - h(t,s,v)| \leq M|u-v| \qquad ((t,s) \in I). \qquad (*)$$

Observe that the class $G^{-} \cdot Lip(I \times \mathbb{R};\mathbb{R})$ is a Banach space equipped with the norm

$$\|h\|_{Lip} = \max\Big\{\|h_0\|_{\infty}, [h]\Big\}, \tag{4.1}$$

where $h_0 : I \longrightarrow \mathbb{R}$ is defined by $h_0(t,s) = h(t,s,0)$ (H_0 is the Nemytskii operator generated by h_0) and

$$\Big[h\Big] = \inf \Big\{ M : M \text{ satisfies } (*) \Big\}. \tag{4.2}$$

Now we are going to prove our main result which is motivated by the technique used in [14].

Main Theorem 4.1 *Suppose that $h(t,s,\cdot) : \mathbb{R} \to \mathbb{R}$ is Lipschitzian for all $(t,s) \in I$. Then the Nemytskii operator H generated by h maps the space $G^-(I;\mathbb{R})$ into itself if and only if $h \in G^-(I;\mathbb{R}) \cdot Lip(I \times \mathbb{R};\mathbb{R})$. Moreover, in this case the operator H is bounded.*

Proof. Observe that if H maps $G^-(I;\mathbb{R})$ into itself then for any function $f : I \to \mathbb{R}$ the operator $H_f : I \to \mathbb{R}$, where $H_f(t,s) = h(t,s,f(t,s))$.

Hence we deduce that $h(t,s,f(t,s))$ is a left-left regularized function for all $f \in \mathbb{R}^I$. Moreover, keeping in mind that $h \in Lip(I \times \mathbb{R};\mathbb{R})$, we get that

$$h \in G^-(I;\mathbb{R}) \cdot Lip(I \times \mathbb{R};\mathbb{R}).$$

Conversely, for all $(t,s) \in I$, we have: (i) If $(t,s) \in (a,b] \times (c,d]$, then

$$H_f(t^-,s^-) - H_f(t,s) = h(t^-,s^-,f(t^-,s^-)) - h(t,s,f(t,s))$$

$$= h_-(t^-,s^-,f(t^-,s^-)) - h_-(t,s,f(t,s))$$

$$= \lim_{(x,y)\to(t^-,s^-)} h(x,y,f(x,y)) - h(t,s,f(t,s)) = 0.$$

(ii) If $(t,s) \in (a,b] \times \{c\}$, then

$$H_f(t^-,s^-) - H_f(t,s) = h(t^-,s^-,f(t^-,s^-)) - h(t,s,f(t,s))$$

$$= h_-(t^-,s^-,f(t^-,s^-)) - h_-(t,s,f(t,s))$$

$$= \lim_{(x,y)\to(t^-,c^+)} h(x,y,f(x,y)) - h(t,s,f(t,s)) = 0.$$

(iii) For $(t,s) \in \{a\} \times (c,d]$ we proceed in a similar way as in (ii).
(iv) If $(t,s) = (a,c)$ the result is trivial and we omit it. Now we will show that H_f is left-left continuous. In fact, we get

$$\lim_{(\tau,\sigma)\to(t^-,s^-)} \Big|H_f(\tau,\sigma) - H_f(t,s)\Big|$$

$$= \lim_{(\tau,\sigma)\to(t^-,s^-)} \Big|h(\tau,\sigma,f(\tau,\sigma)) - h(t,s,f(t,s))\Big|$$

$$= \lim_{(\tau,\sigma)\to(t^-,s^-)} \Big|h_-(\tau,\sigma,f(\tau,\sigma)) - h_-(t,s,f(t,s))\Big|$$

$$= \lim_{(\tau,\sigma)\to(t^-,s^-)} \Big|h(\tau,\sigma,f(\tau,\sigma)) - h(\tau,\sigma,f(t,s)) + h(\tau,\sigma,f(t,s)) - h(t,s,f(t,s))\Big|$$

$$\leq M \cdot \lim_{(\tau,\sigma)\to(t^-,s^-)} \Big|f(\tau,\sigma) - f(t,s)\Big| +$$

$$\lim_{(\tau,\sigma)\to(t^-,s^-)} \Big|h(\tau,\sigma,f(t,s)) - h(t,s,f(t,s))\Big|$$

$$= M \cdot 0 + 0 = 0,$$

where M is the Lipschitz constant from $(*)$. It follows that H maps $G^-(I,\mathbb{R})$ into itself.
Next, we prove that the operator H is bounded.
Let $B_r = \{f \in G^-(I;\mathbb{R}) : \|f\|_\infty \leq r\}$. Then we obtain

$$\big\|H_f\big\|_\infty - \big\|H_0\big\|_\infty \leq \big\|H_f - H_0\big\|_\infty = \sup_{(t,s)\in I} \Big|(H_f)(t,s) - (H_0)(t,s)\Big|$$

$$= \sup_{(t,s)\in I} \Big|h(t,s,f(t,s)) - h(t,s,0)\Big|$$

$$\leq M \cdot \sup_{(t,s)\in I} \Big|f(t,s)\Big| \leq M \cdot r.$$

Next, we derive

$$\begin{aligned} \Big\|H_f\Big\|_\infty &\leq Mr + \big\|H_0\big\|_\infty \\ &= Mr + \big\|h(t,s,0)\big\|_\infty \\ &= Mr + \big\|h_0\big\|_\infty. \end{aligned}$$

This completes the proof. ■

We point out that our results may be extended in different directions. For instance, instead of functions on a rectangle in the plane one may consider functions of several variables on a cube in finite-dimensional Euclidean space. Moreover, all constructions carry over without any change from real valued functions to functions taking their values in a Banach space.

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