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ON THE DESIGN OF MEMBRANES WITH INCREASING FUNDAMENTAL FREQUENCY

SOBRE EL DISEÑO DE MEMBRANAS CON FRECUENCIA FUNDAMENTAL CRECIENTE

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Abstract

By means of a relaxation approach, we study the shape design of a stiff inclusion with given area in a membrane in order to maximize its fundamental frequency. As an eigenvalue control problem, the fundamental frequency is a concave function of the control, which is not described by the membrane shape, but by an element in a function space. First order optimality conditions allow to describe the optimal shape by means of a free boundary value problem.

Keywords: variational methods for eigenvalues, shape optimization, free boundary value problems.

Resumen

Mediante un método de relajación, se estudia la forma de una inclusión rígida de área dada en una membrana de manera que se maximice su frecuencia fundamental. Analizado como un problema de control de valores propios, la frecuencia fundamental es una función cóncava del control, el cual no es descrito por la forma de la membrana, sino por un elemento de un espacio de funciones. Las condiciones de optimalidad de primer orden permiten describir la forma óptima mediante un problema de frontera libre.

Palabras clave: métodos variacionales para valores propios, optimización de forma, problemas de frontera libre.

Mathematics Subject Classification: 35J20, 35R35, 49R05, 49Q10.

1 Introduction

The subject of the present study was motivated by an article due to Payne and Weinberger [22] where the following is stated: suppose a two dimensional membrane, defined on a domain Ω, fixed along its outer boundary, perforated by “holes" with boundaries Γ_i and along them the membrane is free. For a given area $|\Omega|$ and a given perimeter L of the exterior boundary, the highest fundamental frequency is attained when the domain Ω is annular. This classical result, which was proved by means of isoperimetric inequalities, lies at the origin of the following problem: let be given a perforated membrane with uniform density, supposed fixed on the exterior boundary and with the perforation filled by a rigid inclusion. Assuming that its area $|\Omega|$ is a given constant, and the outer boundary is fixed, we look for the location and shape of the inclusion in order to maximize the fundamental frequency of the membrane. This problem falls in two areas: eigenvalue control and optimal shape design, the subject we present is related to studies published among others, by Buttazo and Dal Maso [2], Cox and McLaughlin [6], Egnell [8], Eppler [9], Henrot [15], Jouron [17],Rousselet [23], Tahraoui [26], and Zolesio [30]. We present here a unified perspective by means of an approach similar as the one used for other shape optimization problems (cf. Gonzalez de Paz [11] and [12]). We define a regularized problem, where a concave functional is maximized on a convex set. Within this framework, existence of optimal design, corresponding to the maximal fundamental frequency, so as differentiability and concavity properties are easily obtained. The functional is Gateaux differentiable (even Frechet differentiable) and the analysis of the first order optimality conditions allows us to describe the boundary Γ of the optimal set as a free boundary. If the free boundary is regular enough, the results obtained by our approach in terms of the functional derivative lead to similar properties published elsewhere concerning shape gradients (cf. Eppler [9], Rousselet [23], Simon [24] and Zolesio [30]).

Physically, by adding a small enough regularization term, we will handle a larger membrane defined on the whole domain $D = \Omega \cup \overline{\Omega}_e$ with two components: the original one defined on Ω and a membrane vibrating on Ω_e, which is affected by a stiffness factor. Mathematically, this is described by means of elliptic operators of the type $-\Delta + q$ where the stiffness factor (regularization term) q is a function defined on a certain class. We prove that, by increasing the stiffness factor, in the limit the lowest eigenvalue of the operator is maximized by $q = \lambda\chi_{\Omega_e^*}$ where λ is the first eigenvalue and $\chi_{\Omega_e^*}$ is the characteristic function of the optimal set Ω_e^*. A similar result was obtained by Egnell [8] in a context of quantum mechanics.

Though this problem has been the subject of research during decades, we think that our approach adds some new perspectives concerning the characterization of the optimal inclusion. Besides, it is constructive and well adapted for numerical calculations applying gradient-type algorithms.

2 The regularized problem

Let $D = \Omega \cup \overline{\Omega}_e$ be an open, bounded, star shaped, connected set in $\mathbb{R}^2$, with a piecewise smooth boundary ∂D. Let $\Omega \subset D$ be a subset such that $\partial\Omega = \partial D \cup \Gamma$, where $\Gamma = \partial\Omega_e$ denotes the boundary of the "inclusion" Ω_e.

We recall that for the case that the membrane is fixed along its boundary, the eigenvalue problem for the laplacian on Ω with homogeneous Dirichlet condition on its boundary describes mathematically the vibration problem. Furthermore, for the fundamental frequency λ_0 (lowest or first eigenvalue) the Ritz-Rayleigh principle states that:

$$\lambda_0 = \min_{u \in S(\Omega)} \|\nabla u\|^2.$$

Here, $S(\Omega) = \left\{ u \in H_0^1(\Omega) \mid \|u\|^2 = 1 \right\}$, the double bars denote the usual L^2 norm in Ω, and $H_0^1(\Omega)$ the usual Sobolev space (cf. for example Neças [21]).

For the penalized problem we introduce a "stifness factor" which will be described the following way: let μ be a positive element of the unit ball in $L^\infty(D)$, such that for a given positive constant $A < |D| : |\mu|_{L^1} = A$. Recall that $C = \{\mu \in L^\infty(D) \mid 0 \le \mu \le 1, |\mu|_{L^1} = A\}$ is a convex set.

For a given positive constant β we define the functional $J_\mu : H_0^1(D) \longrightarrow \mathbb{R}$

$$u \mapsto J_\mu(u) = \|\nabla u\|^2 + \beta \left\langle \mu, u^2 \right\rangle. \tag{1}$$

The brackets denote the usual $\left(L^\infty, L^1\right)$ duality. It is well known that by minimizing the functional 1 defined above on the set

$$S = \left\{ u \in H_0^1(D) \mid \|u\|^2 = 1 \right\}$$

the existence of optimal solution $u_\mu \in S$ is a classical fact. The optimal value $J_\mu(u_\mu)$ given by the functional is the lowest eigenvalue of the boundary value problem $P(\mu)$:

$$-\Delta u + \beta\mu u = \lambda u \text{ in } D, \tag{2}$$

$$u = 0 \text{ on } \partial D. \tag{3}$$

For a fixed, positive β we define the functional on C:

$$\mu \to \Lambda_\beta(\mu) = J_\mu(u_\mu) = \lambda_\beta. \tag{4}$$

The mapping 4 is well defined if the "stiffness factor" $\beta\mu$ is a "small" perturbation for the Laplacian. This is precised in the following sense:

Lemma 1 *For the differential operator $-\Delta + \beta\mu$ with first eigenvalue λ_β, as defined for the boundary value problem $P(\mu)$, in the case $\beta < \lambda_\beta$, the corresponding eigenfunction u_β is superharmonic and strictly positive, so that λ_β is a simple eigenvalue.*

Proof. First note that any eigenfunction u is a $C^{1,1}_{loc}$ function (cf. Jensen [16]). The partial differential equation has a sense a.e. in D. The function $v = |u|$ is also a continuous eigenfunction and we have $-\Delta v = (\lambda_\beta - \beta\mu)v > 0$ a.e. in D, i.e. v is superharmonic. Suppose there exists an $x_0 \in D$ such that $v(x_0) =$

0, then there exists a ball centered in x_0 where $v = 0$. The Hopf Maximum principle states that in this case $v = 0$ in D (cf. for example Miranda [20]). It follows: $u = 0$ in D, which contradicts the fact that $u \in S$, so $u > 0$ on D. Consequently, as every function has fixed sign, the first eigenvalue λ_β is simple (let us remark that this property can also be proved by means of the Krein-Rutman Theorem). ■

Remark 2 *Note that the mapping $\mu \to \Lambda_\beta(\mu) = \min_{u\in S} \|\nabla u\|^2 + \beta \langle \mu, u^2 \rangle$ is the lower envelope of affine functions related to μ, which implies it is a concave function respect to μ. This means, the first eigenvalue is a concave function of the "stiffness" factor.*

The next step will be to find the best factor among a certain class in order to maximize the fundamental frequency of the relaxed problem.

For this goal, we consider now the optimization problem:

$$\sup_{\mu\in C} \Lambda_\beta(\mu). \tag{5}$$

Let us recall that the convex set C is compact for the $\sigma\left(L^\infty, L^1\right)$ −topology. In order to obtain existence results for the solution of optimization we prove in a similar way as in Gonzalez de Paz [11]:

Theorem 3 *The mapping 4 $\mu \to \Lambda_\beta(\mu)$ is $\sigma\left(L^\infty, L^1\right)$ −continuous, so that for β small enough there exists an element $\mu_\beta \in C$ such that*

$$\Lambda_\beta\left(\mu_\beta\right) = \max_{\mu\in C} \Lambda_\beta(\mu) = \lambda_\beta. \tag{6}$$

Proof. For a fixed $\beta > 0$, the functional $\mu \to \Lambda_\beta(\mu)$ is bounded on C. We remark that there exists a constant $C_\beta > 0$ such that for every $\mu \in C$ and every $u \in H_0^1(D)$:

$$\|\nabla u\|^2 + \beta \langle \mu, u^2 \rangle \leq C_\beta \|u\|_{H_0^1}^2. \tag{7}$$

For a fixed $u_0 \in S$, we have for every $\mu \in C$:

$$\Lambda_\beta(\mu) \leqslant C_\beta \|u_0\|_{H_0^1}^2. \tag{8}$$

As Λ_β is bounded on C, there exists a ball $B \subset H_0^1(D)$ such that for every $\mu \in C$:

$$\min_{u\in S} J_\mu(u) = \min_{u\in S\cap B} J_\mu(u). \tag{9}$$

Because of the Rellich-Kondrasov injection theorem, the set $W = \left\{w \mid w = u^2, u \in B\right\}$ is strongly compact in $L^1(D)$. We define the set of affine mappings: $\mathcal{F} = \{J_u : \mu \to J_\mu(u) \mid u \in B\}$, it follows that Λ_β is the lower envelope of $\mathcal{F}$.

Let be given $\delta > 0$ and a fixed $\mu_0 \in C$. Furthermore, let be a $\mu \in C$ such that for every $u \in B$:

$$\beta \left|\left\langle \mu - \mu_0, u^2 \right\rangle\right| \leqslant \delta. \tag{10}$$

This means that $\mu - \mu_0 \in \left(\frac{\beta}{\delta}W\right)^0 \subset L^\infty(D)$, which is the polar set of $\frac{\beta}{\delta}W \subset L^1(D)$. It follows that μ lies in a neighborhood of μ_0 for the topology of the uniform convergence on the strong compact sets of $L^1(D)$, noted also as the τ-topology. (see for ex. Bourbaki [1]). For $\varepsilon = \delta$ and for every $u \in B$,

$$\left|J_\mu(u) - J_{\mu_0}(u)\right| \leqslant \varepsilon. \tag{11}$$

This means that the mappings collection $\mathcal{F}$is $\tau-$equicontinuous. Being Λ_β the lower envelope of affine $\tau-$equicontinuous functions, it is also $\tau-$continuous. We remark that the $\tau-$topology and the $\sigma\left(L^\infty, L^1\right)-$topology are equivalent on the unit ball in L^∞ (cf. [1]) which proves our claim. ■

3 Optimality conditions

As we can see, the eigenvalues depend on a perturbation for the Laplacian; assuming a small enough perturbation, the first eigenvalue remains simple and we may proceed to calculate the functional derivative for Λ_β. Other authors have remarked that the eigenvalues are differentiable respective to domain deformations in the case they are simple (cf. for example Rousselet [23], Zolesio [30]).

Similar as in Gonzalez de Paz [11], we have the following theorem.

Theorem 4 *For every $\mu \in C$:the functional $\mu \to \Lambda_\beta(\mu)$ has a Gateaux-derivative, so that for every $\alpha = \mu - \mu_\beta$ and $u_\beta \in S$, solution of $P(\mu_\beta)$:*

$$\Lambda'_\beta(\mu_\beta; \alpha) = \beta \left\langle u_\beta^2, \alpha \right\rangle. \tag{12}$$

A result due to M. Valadier [27] proves that the mapping $\mu \to \Lambda_\beta(\mu)$ is Frechet-differentiable. The gradient is defined as $\bigtriangledown\Lambda_\beta(\mu_\beta) = \beta u_\beta^2 \in L^1(D)$.

Clasically, a necessary optimality condition states that for every $\alpha = \mu - \mu_\beta, \mu \in C$:

$$\Lambda'_\beta(\mu_\beta; \alpha) \leqslant 0$$

which implies that for the optimal μ_β:

$$\int_D \mu_\beta u_\beta^2 d\varpi \geqslant \int_D \mu u_\beta^2 d\varpi. \tag{13}$$

for every $\mu \in C$. The optimality condition 13 is a continuous linear programming problem, its solution is a standard procedure. As it will be shown next, we look how to "place" the integrand in a domain $\Omega_{e,\beta}$ in order to maximize the integral value. To describe the optimal domain $\Omega_{e,\beta}$, we remark first with following Lemma.

Lemma 5 *Let β be a positive constant such that $\beta < \lambda_\beta$ and let $u_\beta > 0$ be the corresponding solution for the boundary value problem $P(\mu_\beta)$. For every constant $p > 0$ such that the level set $S_p = \{x \in D \mid u_\beta(x) = p\}$ is not empty, the Lebesgue measure of S_p is zero.*

Proof. We remark that for ∂D regular enough, $u_\beta \in C^{1,1}_{loc}(D) \cap H^2(D)$. The partial differential equation is solved in the sense almost every where in D. Thus, $-\Delta u_\beta = (\lambda_\beta - \beta\mu)u_\beta > 0$ a.e. in D. On the other side, on every S_p we have $-\Delta u_\beta = 0$, in the sense a.e. in D. So it follows: $meas(S_p) = 0$. ■

We are now able to describe the optimal set $\Omega_{e,\beta}$.

Proposition 6 *Let β be a positive constant such that $\beta < \lambda_\beta$, u_β as above, then there exists a scalar $p > 0$ so that $\mu_\beta = \chi_{\Omega_{e,\beta}}$ and*

$$\Omega_{e,\beta} = \{x \in D \mid u_\beta(x) > p\} \tag{14}$$

$$\partial\Omega_{e,\beta} = \{x \in D \mid u_\beta(x) = p\}. \tag{15}$$

Sketch of the proof: We consider the maximization of the linear mapping $\mu \to \int_D \mu u_\beta^2 d\varpi$ on C. Then there exists a Lagrange multiplier p related to the measure constraint $|\mu|_{L^1} = A$ (cf. Cea-Malanowski [4]) so that

$$\begin{array}{lll} u_\beta(x) > p & \text{implies} & \mu_\beta(x) = 1 \\ u_\beta(x) = p & \text{implies} & \mu_\beta(x) \in [0,1] \\ u_\beta(x) < p & \text{implies} & \mu_\beta(x) = 0. \end{array}$$

The Lebesgue measure of the set S_p is zero, so that $\mu_\beta = \chi_{\Omega_{e,\beta}}$ almost everywhere in D. The structure of the boundary as $\Gamma_\beta = \partial\Omega_{e,\beta} = S_p$ follows from the fact that u_β is continuous and superharmonic. The function μ_β is a characteristic function, so is an extremal point of C. (cf. Castaing-Valadier [3]). Hence, we can constraint our search for maximizing functions among the extremal points of C, in other words, we look a set Ω_e where to place the integrand in order to maximize the integral $\int_{\Omega_e} u_\beta^2 d\varpi$.

Remark 7 *The set $\Omega_{e,\beta}$ is unique.*

Recall that the gradient βu_β^2 is a positive and non constant in every set in D with positive measure. So for every $\alpha = \mu - \mu_\beta \neq 0$ a.e. it follows $\Lambda'_\beta\left(\mu_\beta;\alpha\right) < 0$. Therefore the mapping $\mu \to \Lambda_\beta\left(\mu\right)$ is strictly concave and consequently the maximizing point is unique.

Remark 8 *The domain functional 4 to be optimized can be interpreted in our framework as the first eigenvalue of the membrane defined on Ω_β, i.e. for $\mu_\beta = \chi_{\Omega_{e,\beta}}$:*

$$\lambda_\beta(\Omega_\beta) = \Lambda_\beta\left(\chi_{\Omega_{e,\beta}}\right). \tag{16}$$

Remark 9 *In another context, Delfour [7] and Zolesio [29] calculate the so-called shape derivative of functionals based on the continuous deformations of domains. These authors base their result using the so-called deformation speed θ, which is nothing but the gradient vector field of the function $\varphi_t : D \to D$ describing the continuous deformation of the domain $\varphi_t(\Omega) = \Omega_t$. In our framework, the Gateaux derivative already calculated becomes to the limit a shape-derivative $d\lambda_\beta(\Omega_0;\theta)$ in the sense that, formally, for a boundary Γ_β regular enough:*

$$\begin{aligned} d\lambda_\beta(\Omega_0;\theta) &= \lim_{t\to 0^+}\frac{1}{t}\Lambda'_\beta\left(\chi_{\Omega_0};\chi_{\Omega_t}-\chi_{\Omega_0}\right) \\ &= \lim_{t\to 0^+}\frac{\beta}{t}\left\langle u_\beta^2,\chi_{\Omega_t}-\chi_{\Omega_0}\right\rangle \\ &= \beta\int_{\Gamma_\beta} u_\beta^2\theta_n d\sigma. \end{aligned} \tag{17}$$

In this case, the term θ_n describes the normal component ot the vector field $\theta = D_t\varphi_t$ on the boundary Γ_β (see also Eppler [9]).

4 The free boundary value problem

As a consequence of the optimality conditions, the function $u_\beta \in H_0^1(D) \cap C_{loc}^{1,1}(D)$ solves the free boundary problem:

$$\begin{aligned} -\Delta u_\beta &= \lambda_\beta u_\beta \text{ in } \Omega_\beta = \{x\in D \mid 0 < u_\beta(x) < p\}, && (18)\\ -\Delta u_\beta + \beta u_\beta &= \lambda_\beta u_\beta \text{ in } \Omega_{e,\beta} = \{x\in D \mid p < u_\beta(x)\}, && (19)\\ u_\beta(x) &= p \text{ on } \Gamma_\beta, && (20)\\ u_\beta(x) &= 0 \text{ on } \partial D. && (21) \end{aligned}$$

Physically, we can interpret this system as a membrane with two components defined on $D = \Omega_\beta \cup \overline{\Omega}_{e,\beta}$. In the subdomain Ω_β it has the fundamental frequency λ_β, it is fixed on ∂D and interacts along the free boundary Γ_β with the membrane defined on $\Omega_{e,\beta}$, which vibrates with fundamental frequency $\lambda_\beta - \beta$.

Remark 10 *The constraint $\beta < \lambda_\beta$ allows the corresponding u_β to be a superharmonic function. If the domain D is star-shaped, the level sets are simply-connected (cf. Kawohl [18]) and this implies, the set $\Omega_{e,\beta}$ is a connected domain.*

Remark 11 *Because of the $C^{1,1}_{loc}$ regularity of the solution, the free boundary is locally Lipschitz continuous (cf. Kinderlehrer-Stampacchia [19]). Recalling that the free boundary Γ_β is a level set, we have for every neighborhood of points in Γ_β where $|\bigtriangledown u_\beta| > 0$:*

$$\frac{\partial u^+_\beta}{\partial n} = \frac{\partial u^-_\beta}{\partial n} on \Gamma_\beta. \tag{22}$$

Here the restrictions of u_β to Ω_β and $\Omega_{e,\beta}$ are denoted by u^+_β and u^-_β respectively. This "transmission" condition describes the interaction along the boundary between the vibrating membranes occupying each subdomain.

5 A global existence result

We have seen that for each set $\Omega_e \subset D$ with a given measure A describing an inclusion in the membrane, we are able to calculate a "relaxed" fundamental frequency in the sense that for the domain $\Omega = D \backslash \overline{\Omega}_e$: $\lambda_\beta(\Omega) = \Lambda_\beta\left(\chi_{\Omega_e}\right)$. For a fixed β, the maximization of the mapping $\mu \to \Lambda_\beta(\mu)$ on the convex set C provides a set $\Omega_{e,\beta}$ and the corresponding first eigenvalue $\lambda_\beta(\Omega_\beta)$. We will show in this section that for a family of bounded stiffness factors $(\beta_n)_n$ and a fixed domain $\Omega_e \subset D$, there exists a global optimum.

First we prove a monotonicity property.

Lemma 12 *Let $\Omega_e \subset D$ be a fixed domain, let $\beta' > \beta$ two stiffness factors then we have for the first eigenvalues corresponding to the relaxed problems: $\lambda_{\beta'} > \lambda_\beta$.*

Proof. Recall that for a fixed β and a fixed $\chi_\Omega \in C$:

$$\Lambda_\beta\left(\chi_{\Omega_e}\right) = \|\bigtriangledown u_\beta\|^2 + \beta\left\langle \chi_{\Omega_e}, u^2_\beta \right\rangle$$

with $u_\beta \in S$ solution of $P(\chi_\Omega)$. Then for $\beta' > \beta$:

$$\Lambda_{\beta'}\left(\chi_{\Omega_e}\right) > \left\|\bigtriangledown u_{\beta'}\right\|^2 + \beta\left\langle \chi_{\Omega_e}, u^2_{\beta'} \right\rangle \geqslant \|\bigtriangledown u_\beta\|^2 + \beta\left\langle \chi_{\Omega_e}, u^2_\beta \right\rangle = \Lambda_\beta\left(\chi_{\Omega_e}\right).$$

That is, $\lambda_{\beta'} > \lambda_\beta$. ■

This means, as the stiffness factor grows, so it does the eigenvalue. Now, what we aim to show is that the fundamental frequency corresponding to the vibrating membrane with a stiff inclusion on Ω_e bounds the relaxed frequencies we studied before.

Lemma 13 *Let $\Omega_e \subset D$ be a given, simply connected set with a piecewise smooth boundary Γ. Let be given a sequence of increasing stiffness factors β such that $\beta < \lambda_\beta = \Lambda_\beta(\chi_{\Omega_e})$. Then there exists a $\widetilde{\lambda} > 0$, such that for some $\widetilde{u} \in H_0^1(D)$,*

$$-\Delta\widetilde{u} = \widetilde{\lambda}\widetilde{u} \text{ in } \Omega = D\backslash\overline{\Omega}_e. \tag{23}$$

$$\Delta\widetilde{u} = 0 \text{ in } \Omega_e \tag{24}$$

in the weak sense, and for every λ_β:

$$\widetilde{\lambda} \geqslant \lambda_\beta. \tag{25}$$

Proof. Let λ_0 be the fundamental frequency of a membrane defined on Ω and fixed on its boundary $\partial\Omega = \partial D \cup \Gamma$ (i.e. a homogenous boundary condition of the Dirichlet type is prescribed for the corresponding eigenfuntion). We remark that, for every λ_β as defined above: $\lambda_\beta \leqslant \lambda_0$. This implies that the relaxed eigenfrequencies remain bounded and this allows to define: $\widetilde{\lambda} = \sup_\beta \lambda_\beta < \infty$.

Let be given an increasing sequence of stiffness factors $(\beta_n)_n$, let λ_n be the corresponding relaxed eigenfrequency. The eigenvalue sequence is monotone increasing and it follows: $\lambda_n \to \widetilde{\lambda}$. Recalling the fact that $\beta_n < \lambda_n$, let the sequence $(\beta_n)_n$ be such that $\beta_n \to \widetilde{\lambda}$, if $n \to \infty$.

For the corresponding eigenfunctions $u_n \in H_0^1(D)$ it follows that the sequence $(\|\bigtriangledown u_n\|)_n$ is bounded. This implies the existence of a weakly convergent subsequence in $H_0^1(D)$, noted also $(u_n)_n$, i.e. $u_n \rightharpoonup \widetilde{u} \in H_0^1(D)$.

We have therefore in the weak sense for every test function $\varphi \in \mathcal{D}$ and every n:

$$(\bigtriangledown u_n, \bigtriangledown\varphi) + \beta_n\left(\chi_{\Omega_e} u_n, \varphi\right) = \lambda_n (u_n, \varphi).$$

To the limit it becomes,

$$(\bigtriangledown\widetilde{u}, \bigtriangledown\varphi) + \widetilde{\lambda}\left(\chi_{\Omega_e}\widetilde{u}, \varphi\right) = \widetilde{\lambda}(\widetilde{u}, \varphi),$$

which is nothing but the weak formulation of the partial differential equation system 23 and 24 stated above. ■

Physically, to the limit we have a vibrating membrane on Ω fixed on the outer boundary ∂D and free along Γ.

Theorem 14 *Let D be a star-shaped domain with a piecewise smooth boundary, then there exists a simply connected set $\Omega_e^* \subset D$ with meas$(\Omega_e^*) = A$ such that the fundamental frequency $\lambda(\Omega^*)$ for the membrane defined on $\Omega^* = D\backslash\overline{\Omega}_e^*$, fixed on ∂D and an inclusion defined in Ω_e^* has the property:*

$$\lambda(\Omega^*) \geqslant \lambda(\Omega) \tag{26}$$

for every fundamental frequency $\lambda(\Omega)$ corresponding to a domain $\Omega = D\backslash\Omega_e$, with a stiff inclusion defined on Ω_e, with measure A.

Proof. Let us consider an increasing sequence of $\beta' s.$ such that $\beta < \lambda_\beta$. Remark that for $\beta' > \beta$:

$$\Lambda_{\beta'}\left(\mu_{\beta'}\right) \geqslant \Lambda_{\beta'}\left(\mu_\beta\right) > \Lambda_\beta\left(\mu_\beta\right).$$

The corresponding $\left\{\Lambda_\beta\left(\mu_\beta\right)\right\}_\beta$ build therefore an increasing sequence.

The sequence of optimal control functions $(\mu_\beta)_\beta$ is L^2-bounded, so there exists a subsequence weakly convergent to $\mu^* \in C$.

Let $W = \left\{u \in H_0^1(D) \mid \left\langle \mu^*, u^2 \right\rangle = 0\right\}$ which is closed in $H_0^1(D)$. Then there exists an element $u_0 \in S \cap W$ such that:

$$\|\triangledown u_0\|^2 = \min_{u\in S\cap W} \|\triangledown u\|^2.$$

Thus, for every β: $\Lambda_\beta\left(\mu_\beta\right) \leqslant \|\triangledown u_0\|^2$, i.e. the sequence of optimal $\Lambda_\beta\left(\mu_\beta\right) = \lambda_\beta(\Omega_\beta)$ is bounded.

We note: $\lambda^* = \sup_\beta \Lambda_\beta\left(\mu_\beta\right)$, remark that $\lambda_\beta(\Omega_\beta) \to \lambda^*$ for any sequence of increasing β, provided that $\beta < \lambda_\beta$

Let us choose a subsequence $(\beta_k)_k$ such that $\beta_k \to \lambda^*$, for $k \to \infty$. For the corresponding eigenfunctions u_k we have:

$$\|\triangledown u_k\|^2 < \Lambda_\beta\left(\mu_{\beta_k}\right) < \lambda^*.$$

Choosing a suitable subsequence noted also $(u_k)_k$, it converges weakly in $H_0^1(D)$. Besides, for every test function $\varphi \in \mathcal{D}$:

$$(\triangledown u_k, \triangledown\varphi) + \beta_k\left(\mu_{\beta_k} u_k, \varphi\right) = \lambda_k\left(u_k, \varphi\right).$$

Because of the Rellich-Kondrasov injection theorem, the sequence $(u_k)_k$ converges also in the strong topology in $L^2(D)$ so that to the limit we obtain the variational equation:

$$(\triangledown u^*, \triangledown\varphi) + \lambda^*\left(\mu^* u^*, \varphi\right) = \lambda^*\left(u^*, \varphi\right).$$

Setting $\varphi = u^* \in S$:

$$\left\|\nabla u^*\right\|^2 + \lambda^* \left\langle \mu^*, u^{*2} \right\rangle = \lambda^*.$$

In a weak sense:

$$-\Delta u^* + \lambda^* \mu^* u^* = \lambda^* u^* \text{ in } D.$$

Note that $u^* \in C^{0,1}_{loc}(D) \cap H^2(D)$, so the equation above has a sense also almost everywhere in D.

It follows:

$$\mu_\beta u_\beta \to \mu^* u^* \text{ a.e. in } D.$$

As for every $\beta_k : u_k > 0$ a.e. in D, this implies $\mu_\beta \to \mu^*$ a.e. in D and consequently:

$$\mu^* = \chi_{\Omega_e^*} \text{ a.e. in } D \text{ for a set } \Omega_e^* \subset D.$$

Up to a null measure set $\Omega_e^* = \liminf_\beta \Omega_{e,\beta}$. As all sets $\Omega_{e,\beta}$ are in a metric space, the limit set Ω_e^* is closed and connected.

In order to obtain more information on the set Ω_e^*, we apply the first order optimality condition for μ_β. We know that for each β_k as defined before we have the condition 13:

$$\int_D \mu_{\beta_k} u_k^2 d\varpi \geqslant \int_D \mu u_k^2 d\varpi \text{ for every } \mu \in C.$$

To the limit $k \to \infty$ we obtain:

$$\int_D \mu^* u^{*2} d\varpi \geqslant \int_D \mu u^{*2} d\varpi \text{ for every } \mu \in C.$$

Similar as before, there exists a Lagrange multiplier p^* related to the measure constraint $|\mu^*|_{L^1} = A$ such that

$$\begin{array}{lll} u^* \geqslant p & \text{a.e. on} & \Omega_e^* \\ u^* < p & \text{a.e. on} & \Omega_* = D \backslash \Omega_e^*. \end{array}$$

As u^* is continuous on D, the boundary $\partial\Omega_e^* = \Gamma^*$ is included in the level set $S_{p^*} = \{x \in D \mid u^*(x) = p^*\}$. Recall that for star-shaped D, the subdomain Ω_e^* is simply connected. The function u^* is harmonic in the interior of Ω_e^*, applying the maximum principle for harmonic functions we obtain:

$$u^* = p \in \Omega_e^* \tag{27}$$

and

$$-\Delta u^* = \lambda^* u^* \in \Omega^* = D\backslash\Omega_e^* = \{x \in D \mid 0 < u^*(x) < p^*\}\,. \qquad (28)$$

Physically in the domain Ω^* we have a vibrating membrane with fundamental frequency λ^*. For the subdomain Ω_e^* the function u^* describes the corresponding vibration mode for the fundamental frequency of a free membrane defined on Ω_e^*. It is a classical result that in this case, the fundamental frequency is equal to zero with associated eigenfunctions $u = const.$ (cf. Courant-Hilbert [5]). In other words, $\lambda^* = \lambda(\Omega^*)$ is the fundamental frequency of a membrane fixed along the outer boundary ∂D with a stiff inclusion fixed on the free boundary Γ^*.

Finally, this means that for every simply connected set $\Omega_e \subset D$ with $meas(\Omega_e) = A$ and for every $\beta < \lambda_\beta$:

$$\lambda(\Omega^*) \geqslant \Lambda_\beta\left(\mu_\beta\right) \geqslant \Lambda_\beta\left(\chi_{\Omega_e}\right).$$

Letting $\beta \rightarrow \widetilde{\lambda}$ and applying previous Lemma, we conclude for every $\Omega = D\backslash\overline{\Omega}_e$:

$$\lambda(\Omega^*) \geqslant \widetilde{\lambda}(\Omega) \geqslant \lambda(\Omega). \quad \blacksquare \qquad (29)$$

Remark 15 *In another context, assuming that Γ is a regular enough, Rousselet [23], Simon [24], Sokolowski and Zolesio[25] calculate the so-called shape derivative of the fundamental frequency of a membrane with free boundary Γ (in other words, the classical Hadamard formula), so that for the corresponding eigenfunction u and the gradient vector field θ of the function $\varphi_t : D \rightarrow D$ describing the continuous deformation of the domain $\Omega \rightarrow \varphi_t(\Omega) = \Omega_t$:*

$$d\lambda(\Omega;\theta) = -\frac{1}{2}\int\limits_{\Gamma}\left|\frac{\partial u}{\partial n}\right|^2 \theta_n d\sigma. \qquad (30)$$

Here, θ_n describes the normal component of the vector field $\theta = D_t\,\varphi_t$ on Γ.

Several authors have obtained as optimality condition (provided that the boundary Γ of the optimal domain is regular enough) an additional boundary condition of the Neumann type on the free boundary:

$$\left|\frac{\partial u}{\partial n}\right| = \text{ const. on } \Gamma. \qquad (31)$$

Thus, the optimal domain Ω^* and the corresponding eigenfunction u^* solve an overdetermined boundary value problem of the Cauchy type for the eigenvalue equation.

6 Some remarks on the numerical analysis of the optimization problem

6.1 A sketch of the algorithm:

The approach we have proposed seems well suited for numerical implementation, as the functional $\mu \to \Lambda_\beta(\mu)$ is differentiable and strictly concave, a gradient method of the Frank-Wolfe type is proposed to solve a finite dimensional approximation. (cf. Valadier [28], Gonzalez de Paz-Tiihonen [13]).

1. For a sequence $(\beta_k, \mu_k)_k$, generate the corrresponding $(u_k, \lambda k)$ solving the eigenvalue problem:

$$\begin{aligned} -\Delta u_k + \beta_k \mu_k u_k &= \lambda_k u_k \text{ in } D \\ u_k &= 0 \text{ on } \partial D. \end{aligned}$$

2. Define the set $S_{k+1} = \{x \in D \mid u_k(x) \geqslant p_k\}$. The multiplier p_k being such that $meas(S_{k+1}) = A$.

3. Define μ_{k+1} by $\mu_{k+1}(x) = \begin{cases} 1 & \text{if } x \in S_{k+1} \\ 0 & \text{elsewhere.} \end{cases}$

4. If $|\lambda_{k+1} - \lambda_k| > \epsilon_{tol}$, set $\beta_{k+1} = \lambda_k$ and go to step 1. Else, if $|\lambda_{k+1} - \lambda_k| \leq \epsilon_{tol}$ set $\Omega_e^* = S_{k+1}$.

6.2 A test problem

As a matter of illustration, we consider a membrane defined on a rectangular domain $D = [0, 2] \times [0, 2]$. Numerical calculations were carried out using a finite difference approximation (discretization parameter $h = 0.1$) for the laplacian operator. As initial domain Ω_e the best choice seems to be given by the set bounded by the equipotential curve of the eigenfunction for $\beta = 0$ satisfying the measure constraint. In fact, in this case there is no need to change the domain again. Convergence was achieved quite fast, after three iterations the vibration model remains constant on the domain Ω_e^*. We stopped short before the tolerance threshold was achieved, because the numerical stability of the limit case when $\beta = \lambda_\beta$ was affected. The numerical results are presented in the next table:

#Iter.step	β	λ_β
1	0	4.92
2	4.92	5.78
3	5.78	5.92

We present in Figures 1, 2 and 3 some ilustrations with the equipotential lines and the profile of the first vibration mode corresponding to the results calculated by the three iterations. The optimal inclusion in Ω_e^* is well identified in the third iteration step as the set with positive measure where the function u^* is constant.

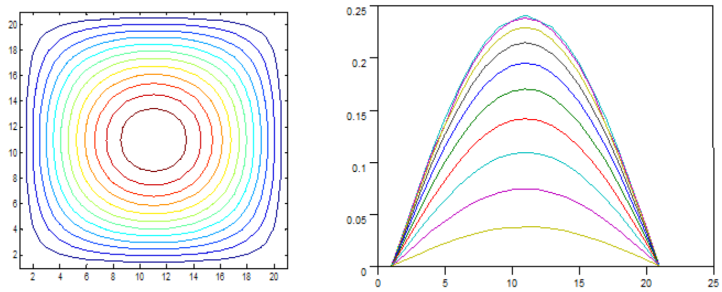

Figure 1: First iteration, $\beta = 0$, $\lambda = 4.92$.

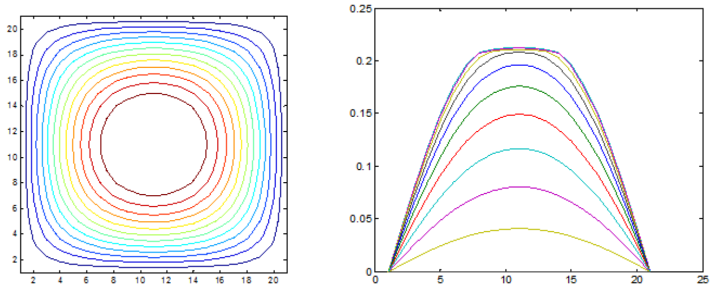

Figure 2: Second iteration, $\beta = 4.92$, $\lambda = 5.78$.

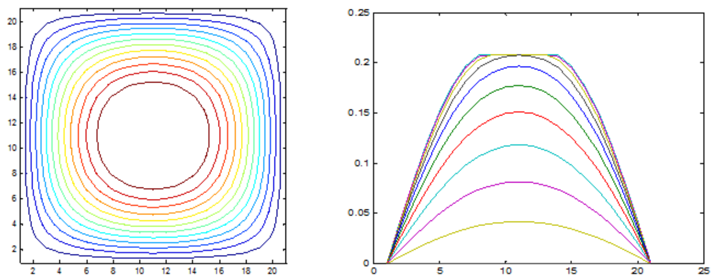

Figure 3: Third iteration, $\beta = 5.8$, $\lambda = 5.92$.

7 Conclusions

We have performed an analysis on the shape and location of stiff inclusions for membranes with maximal fundamental frequency. This is done by means of a regularization approach in a Convex Analysis framework. The functional to be maximized, which is equivalent to the lowest eigenvalue of the regularized problem, is concave and differentiable, and our results concerning the existence and description of the optimizing elements are related to other research already known (cf. Buttazzo and Dal Maso [2] and Egnell [8]). The regularization approach allows these results to be presented in a unified way. As we develop further the analysis of first order optimality conditions, we have shown that the functional derivative calculated in this context has a relation with the shape-derivative given by other authors, which under suitable regularity assumptions, can be interpreted as a limit case. The characterization obtained for the description and location of the optimal inclusions seems related to work due to Harrel,Kröger,Kurata [14]. Further applications to the analysis of conjectures raised by these authors and Henrot (cf. chapter 3 of [15]) could be worthwhile.

From the numerical analysis perspective, due to the structure of the functional derivative, the approach presented is well suited for numerical calculations and a gradient method is proposed. A main advantage seems to be the stability of the algorithm, as calculations are performed with a fixed grid.

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