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# local regularity analysis of market index for the 2008 economical crisis

# regularidad local del mercado de índices para la crisis económica de 2008

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**Abstract**

There is evidence that signals from financial markets, such as stock indices, interest rates or commodities, have a multifractal nature. In recent years, many efforts have been made to relate the inefficiency of markets with the multifractal characteristics of this corresponding signals. These characteristics are summarized in the knowledge of the *spectrum of singularities* or *multifractal spectrum*

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that relates to the set of singular points of the signal with its corresponding Hausdorff dimension. The novel approach proposed in this paper, to study the dynamics of financial markets, is to analyze the evolution of the set of singular points or *Hölder exponents* of the series of exchanges, measured daily. We examined the "logarithmic returns" of stock indices from 9 countries in developed markets and 12 belonging to emerging markets from February 2006 to March 2009. The analysis reveals that the temporal variation of the local Hölder exponent point reflects the evolution of the crisis and identifies the historical events which have occurred during this phenomenon, from the minimum values of the Hölder exponent.

**Keywords:** Local Regularity, Pointwise Hölder Exponent, Wavelet Analysis, Stock Market Indices.

**Resumen**

Existe evidencia de que señales provenientes de los mercados financieros, tales como índices bursátiles, tasas de interés, variaciones de precios de productos básicos, tienen naturaleza multifractal. En los últimos años se han hecho esfuerzos para relacionar la ineficiencia de los mercados con las características multifractales de sus correspondientes señales. Estas características se resumen en el conocimiento del *espectro de singularidades* o *espectro multifractal* que relaciona al conjunto de puntos singulares de la señal con su correspondiente dimensión de Hausdorff. La novedosa aproximación que se propone en este trabajo, para el estudio de la dinámica de los mercados financieros, es el estudio de la evolución de los puntos singulares o *exponentes Hölder* locales de las series de índices bursátiles, medidos diariamente. Se analizaron los "retornos logarítmicos" de los índices bursátiles de 9 países pertenecientes a mercados desarrollados y 12 pertenecientes a mercados emergentes, desde febrero de 2006 hasta marzo de 2009. El análisis revela que la variación temporal del exponente Hölder puntual refleja la evolución de la crisis y detecta los eventos históricos que se desarrollaron durante este fenómeno, a partir de los valores mínimos del exponente Hölder puntual.

**Palabras clave:** Regularidad Local, Exponente Hölder Puntual, Análisis Wavelet, Wavelet Leaders, Indices Bursátiles

**Mathematics Subject Classification:** 65T60, 94A12, 26A16, 37M10.

# 1 Introduction

The multifractal nature of empirical data have been shown in financial markets, such as stock market indices, foreign exchange markets, commodities, traded volumes and interest rates. Many efforts have been made to adapt the multifractal formalism to the dynamics of the financial variables. In the last years, many authors provide evidence that the multifractality degree for a broad range of stock markets is associated with the stage of their development. To evaluate, numerically, the multifractality degree (MD), the authors employ different ways to the estimation of multifractal spectrum, such as the *Multifractal Detrended Fluctuation Analysis* (MFDFA) [5, 14].

Another approach is to study the local regularity of the signals. Local regularity analysis is useful in many fields, such as fluid mechanics, PDE theory or signal and image processing [1]. Different quantifiers have been proposed to measure the local regularity of a function [11, 2, 12]; the simplest one is the pointwise Hölder exponent estimated at each point where a locally bounded function is defined. A highly irregular point in a function is characterized by a lower value of Hölder exponent, while the smooth portions of a function have higher exponents. This exponent is a useful tool in signal and image processing which is used to detect contours in images, perform data interpolation, denoise signals and images, among other applications [7, 6]. In addition, the statistical distribution of the Hölder exponents is used to characterize natural signals through multifractal analysis. There are several techniques to estimate numerically the pointwise Hölder exponent: in this work we used the *Wavelet Leaders method*, formulated by S. Jaffard, which is effective in the estimation, [4].

## 2 Local regularity and wavelet leaders

### 2.1 Pointwise Hölder regularity

Singularities and irregular structures often carry essential information in a signal or image. For example, in image processing, the contour of objects are related with discontinuities and singularities in the image. To characterize these localized singular structures it is necessary to quantify the local regularity of a signal $f(t)$. An alternative is to study the pointwise Hölder exponent which quantifies how "ruguos" or spiky is the graph of a function; a low pointwise Hölder exponent value indicates an irregular point while the smooth portions of a function have higher exponents.

The pointwise Hölder exponent is defined, in each $x_0 \in R$ where a locally bounded function is defined, as

$$H_f(x_0) = \sup_{0 \leq \alpha < +\infty} \{\alpha : f \in C^{\alpha}(x_0)\}, \tag{1}$$

where a function $f$ is in the class $C^{\alpha}(x_0)$ if there exists $C > 0$ and a polynomial $P_{x_0}(x)$ of degree less than $\alpha$ such that $|f(x) - P_{x_0}(x)| <$

$C\,|x - x_0|^{\alpha}$ near the point $x_0$. In the case $\alpha = 0$ we adopt, for convention, that the null polynomial has degree $-\infty$ and $|f(x)| < C$.

If $0 < H_f(x_0) < 1$ then $f$ is not differentiable and $P_{x_0}(x) = f(x_0)$. For example, the functions $f(x) = |x|^{\alpha}$ and $f(x) = |x|^{\alpha} \sin\left(\frac{\omega}{|x|^{\beta}}\right)$ continuously extended, with $0 < \alpha \leq 1$, $\omega >> 1$, $\beta > 0$ have $H_f(0) = \alpha$ and $H_f(x) = +\infty$ otherwise. This fact shows that the pointwise Hölder exponent captures a singularity in a regular environment. However we note that both functions having the same regularity exhibit different singularity structures; for the first one $x_0$ is a cusp point and for the other we can see a local oscillating behavior of the graph at $x_0$, known as a *chirp-like* singularity.

Furthermore, the function $f(x) = |x|^{\alpha}\, 1_Q$, where $1_Q$ is the characteristic function over the rational number set, has $H_f(0) = \alpha$ and $H_f(x) = 0$ otherwise. Then, the pointwise Hölder exponent also detects a regular point in an irregular environment.

There are several techniques to estimate numerically the pointwise Hölder exponent; among them heuristic methods as those presented in [8, 13]. On the other hand, the study of the amplitude of wavelet coefficients reveals the signal regularity, among these methodologies is the Wavelet Leaders method which is very effective in the estimation.

At last we recall that the Hölder pointwise exponent is the usual and most preferable but not the unique option to describe the regularity. Other exponents and parameters can be used, we refer to [11, 12] for specific details.

### 2.2 Wavelet tools

The wavelet transform is used to represent signals in both the time and frequency domains. The wavelet representation provides precise measurements regarding when and what degree of transient and singularity events occur and when and how the frequency content of a signal waveform changes over time. This is achieved by using a family of functions generated from a single function (a basic wavelet) by the operation of scaling (stretching or shrinking the basic wavelet) and translation (moving the basic wavelet to different time positions at any scale without changing its shape).

In this way, a time-frequency decomposition of the signal is performed, at each scale, corresponding to a given frequency band, and each time position, the basic wavelet function is correlated with the shape of the waveform at that position. This correlation, known as a wavelet coefficient, measures how much of the basic wavelet, at that scale and position, is included in the waveform. More precisely, a basic wavelet is a quickly vanishing oscillating function with zero average, well-localizated in time and frecuency and a wavelet family $\{\psi_{a,b}\}$ is the set of elemental functions generated by the dilation and translation of a unique admissible mother wavelet $\psi(x)$

$$\psi_{a,b}(x) = a^{-1/2}\psi\left(\frac{x-b}{a}\right), \tag{2}$$

where $a, b \in \mathbb{R}$, $a > 0$ are the dilation and translation factors. If $\psi(x)$ is a real wavelet belonging to the space of signals having a finite energy $L^2(\mathbb{R})$ (the square integrable functions), the *continuous wavelet transform* of $f \in L^2(\mathbb{R})$ at scale $a$ and time $b$ is:

$$Wf(a,b) = a^{-1/2}\int_{-\infty}^{+\infty} f(x)\psi\left(\frac{x-b}{a}\right)dx = \langle f, \psi_{a,b}\rangle \tag{3}$$

and it measures the variation of $f$ in a neighborhood of $b$ proportional to $a$. The information displayed at closely space scales or at closely spaced time points is highly correlated. In consequence, the *continuous wavelet transform* provides a redundant representation of the signal under analysis. In this view, the *discrete wavelet transform* is defined, giving a non redundant, highly efficient wavelet representation that can be implemented with a simple recursive filter scheme and the original signal reconstruction can be obtained by an inverse filtering operation. Then for the discrete set of parameters, $a_j = 2^{-j}$ and $b_{j,k} = 2^{-j}k$ with $j, k \in \mathbb{Z}$ and an appropriated election for the wavelet mother $\psi(x)$, the family $\mathcal{F} = \left\{\psi_{|,\|}(\S) = \in^{|/\in}\psi(\in^{|}\S - \|)\right\}$ constitutes an orthonormal wavelet basis of $L^2(\mathbb{R})$. Such a function $\psi(x)$ exists, see [11, 9] for more details. Each scale $a_j = 2^{-j}$ is related with a given frequency band and $j$ is known as the resolution level. Then, the signal $f \in L^2(\mathbb{R})$ can be recovered by:

$$f(x) = \sum_{j\in Z}\sum_{k\in Z} C_{j,k}\psi_{j,k}(x) \tag{4}$$

in the $L^2(\mathbb{R})$ sense, where the wavelet inner-product coefficients

$$C_{j,k} = \langle f, \psi_{j,k}\rangle = \int_{-\infty}^{+\infty} f(x)\psi_{j,k}(x)dx \tag{5}$$

are the wavelet coefficients. These wavelet coefficients provide full information in a simple way and a direct estimation of local energies at the different relevant scales, furthermore the amplitude of coefficients reveals the signal regularity. The information can be organized into multiresolution signal approximations which were formulated by Mallat and Meyer

[9], who were inspired by original ideas developed in computer vision to analyze images at several resolutions. The properties of orthogonal wavelets and the multiresolution scheme brought to light a link with filter banks and a fast wavelet transform algorithm decomposing signals of $M$ samples with $O(M)$ operations.

In the present study, an orthogonal decimated discrete wavelet transform is applied and the sequences of wavelet coefficients from different resolution levels information is organized in a hierarchical scheme, [10]. Among several alternatives orthogonal $\beta$-cubic spline functions are used as mother wavelets in this paper. They combine in suitable proportion smoothness with numerical advantages.

## 2.3 Wavelet leaders

The notion of wavelet leaders were introduced by Jaffard in [3], finding a formula which yields the upper box dimension of a graph of a function. Then, in 2004, he gives a new characterization of the local regularity using the wavelet leaders [4].

In the context of studying the local regularity additional properties are required for the wavelet mother $\psi$. More precisely, we suppose that the admissible wavelet mother $\psi$ is $C^r$, $r \in N$, with derivatives that have a fast decay which implies that $\psi$ has $r$ vanishing moments, that is,

$$\int_{-\infty}^{+\infty} x^k \psi(x) dx = 0 \quad for \quad 0 \leq k < r. \tag{6}$$

To measure the local regularity of a signal, vanishing moments are crucial. If the wavelet has $r$ vanishing moments, the wavelet transform can be interpreted as a multiscale differential operator of order $r$. This yields a first relation between the differentiability of $f$ and its wavelet transform decay at fine scales [10]. Following the formula 4, for $f$ a signal in the space of signals having a finite energy $L^2(\mathbb{R})$,

$$f(x) = \sum_{j\in Z} \sum_{k\in Z} c_{j,k} \psi(2^j x - k) \tag{7}$$

where $c_{j,k} = 2^{j/2} \langle f, \psi_{j,k} \rangle$ are the wavelet coefficients of $f$ instead the usual $C_{j,k}$. There is a direct correlation between the wavelet coefficients $c_{j,k}$ and the pointwise Hölder regularity. If $f$ is in the class $C^\alpha(x_0)$, it is proved in [3] that the wavelet coefficients of $f$ satisfy, for all $j \geq 0$,

$$|c_{j,k}| \leq C\ 2^{-j\alpha}(1 + |2^j x_0 - k|)^\alpha \tag{8}$$

for some constant $C$. Moreover, when $f$ has a non-oscillating singularity in $x_0$, like a cusp point, the significance coefficients are "localized" near the point $x_0$ and

$$|c_{j,k}| \approx C\ 2^{-j\alpha}. \tag{9}$$

But this is not the case when the function has oscillating singularities, like a chirp. Then, the maxima coefficients may be "placed" far from the singular point and the last property fails. In this view, Jaffard gives a new formulation for this property, characterizing the local regularity in terms of the local suprema of the wavelet coefficients, the *wavelet leaders*.

We can suppose that $\psi$ is essentially localized on the interval $[0,1]$ and then $\psi_{j,k}$ is localized on the dyadic interval $I_{j,k} = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right)$ which means that $c_{j,k}$, the wavelet coefficient of $f$ in $I_{j,k}$, has information related to this interval. Then, *wavelet leaders* of $f$, are defined as follows,

$$d_{j,k} = \sup_{I_{l,h} \subset 3I_{j,k}} |c_{l,h}| \tag{10}$$

where $3I_{j,k} = \left[\frac{k-1}{2^j}, \frac{k+2}{2^j}\right)$ is the dilated interval. We denote $I_j(x_0)$ the unique dyadic interval $I_{j,k}$ containing $x_0 \in R$ for the level $j$. Then the wavelet leader for $x_0$ in the level $j$ is defined as

$$d_j(x_0) = \sup_{I_{l,h} \subset 3I_j(x_0)} |c_{l,h}|\,. \tag{11}$$

Then, concentrating the wavelet coefficients information in the wavelet leaders, Jaffard proved the following general result characterizing the "leaders" coefficients decay [4].

Let $f$ be a locally bounded function in $C^\alpha(x_0), \alpha > 0$. Then for all $j > 0$,

$$d_j(x_0) \leq C\ 2^{-j\alpha} \tag{12}$$

for some constant $C$. Furthermore, if $f$ is uniformly Hölder, the pointwise Hölder exponent of $f$ can be computed using

$$H_f(x_0) = \liminf_{j \to +\infty} \frac{\log(d_j(x_0))}{\log(2^{-j})}. \tag{13}$$

This property is independent of the wavelet mother election as long as $\psi$ has the required conditions and $r$ vanishing moments with $r > \alpha$. Then, from the sample values of a signal and using this one result, the pointwise Hölder exponent can be estimated numerically by linear regression.

## 3 Results and conclusions

We study daily data beginning in February 22, 2006 and ending in March 1, 2009 from the Bloomberg database, with 789 observations. All country indices were studied for the same time period. We analyzed 9 developed (Australia, Canada, France, Germany, Hong Kong, Japan, United Kingdom and United States) and 12 emerging stock markets (Argentina, Brazil, Chile, Mexico, China, India, Korea, Malaysia, Philippines, Taiwan, Russia and South Africa). This classification is obtained following the *Morgan Stanley Capital Index* methodology to define developed and emerging stock markets.
Let $x(t)$ be the equity index of a stock on a time $t$, the equity index returns, $rt$, are calculated as its logarithmic difference, $rt = log(x(t+1)/x(t))$.

Applying the formula 13 to $rt$ series we obtain the pointwise Hölder exponent(PHE) for every country. We will display the results of four paradigmatic examples, two from emerging markets: Brazil and China and two from developed markets: Japan and U. S. The corresponding indices for these countries are: IBOV (Brazil Bovespa Stock Index) for Brazil, SHCOMP (Shangai Stock Exchange Composite Index) for China, TPX (Tokio Stock Price Index) for Japan and SPX (S&P 500 Index) for U. S.

We note on the figures some important dates in the evolution of the crisis, these are: a) August 12, 2007 which corresponds to the subprime mortgage crisis that spans the financial markets, mainly, from Thursday 9 August 2007, although its origins go back to previous years; b) On Monday 21 January, produced a historic stock market crash, dragging bags to all the world except the U. S., which is closed by being festive; c) In May 2008, there is an apparent recovery of markets and reaches its maximum on 22 May and d) Period of maximum intensity of the crisis, between September 2 and December 15, 2008: on September 22 the U.S. Federal Reserve approved the conversion of the last two independent investment banks remaining, Goldman Sachs and Morgan Stanley into commercial banks, allowing greater control and regulation by the authorities. This is just a banking model with 80-year history and the largest bank failure in the history of the United States.

Values of the PHE near zero indicate an irregularity in the signal, to which it may be interpreted as an indicator to of financial market instability. So, from observing the graphs, we can conclude that:

- The analysis reveals that the temporal variations of PHE reflects the crisis evolution, detects historical events and precursors of the phenomenon from the minimum values of PHE.

- Temporal evolution of the PHE provides timely information about the phenomenon that is not evident in the data series. An example of this is seen in the evolution BOVESPA index which shows a rising trend after the subprime mortgage crisis, while values of exponents hold, on average, the same value indicating that the dynamics of the system remains the same precursor characteristics of this crisis,(see Fig. 1 and 2). Also in the evolution of TPX from Japan and SPX from U. S., (see Figs. 5 to 8).

- The Hölder exponent time evolution of the China SHCOMP index has a stable behavior with a moderate decline during the most critical period of the financial crisis, which indicates that the global economic crisis had more influence moderate in this market, coinciding with the events, (see Figs. 3 and 4).

In summary, the method PHE is an interesting alternative for studying the transitions of a signal, through the change of the local regularity of the data. We hope that in future works we'll find new applications for this methodology.


Figure 1: *Daily series of the Brazil Bovespa Stock Index (IBOV)*

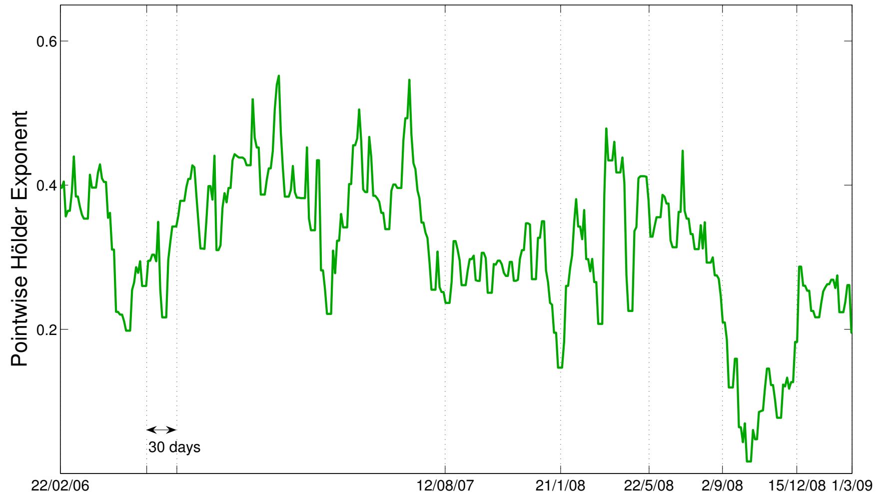


Figure 2: *Pointwise Hölder exponent of the Brazil Bovespa Stock Index*


Figure 3: *Daily series of the China Shangai Stock Exchange Composite Index (SHCOMP)*

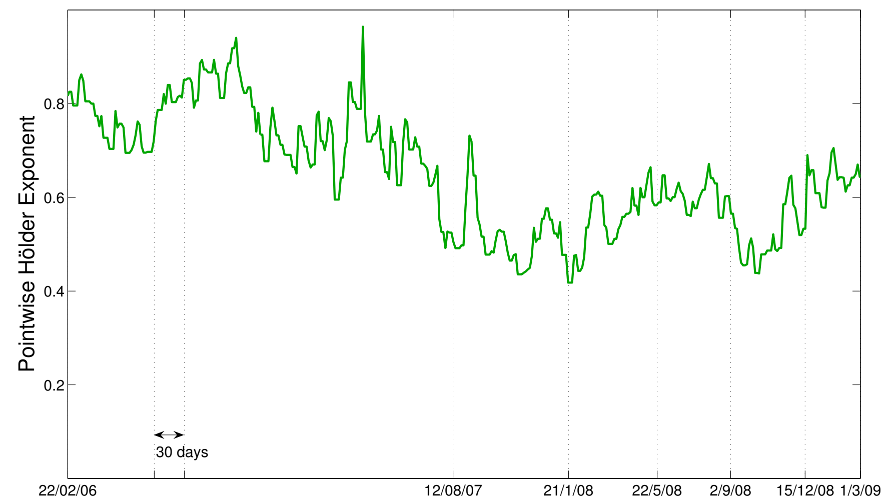


Figure 4: *Pointwise Hölder exponent of the Shangai Stock Exchange Composite Index*

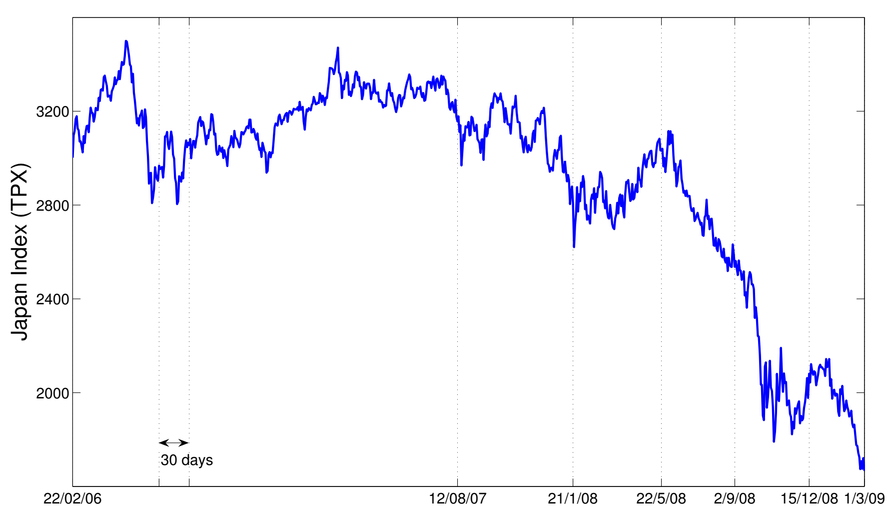


Figure 5: *Daily series of the Japan Tokio Stock Price Index (TPX)*

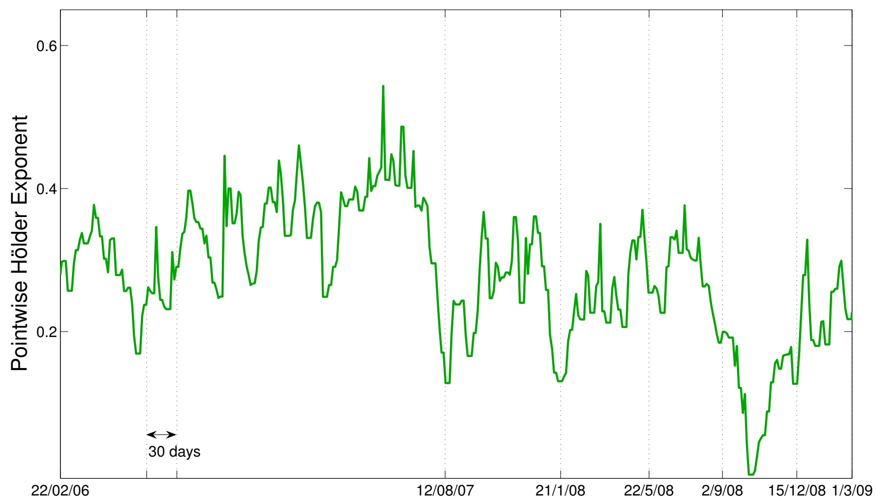


Figure 6: *Pointwise Hölder exponent of the Tokio Stock Price Index*

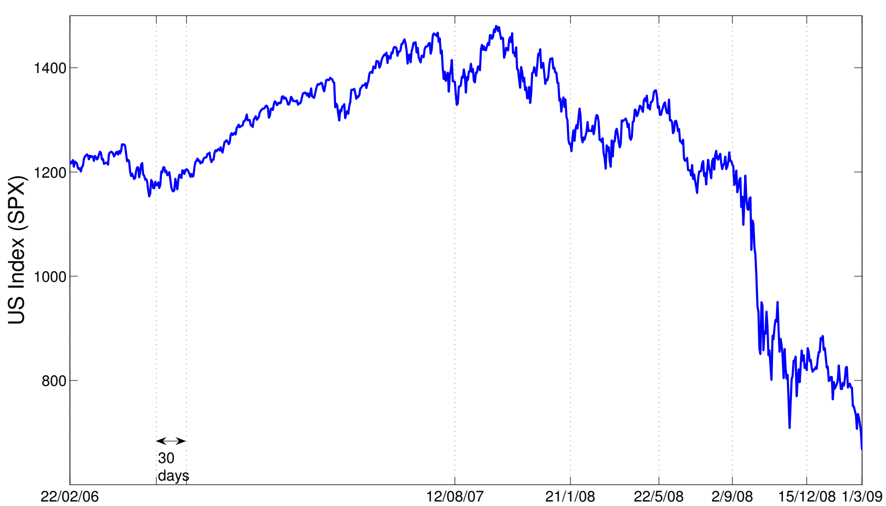


Figure 7: *Daily series of the United Stated S&P 500 Index (SPX)*

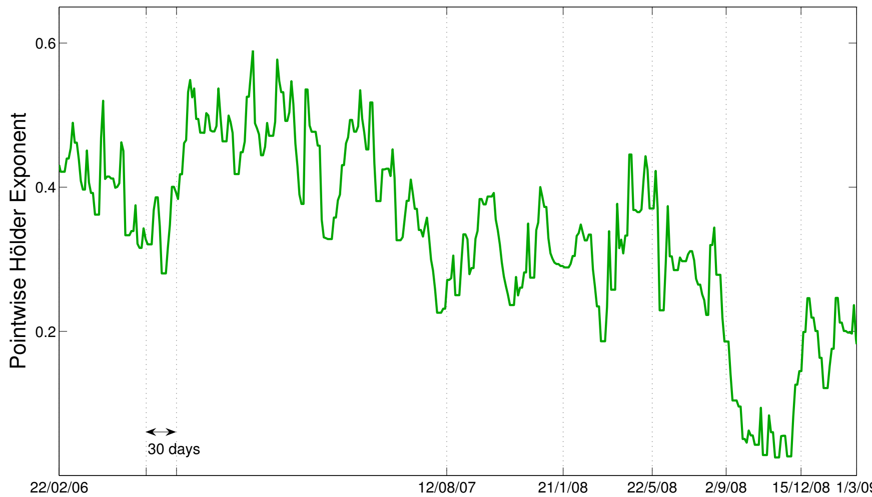


Figure 8: *Pointwise Hölder exponent of the S&P 500 Index*

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