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# On the right commutativity of loop rings

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**Abstract:** It is proved that extra loops and central loops are conjugated closed, consequently they are G-loops. It is also proved that extra loops and central loops are M- loops.

**Subject headings**: loops, G-loops, M-loops

**Resumen:** Se prueba que anillos extra y anillos centrales son conjugadamente cerrados, consecuentemente ellos son anillos G. También se prueba que anillos extra y anillos centrales son anillos M.

**Encabezados de materia**: anillos, anillos G, anillos M

## 1. Introduction

In this paper we introduce the notion of right commutativity in loop rings study and obtain some interesting results about them. For more about loops in general please refer [1].

**Definition:**

( [1] Bruck ) A non empty set $L$ is called a loop if on $L$ is defined a binary operation '.' satisfying the following conditions:

( i ) For all $a, b \in L,\ a.b \in L$

( ii ) For every pair $a, b \in L$ there exists a unique $x$ in $L$ such that $ax = b$ and a unique $y$ in $L$ such that $ya = b$.

( iii ) For every $a \in L$; $a.e = e.a = a$ where $e$ in $L$ is called the identity with respect to '.'.

The operation '.' defined on $L$ in general is not associative. For more about loops in general please refer [l].

**Definition 1**

Let $L$ be a loop and $R$ a commutative ring with unity. The loop ring; $RL$ of the loop $L$ over the ring $R$ consists of all finite formal sums of the form $\sum \alpha(m_i)\, m_i$, $m_i \in L$ and $\alpha(m_i) \in R$ satisfying the following condition:

(1) $\sum \alpha(m_i)\, m_i = \sum \beta(m_i)\, m_i \iff \alpha(m_i) = \beta(m_i)\ \forall\ m_i \in L$

(2) $\sum \alpha(m)\, m + \sum \beta(m)\, m = \sum(\alpha(m) + \beta(m))m\ \forall\ m \in L$

(3) $(\sum \alpha(m)\, m)\,(\sum \beta(m)\, m) = \sum_{xy=n} \gamma(n)\, n,\ \textit{where}\ \gamma(n) = \sum_{xy=n} \alpha(x)\beta(y)$

(iv) Since $1 \in L$, we have $R \subset RL$ by the natural embedding $r \longrightarrow r.1$ where 1 is the identity of $L$.

As we take only finite formal sums without loss in generality we can assume $\sum_{m \subset L} \alpha(m)\, m = \sum_{i=1}^{n} \alpha_i\, m_i$ where $n$ is a finite positive number. Further as $l \in R$ we have $l.L \subseteq RL$. Clearly loop rings are non associative structures with respect to the operation of multiplication.

**Definition 2**

Let $RL$ be the loop ring of the loop $L$ over a commutative ring $R$ with unity. If for every triple $\gamma, \alpha, \beta \in RL$ we have $(\alpha\beta)$ or $\alpha(\beta\gamma) = \alpha(\gamma\beta)$, or, $\alpha\gamma)\beta$ then we call the loop ring $RL$ to be strongly right commutative; ie. for every triple $\gamma, \alpha, \beta \in RL$ we must have

( i ) $\alpha(\beta\gamma) = \alpha(\gamma\beta)$, or, $\alpha\gamma)\beta$, or,

( ii ) $(\alpha\beta)\gamma = (\alpha\gamma)\beta$, or, $\alpha(\gamma\beta)$, or,

then we say $RL$ is strongly right commutative. The term or in general is not used in the mutually exclusive sense.

**Proposition 3**

Every strongly right commutative loop ring $RL$ is commutative.

**Proof**

Obvious from the fact for every triple $\gamma, \alpha, \beta \in RL$ we have $\alpha\beta = \beta\alpha$, $\alpha\gamma = \gamma\alpha$ and $\beta\gamma = \gamma\beta$ where one of $\alpha$, or, $\beta$, or $\gamma$ is assumed to be the unit of the loop ring $RL$. Hence every pair is commutative.

**Proposition 4**

Let $RL$ be a strongly right commutative loop ring of the loop $L$ over a ring $R$. Then the loop $L$ is commutative.

**Proof**

Obvious from the fact $RL$ is commutative so $L$ $R$L must also be commutative.

**Remark**

On similar lines one can define strongly left commutative loop rings as $(\alpha\beta)\gamma = (\beta\alpha)\gamma$, or, $\beta(\alpha\gamma)$ for every triple $\gamma, alpha, beta \in L$.

**Definition 5**

Let $RL$ be the loop ring of the loop $L$ over the ring $R$. If for every pair of elements $\alpha, \beta \in R$ there exists an element $\gamma \epsilon R$ 0, 1 such that $\gamma(\alpha\beta)$, or, $(\gamma\alpha)\beta = (\gamma\beta)\alpha$, or, $\gamma(\beta\alpha)$ then the loop ring $RL$ is said to be right commutative.

**Remark**

( i ) $\gamma(\alpha\beta) = \gamma(\beta\alpha)$, or, $(\gamma\beta)\alpha$ or

( ii ) $(\gamma\alpha)\beta) = \gamma(\beta\alpha)$, or, $(\gamma\beta)\alpha$. Similarly one can define left commutative loop rings.

**Proposition 6**

Let $R$ be a commutative ring with unity and $L$ a non commutative loop . If the loop ring $RL$ is a right commutative loop ring then $RL$ has non trivial divisor of zero.

**Proof**

Clearly since $RL$ is a non commutative loop ring we have $\alpha\beta = \beta\alpha$ for some pair of elements $\alpha, \beta \in L$. If in addition $RL$ is right commutative we have $a\gamma \in RL$ ( $\gamma \neq 0$, *and* $\gamma \neq 1$) such that $\gamma(\alpha\beta) = \gamma(\beta\alpha)$ so that $\gamma(\alpha\beta - \beta\alpha) = 0$. Hence the claim.

**Proposition 7**

Let $RL$ be the loop ring of the loop $L$ over the ring $R$. If $L$ is a strongly right commutative ring then $RL$ is a right commutative ring.

**Proof**

Obvious from the definition ; if $RL$ is strongly right commutative then it is right commutative.

**Theorem 8**

Let $RL$ be a right commutative loop ring having no zero divisor. Then $RL$ is commutative.

**Proof**

If $Rl$ is right commutative loop ring then $\gamma(\alpha\beta - \beta\alpha) = 0$, $\gamma \neq 0$, or, 1 have $\alpha\beta = \beta\alpha$ if $RL$ has no divisor of zero. Hence $RL$ must be commutative.

**Definition 9**

Let $RL$ be the loop ring. $RL$ is said to be weakly right commutative if for every pair of elements $\alpha$, $\beta \epsilon RL$ there exists $a\gamma \epsilon RL$ such that

( i ) $\gamma(\alpha\beta) = (\gamma\beta)\alpha$, or, $\gamma(\beta\alpha)$

( ii) $(\gamma\alpha)\beta = \gamma(\beta\alpha)$, or, $(\gamma\beta)\alpha)$.

**Theorem 10**

Let $RL$ be a right commutative loop ring then $RL$ is weakly right commutative.

**Proof**

Obvious from the definition.

**Remark**

Converse is not true in general.

**Theorem 11**

Every loop ring $RL$ is weakly right commutative.

**Proof**

Clearly for every pair $\alpha$, $\beta \epsilon RL$ one can choose $\gamma = 0$ in $RL$ so that $\gamma.\alpha\beta = \gamma\beta\alpha = 0$. Hence the claim.

**Definition 12**

Let $RL$ be the loop ring of the loop $L$ over $R$. $RL$ is said to be strictly right commutative if for every pair of elements $\alpha$, $\beta \epsilon RL$ we have $a\gamma \epsilon RL \setminus 0, 1$ such that $\gamma(\alpha\beta) = \gamma(\beta\alpha)$, or, $(\gamma\alpha)\beta = (\gamma\beta)\alpha$.

**Theorem 13**

Every strictly right commutative loop ring is right commutative.

**Proof**

Obvious.

**Theorem 14**

Every strongly right commutative loop ring need not be strictly right commutative ring.

**Proof**

By the very definition of both notions the result is obvious.

**Problem**

Characterize those loop rings which are

(i) strictly right commutative;

(ii) right commutative;

(iii) strongly right commutative.

## 2. References

[1]) R.H. Bruck, A survey of Binary systems, Springer Verlag, 1968.