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Semi-Orthogonalities of a Class of Gauss Hypergeometric Functions

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Abstract: We present three semi-orthogonalities for a class of Gauss hypergeometric functions. We further employ the semi-orthogonalities to generate a theory concerning finite series expansion involving our hypergeometric functions..

Subject headings: Hypergeometric functions, Saalschutz's theorem, Semi-orthogonalities

Resumen: Se presentan tres semi ortogonalidades de una clase de funciones hipergeométricas. Adicionalmente, se utilizan las semi ortogonalidades para generar una teoría relacionada con expansiones finitas en serie que envuelven a este tipo de funciones hipergeométricas.

Encabezados de materia: funciones hipergeométricas, teorema de Saalschutz, semi ortogonalidades

1. Introduction

The object of this paper is to present three semi-orthogonalities of a class of Gauss hypergeometric functions:

$$ {}_2F_1\left(-m, b; 1-a+m; \frac{1}{x}\right) = \sum_{n=0}^{m} \frac{(-m)_n (b)_n}{(1-a+m)n!} \left(\frac{1}{x}\right)^n, \quad m = 0, 1, 2, \cdots \tag{1.1} $$

We further apply semi-orthogonalities to develop a theory regarding the finite series expansion involving our hypergeometric functions. The following three integrals are required in the proofs.

1.1. First Integral

$$ \int_0^1 x^{h+n}(1-x)^{a+b-2n-1} {}_2F_1\left(-m, b; 1-a+m; \frac{1}{x}\right) dx = $$

$$ = (-1)^n \frac{(1+b+h)_n \Gamma(1+h)\Gamma(a+b-n)}{(1-a+n)_n \Gamma(a+b+h-n+1)}, \quad n = 0, 1, 2, \cdots \tag{1.2} $$

where $Re(h) > -1$, $Re(a+b) > n$, $Re(b+h) > -1$.

Proof

The integral (1.2) is established by expressing the hypergeometric functions in the integrand as its series representation (1.1) interchanging the order of integration and summation, evaluating the resulting integral with the help of the Beta integral (1,p.9):

$$ \int_0^1 x^{p-1}(1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p > 0, \ q > 0 \tag{1.3} $$

and simplifying, we get

$$ = \frac{\Gamma(1+h+n)\Gamma(a+b-2n)}{\Gamma(a+b+h-n+1)} {}_3F_2\left(-n, b, n-a-b-h; 1, 1-a+m, -h-n\right) \tag{1.4} $$

It can easily be verified that the generalized hypergeometric series (1.4) is Saalschutzian. Therefore, applying the Saalschuz's theorem (1,p.188, (3)):

$$ {}_3F_2\left(-n, a, b; 1; c, 1-c+a+b-n\right) = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} \tag{1.5} $$

and simplifying with the help of the following form of the formula (1,p.3,(4)):

$$\Gamma(1+a-n) = \frac{(-1)^n \Gamma(1+a)}{(-a)_n} \tag{1.6}$$

the right hand side of (1.2) is obtained.

1.2. Second Integral

$$\int_1^\infty x^{h+n}(x-1)^{a+b-2n-1} {}_2F_1\left(-n, b; 1-a+n; \frac{1}{x}\right) dx =$$

$$= \frac{(1+b+h)_n \, \Gamma(n-a-b-h)\, \Gamma(a+b-n)}{(1-a+n)_n \, \Gamma(-h)}, \quad n = 0, 1, 2, \cdots \tag{1.7}$$

where $Re(a+b+h) < 0$, $Re(a+b) > n$, $Re(b+h) > -1$.

Proof

The integral (1.7) is established on following the technique employed to establish (1.2) except instead of (1.3) using the integral (2,p.201, (7)

$$\int_1^\infty x^{-v}(x-1)^{w-1} dx = \frac{\Gamma(v-w)\Gamma(w)}{\Gamma(v)}, \quad Re(v) > Re(w) > 0 \tag{1.8}$$

1.3. Third Integral

$$\int_0^\infty x^{h+n}(1+x)^{a-b-2n-1} {}_2F_1\left(-n, b; 1-a+n; \frac{1}{x}\right) dx =$$

$$= \frac{(1+b+h)_n \, \Gamma(n-a-b-h)\, \Gamma(1+h)}{(1-a+n)_n \, \Gamma(1-a-b+n)}, \quad n = 0, 1, 2, \cdots \tag{1.9}$$

where $Re(h) > -1$, $Re(a+b+h) < 0$, $Re(b+h) > -1$.

Proof

The integral (1. 9) is established on following the technique employed to establish (1. 2) except instead of (1. 3), using the integral (2,p. 233,(8)):

$$\int_0^\infty x^{v-1}(1+x)^{-w} dx = \frac{\Gamma(v)\Gamma(w-v)}{\Gamma(w)}, \quad Re(w) > Re(v) > 0 \tag{1.10}$$

2. SEMI-ORTHOGONALITIES

The semi-orthogonalities to be established are

$$\int_0^\infty x^{-1-b+n}(1-x)^{a+b-2n-1}\,{}_2F_1\left(-m,b;1-a+m;\frac{1}{x}\right)\,{}_2F_1\left(-n,b;1-a+n;\frac{1}{x}\right)dx =$$

$$= \begin{cases} 0 & if\ m<n \\ \frac{(b)_n\, n!\,\Gamma(-b)\,\Gamma(a+b-n)}{(1-a+n)_n\,(1+b)_n\,\Gamma(a-n)} & if\ m=n \end{cases} \tag{2.1}$$

where $Re(b) > -1-n$, Re(b) ¡ -1-n,Re(a+b) ¿0.

$$\int_0^\infty x^{-1-b+n}(x-1)^{a+b-2n-1}\,{}_2F_1\left(-m,b;1-a+m;\frac{1}{x}\right)\,{}_2F_1\left(-n,b;1-a+n;\frac{1}{x}\right)dx =$$

$$= \begin{cases} 0 & if\ m<n \\ \frac{(b)_n n!\Gamma(a+b-n)\Gamma(1-a+n)}{(1-a+n)_n\Gamma(1+b+n)} & if\ m=n \end{cases} \tag{2.2}$$

where $Re(a+b) > 0$, Re(a) ¡ -n ,Re(b) ¿ -n,Re(b) ¡ -m.

$$\int_0^\infty x^{-1-b+n}(1+x)^{a+b-2n-1}\,{}_2F_1\left(-m,b;1-a+m;\frac{-1}{x}\right)\,{}_2F_1\left(-n,b;1-a+n;\frac{-1}{x}\right)dx =$$

$$= \begin{cases} 0 & if\ m<n \\ \frac{(b)_n n!\Gamma(1-a+n)\Gamma(-b)}{(1-a+n)_n\Gamma(1-a-b+n)\Gamma(1+b)_n} & if\ m=n \end{cases} \tag{2.3}$$

where $Re(a) < 1$m, $Re(b) < -m$, $Re(b) > -n$.

Proof

To prove (2.1), we write its left hand side in the form:

$$\sum_{r=0}^{m} \frac{(-m)_r(b)_r}{r!(1-a+m)_r} \int_0^1 x^{-1-b+n-r}(1-x)^{a+b-2n-1}\,{}_2F_1\left(-n,b;1-a+n;\frac{1}{x}\right) \tag{2.4}$$

On evaluating the integral in (2.4) with the help of (1.2), it reduces to the form

$$\sum_{r=0}^{m} \frac{(-m)_r(b)_r}{r!(1-a+m)_r}\,\frac{\Gamma(-b-r)\Gamma(a+b-n)(-r)_n}{(1-a+n)_n\Gamma(a-n-r)} \tag{2.5}$$

If $r < n$, the numerator of (2.5) vanishes, and since r runs from 0 to m, it follows that (2.5) also vanishes, when $m < n$. Now, it is clear that for $m < n$ all terms of (2.5) vanish, which proves (2.1a).

When $m = n$, using the standard result

$$(-r)_n = \begin{cases} \frac{(-1)^n r!}{(r-n)!} & if\ 0 \le n \le r \\ 0 & if\ n > r \end{cases} \tag{2.6}$$

and simplifying, the right hand side of (2.1a)follows from (2.5).

To prove (2.2), first we reduce its left hand side to the form similar to (2.4), then evaluate the last integral with the help of the integral (2.7) to obtain

$$\sum_{r=0}^{m} \frac{(-m)_r(b)_r}{r!(1-a+m)_r} \frac{(-r)_n\Gamma(1-a+r+n)\Gamma(a+b-n)}{(1-a+n)_r\Gamma(1+b+r)} \tag{2.7}$$

We see that (2.7) is of the same form as (2.5). Therefore, for $m < n$ all terms of (2.7) vanish, which proves (2.2a).

When $m = n$, using the standard result (2.6) and simplifying, the right hand side of (2.2b) follows from (2.7).

To prove (2.3), we first reduce its left hand side to the form similar to (2.4), then evaluate the last integral with the held of the integral (1.9) to get

$$\sum_{r=0}^{m}(-1)^r \frac{(-m)_r(b)_r}{r!(1-a+m)_r} \frac{(-r)_n\Gamma(1-a+r+n)\Gamma(-b-r)}{(1-a+n)_r\Gamma(1-a-b+n)} \tag{2.8}$$

Since (2.8) is of the same form as (2.5). Therefore, for $m < n$ all terms of (2.8) vanish, which proves (2.3a).

When $m = n$, using the standard result (2.6) and simplifying, the result (2.3a) follows from (2.8).

Note

On continuing as above, we can find the values of the integrals (2.1), (2.2) and (2.3) for $m = n+1, n+2, n+3, \cdots$.

3. Finite Series Expansion Involving the Hypergeometric Functions

Based on the relations (2.1a) and (2.1b), (2.2a) and (2.2b), and (2.3a) and (2.3b), we can generate a theory concerning the expansion of arbitrary polynomials, or functions in general, in a fine series expansion s of the hypergeometric functions. Specially if $F(x)$, $G(x)$, and $H(x)$ are suitable functions defined for all x, we consider for expansion of the general form

$$F(x) = \sum_{m=0}^{n} A_m \, {}_2F_1\left(-m, b; 1-a+m; \frac{1}{x}\right), \quad 0 \le x \le 1, \quad m \le n \tag{3.1}$$

where the expansion coefficients A_m are given by

$$A_m = \frac{(1-a+m)_m (1+b)_m \Gamma(a-m)}{m!(b)_m \Gamma(-b)\Gamma(a+b-m)}$$

$$\cdot \int_0^1 F(x) x^{-1-b+m} (1-x)^{a+b-2m-1} \, {}_2F_1\left(-m, b; 1-a+m; \frac{1}{x}\right) \tag{3.2}$$

$$G(x) = \sum_{m=0}^{n} B_m \, {}_2F_1\left(-m, b; 1-a+m; \frac{1}{x}\right), \quad 1 < x < \infty, \quad m \le n \tag{3.3}$$

where the expansion coefficients B_m are given by

$$B_m = \frac{(1-a+m)_m \Gamma(1+b+m)}{m!(b)_m \Gamma(a+b-m)\Gamma(1-a+m)}$$

$$\cdot \int_0^1 G(x) \, x^{-1-b+m} \, (x-1)^{a+b-2m-1} \, {}_2F_1\left(-m, b; 1-a+m; \frac{1}{x}\right) \tag{3.4}$$

$$H(x) = \sum_{m=0}^{n} C_m \, {}_2F_1\left(-m, b; 1-a-m; \frac{-1}{x}\right), \quad 0 < x < \infty, \quad m \le n \tag{3.5}$$

where the expansion coefficients C_m are given by

$$C_m = \frac{(1-a+m)_m (1+b)_n \Gamma(1-a-b+m)}{m!(b)_m \Gamma(1-a+m)\Gamma(-b)}$$

$$\int_0^1 H(x) \, x^{-1-b+m} \, (1+x)^{a+b-2m-1} \, {}_2F_1\left(-m, b; 1-a+m; \frac{-1}{x}\right) \tag{3.6}$$

4. References

1) Erdélyi, A.et al. Higher transcendental functions, Vol. 1. McGraw-Hill, New York (1953)

2) Erdélyi, A. et al. Tables of integral transforms, Vol. 2. McGraw-Hill, New York (1954)