Competencies in mathematics education – potentials and challenges What's the point? What's new? What do we gain? What are the pitfalls?[1]

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Abstract[2]

The present paper focuses on characterising what it means for an individual to be mathematically competent in terms that go across – are independent of – mathematical content and educational levels. Based on the Danish KOM project, eight mathematical competencies, which together are meant to constitute mathematical competence, and three forms of overview and judgment concerning mathematics as a discipline are presented. The normative and descriptive uses of this system of competencies are outlined and discussed as are the challenges encountered when putting the competencies to use in different contexts of research and practice.

Key words

competencies, mathematics education, didactics.

Resumen

El presente trabajo se centra en la caracterización de lo que significa para un individuo ser matemáticamente competente en términos –independientes– que atraviesan el contenido matemático y los niveles educativos. Se presentan, basado en el proyecto danés KOM, ocho competencias matemáticas que en conjunto tienen el propósito de constituir la competencia matemática, y tres formas de visión y juicio relativas a las matemáticas como una disciplina. Se describen y analizan los usos normativos y descriptivos de este sistema de competencias, al igual que las dificultades encontradas al utilizar las competencias en diferentes contextos de la investigación y la práctica.

Palabras clave

competencias, Educación Matemática, didáctica

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[2] El resumen y las palabras clave en español fueron agregados por los editores.

1 Introduction

In the mathematics and mathematics education communities we have always invoked notions of what it means for a person at some educational level to be mathematically able. The notions of mathematical ability have been tacit more often than explicit, but we have invoked them nevertheless, when we have assessed, tested, and examined students in schools and universities, and when we have decided whom to hire or promote as mathematics teachers, lecturers or researchers in public or private institutions or to fill mathematics intensive jobs in agencies or companies. When the notions of mathematical ability are tacit they are not subject of critical analysis and discussion amongst stakeholders. So, the notions tend to be taken for granted even when implicit and they are likely to be rather varying because of their implicitness.

In spite of this, from time to time attempts have been made to characterise aspects of mathematical ability, in particular in the goals or outcomes sections of curricula or in the frameworks of international comparative studies such as PISA (OECD, 2003). It appears that the task of characterising mathematical ability is a difficult one. In a historical perspective, it has been dealt with in a variety of mostly implicit ways. In some contexts mathematical ability has been identified with the ability to correctly state facts concerning specific mathematical domains and carry out certain rule-based procedures in routine situations. In other contexts, it has been identified with the ability to solve particular classes of more or less open-ended, purely mathematical problems. In yet other contexts, it has been equated with the ability to put mathematics to use in certain kinds of extra-mathematical domains or situations. Sometimes mathematical ability has been perceived as the ability to reproduce and / or explain proofs of specific mathematical statements, including theorems, within some theoretical framework, e.g. Euclidean geometry. Sometimes it has been seen as the ability to solve novel or open (not to be confused with open-ended) problems or even to prove new theorems so as to reclaim new land for mathematics. These facets of mathematical ability carry different weights at different educational levels and systems, especially as regards schools and tertiary institutions.

The mathematical abilities just outlined are very different in nature and scope. There is a world of difference in outcome as a function of which (combination) description(s) is chosen as the basis for defining and - assessing – mathematical ability. This suggests potential disagreement or outright conflict amongst quarters pleading for different approaches to mathematical ability.

The above remarks suffice to suggest that there is a significant task and a significant challenge in coming to grips with, defining and characterising mathematical ability, mastery, proficiency, competence, or whatever terms we would like to use. In other words, the overarching question that is going to preoccupy us in this paper is *What does it mean for a person to be mathematically competent?*, as competence is the term we prefer to use. Once we have addressed this question we shall turn to the other questions mentioned in the title, *What's the point?*, *What's new?*, *What do we gain?*, and *What are the pitfalls?*

2 What does it mean for a person to be mathematically competent?

In trying to answer this question there two opposite extreme traps that we want to avoid. The first trap is to answer by saying "to be mathematically competent means to know and be able to do mathematics". This answer is, of course, absolutely correct but also almost void, since it is a circular reformulation of the question into a positive statement. (We might say, though, that a little extra is added by pointing to "knowing" and "being able" as two components of being competent, but that's next to trivial.). The trap at the other extreme is to answer by producing an endless list of facts, i.e. concepts, terms, conventions, rules, results, theories etc., which a mathematically competent person has to know, e.g. being able to state or cite, and a similarly endless list of skills that a mathematically competent person has to possess, i.e. methods, procedures, techniques etc. that he or she is able to carry out, e.g. successfully dealing with specific kinds of tasks, including solving specific kinds of problems. Needless to say, the items on such lists are indeed important and relevant ingredients, atoms, in mathematical competence, but since the lists are endless they do not provide comprehensive information about essential features that are common to the items. It often is, as we know, all the trees that make it difficult to see the forest. Moreover, experience shows that when people actually engage in establishing and discussing such lists they soon run into substantial disagreement of what should be on the lists and would should not. If mathematical competence is defined by lists on which mathematically competent people disagree, the lists can hardly be said to capture the essence of mathematical competence. An analogy: If we were to characterise linguistic competence with respect to some language, say English or Portuguese, no one would solve the task by listing all the words and all the grammatical rules you would have to know. Again, not because the words and the rules are unimportant, of course not, but because listing them misses the point.

If we look at the mathematics that is on the agenda at different levels in different institutions and in different educational systems, it immediately becomes clear that the differences between what we see are dramatic, in terms of mathematical and extra-mathematical content, methods, justification of statements, the nature of tasks and activities that students are involved in, the kinds of things students are expected to be able to do, historico-philosophical perspectives on mathematics as a discipline and so forth and so on. The differences are so huge that one may wonder why we dare to use the same name, mathematics, for all this, across levels, curricula, institutions, systems, and countries. This gives rise to a major challenge in attempting to answer our overarching question: We wish to provide the same characterisation of mathematical competence for any educational level, from kindergarten to PhD studies at university, and for any mathematical content, while at the same time avoiding the two extreme traps outlined above, "excessive, hence empty, generality" and "endless detail and atomisation".

Again, the language analogy comes in handy: Being linguistically competent with a given language means to be able to *understand and interpret* what other people *say* and *write* in that language, in a variety of different contexts, genres and registers, as well as being able to *express oneself orally* and *in writing* so as to make oneself understood by others, again, in a variety of different contexts, genres and registers.

In other words, linguistic competence is constituted by four linguistics competen*cies*, irrespective of age, level, institution, and specific content. It goes without saying that what six year olds hear, read, say and write about, by way of their language, is likely to be very far from what a university professor of literature hears, reads, says and writes about, but the fundamental components are, nevertheless, the same. Our project is to identify comparable components – competen*cies* - in mathematical competence. What we are after are sufficiently large *molecules* (polymers), of course built of numerous atomic facts and processes, that constitute mathematical competence.

As a result of work done by myself and a number of close colleagues in the late 1990's and the early 2000s, the Danish so-called *KOM project* directed by me (KOM is an acronym for the Danish counterpart of *Competencies and the Learning of Mathematics*), the report of which was published in 2002 (Niss & Jensen, 2002), put forward *eight mathematical competencies*, which together are the components that are meant to constitute mathematical competen*ce*. In addition we put forward three kinds of *overview and judgement* concerning *mathematics as a discipline*.

The eight competencies are derived from an empirico-theoretical analysis of what a mathematically competent person actually does / is able to do when dealing with mathematics in a broad sense. The method employed in conducting the initial analysis was rather close to the one adopted by Hadamard when he tried to capture the psychology of mathematical invention (Hadamard, 1945): He conducted reflection and introspection of his own research and asked a number of mathematicians a set of questions about theirs. In the present context the analysis also took advantage of numerous studies in mathematics education research on students' problem solving and modelling behaviour. But now to a definition. *By a mathematical competency we understand an individual's insight-based capability to purposefully and successfully deal with situations that (re)present a particular kind of mathematics-laden challenge.*

Now, the overarching capability is to be able to *pose and answer questions within* and *by means of mathematics*. For that to be possible one must master the *language* and *tools* of mathematics. Against this background, the mathematical competencies are organised into overlapping clusters, as follows:

Posing and answering questions within and by means of mathematics:

Mathematical thinking competency

Including *posing questions* that are characteristic of mathematics, and *knowing the* kinds *of answers* that mathematics can offer; relating to *abstraction* and *generalisation;* distinguishing between different *kinds of mathematical statements* such as definitions, assumptions, theorems, conjectures, cases; understanding and handling the *scope* and *limitations* of a given *concept*.

Problem handling competency

Including identifying, posing, and specifying different kinds of mathematical problems – pure or applied; open-ended or closed; solving different kinds of mathematical problems; checking proposed solutions to problems.

Mathematical modelling competency

Including analysing foundations and properties of existing models; decoding such models, i.e. translating and interpreting model elements in terms of the "reality" modelled and assessing the range and validity of models; performing active modelling in a given context, i.e. structuring the domain to be modelled, mathematising the domain, working with(in) the model, including solving the problems it gives rise to, interpreting and validating the model.

Mathematical reasoning competency

Including following and analysing others' justification of claims; devising formal and informal mathematical arguments to justify a mathematical claim; knowing what a mathematical proof is (not), ands how it differs from other kinds of mathematical reasoning, e.g. heuristics.

Mastering the languages and tools of mathematics:

Representation competency

Including understanding and utilising (decoding, interpreting, distinguishing between) different sorts of representations of mathematical objects, phenomena and situations; choosing, translating between and utilising different representations of the same entity, including knowing about their relative strengths and limitation.

Symbols and formalisms competency

Including decoding and interpreting symbolic and formal mathematical language, and understanding its relations to natural language; understanding the nature and rules of formal mathematical systems (both syntax and semantics); handling and manipulating statements and expressions containing symbols and formulae.

Communicating in, with, and about mathematics

Including understanding others' written, visual or oral 'texts', in a variety of linguistic registers, about matters with a mathematical content; expressing oneself, at different levels of theoretical and technical precision, in oral, visual or written form, about such matters.

Mathematical aids and tools competency

Including knowing the existence and properties of various material tools and aids (ICT included) for mathematical activity, as well as their range and limitations; being able to reflectively use such aids and tools.

These competencies do not form a partition of mathematical competence in disjoint subsets. Yest, the competencies are distinct, each which a well-defined core – a "centre of gravity" – but they all overlap. The interrelations amongst the competencies are depicted by the so-called competency flower (please note that the terms are slightly outdated in this picture):

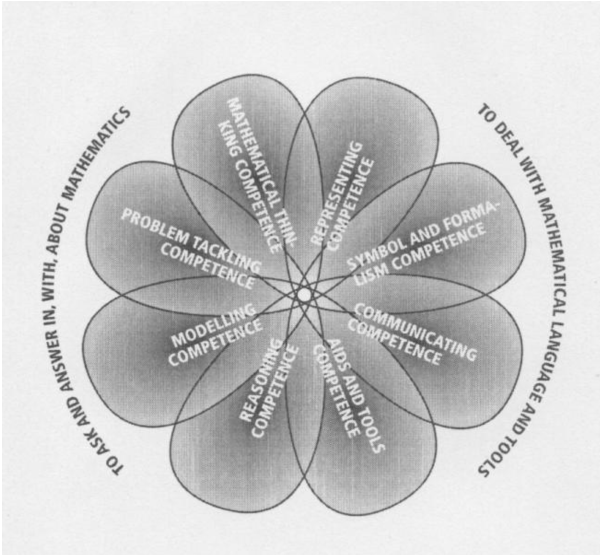

Figure 1: The competency flower.

A few remarks are in order here. Possessing a mathematical competency is not, of course, a 0-1 issue, either you possess this competency or you don't. Competency possession is a continuum unbounded above - the set of positive real numbers may serve as a metaphor – just like the linguistic competency "being able to write". Moreover, as any competency is expressed in dealing with mathematics-laden situations, hence with various kinds of mathematical subject matter and content, possessing and exercising a given competency presupposes knowledge and skills ("atoms") pertaining to the content at issue, exactly as the ability to write an academic essay in English requires knowledge of words and grammar. Finally, even though most of the words (thinking, problem solving, modelling, reasoning, justification, representation, symbols, communication, tools) occurring in the titles of the competencies are of a general nature and make perfect sense in most other subjects, it is essential to insist that the terms in this context should be interpreted as specifically oriented towards *mathematics*. We are not making any claims for generality beyond mathematics. Whether or not competencies with analogous formulations may be activated in relation to other disciplines, say history or physics, is entirely a matter for discipline-specific analyses to decide.

Whilst the mathematical competencies come in to play when people deal with different sorts of mathematics-laden situations, these competencies do not suffice for individuals to come to grips with *mathematics as a whole*, as a multi-faceted discipline (a pure science, an applied science, a system of tools for practice in culture and society, a field of aesthetics, and the world's largest teaching subject). Therefore, in addition to possessing mathematical competencies, a mathematically educated person also possesses overview and judgment concerning mathematics as a discipline. We have identified three such kinds of overview and judgment, concerning:

The ***actual application*** *of mathematics in society* (who uses mathematics in what extra-mathematical contexts, for what purposes, and under what conditions?)

The ***historical development*** *of mathematics in culture and society* (what are the internal and external forces that have driven the development of mathematics in different cultures and societies at different times; under what circumstances did the development take place, and who were the protagonists in it?)

The ***nature of mathematics*** *as a discipline* (what are the characteristics of mathematics, what are its essential commonalities and differences vis-à-vis other disciplines, and what are the features that are responsible for these commonalities and differences?)

While the competencies regard posing and answering questions *within* and *by means* of mathematics and mastering the language and tools of mathematics in challenging situations, overview and judgment concerning mathematics as a discipline rather regard posing and answering question *about* mathematics as a whole.

When the mathematical competencies are meant to be the same at any educational level, it is clear that the competencies cannot be employed to determine the mathematical content – topics - to be on the agenda in a given educational context. This fact gives rise to a pertinent question: What is the relationship between mathematical competencies and mathematical content? The answer is that they constitute two different "orthogonal" dimensions as depicted in the matrix below. The columns are the eight competencies introduced above and the rows are the mathematical topics included in the curriculum at a given level, for example numbers and number domains, algebra, geometry, functions, probability and statistics, etc. to mention just a few typical topics.

Table 1
Topics and competencies.

Competencies / Topics	Competency 1	Competency 2	···	Competency 8
Topic 1				
Topic 2				
···				
Topic n				

Thus, in a specific educational context, the cells in the i'th row represent the ways in which the eight competencies are involved in dealing with Topic i in that context, whereas the cells in the j'th column represent the ways in which each topic draws upon as well as feeds into Competency j. One consequence of this approach is that different educational contexts are represented by different realisations of this generic matrix, in that the topics as well as the cells most likely differ from context to context.

3 What's the point? What's new? What do we gain?

The previous section may primarily be seen as an intellectual exercise focused on the task of characterising mathematical competence independently of educational level and mathematical content domains. Apart from the possible intellectual outcomes what sense does the exercise make for mathematics education?

One of the important points driving the work on competencies was to find a way to define and describe *progression* and *development* of mathematics teaching and learning throughout the educational system. The characterisation should be intrinsic in the sense that it should neither be dependent on aggregation and accumulation of subject matter nor on educational levels. In other words mathematics education should be seen as a continuum evolving throughout the educational system. The notion of competencies offers a solution to this problem. Progression in student's learning of mathematics can then be defined as progression in his or her possession of the mathematical competencies. More specifically, three dimensions are attributed to an individual's possession of a given competency: The individual's *degree of coverage* of the set of aspects involved in the specification of the competency; the *radius of action*, i.e. the range of situations and contexts in which the individual can activate the competency; and finally the *technical level* (in a mathematical sense) on which the individual can exercise the competency. Progression in a student's possession of the competency can be perceived as extension of one or more of these three dimensions, and progression in the student's mathematical competen*ce* then is progression in one or more of the eight competencies. In line with this there is progression in mathematics teaching to the extent it fosters progression in sufficiently many students' mathematical competence. Similarly, we can also speak of progression of a student's overview and judgment concerning mathematics as a discipline in terms of a deepened insight into the actual application of mathematics, the historical development of mathematics, or the specific nature of mathematics as a discipline.

Another point is closely tied to the notion of progression and development, namely transition between institutional segments of the educational system. It is well-known in most countries that transition from, say, primary to lower secondary or from lower to upper secondary mathematics education, or from school to university, is associated with problems and sometimes even gaps or barriers. Looking at such transitions through competency lenses provides us with means for understanding the nature of the problems and gaps that student encounter and hence, eventually, with means for remedying problems in the transition.

The system of mathematical competencies can be exploited in two rather different ways. As a *normative tool*: for specifying the competencies on which emphasis should be placed in a given educational context and for designing corresponding mathematics curricula and teaching-learning activities to implement the specification, and for constructing assessment modes, instruments and tasks. And as a *descriptive-analytic* tool for investigation and analysis of curricula, textbooks, teaching, student activities and tasks, classroom interaction, teachers mathematics backgrounds and so on and so forth. Differently put, the set of competency lenses is a powerful tool for research. As an example, a modified and slightly compressed version of the competencies has proved instrumental in successfully capturing the intrinsic mathematical demand and difficulty (not to be confused with the statistical difficulty) of a large pool of PISA mathematics items.

It is notoriously difficult to communicate with interested parties outside the mathematics and mathematics education communities about what mathematics is, what it means to master mathematics, and what mathematics education is in non-technical terms. Making use of the competencies has proved to be helpful in this respect, especially because it gives rise to interesting exchanges on the nature and interpretation of the different competencies, and their possible relevance with regard to other subjects. We have experiences with this in Denmark where we have also taken things one step further by inspiring colleagues in other disciplines – particular in the sciences and linguistics – to establish analogous competency-based descriptions of mastery in their disciplines. This has allowed us to compare and contrast disciplines in a much deeper way than by just indicating the differences in content and subject matter.

What's new, then, in establishing and using competencies to characterise mathematics teaching and learning? Well, traditionally, in many countries, a given mathematics curriculum is specified by means of (at most) four components: (a) Statements of the *purposes and goals* that are to be pursued in teaching and learning; (b) specification of mathematical *content*, given in the form of a *syllabus*, i.e. lists of the mathematical topics, concepts, theories, methods and results to be covered; (c) *activities* that students are supposed to engage in; and (d) forms and instruments of assessment *and testing* to judge to what extent students have achieved the goals set for the syllabus as established under (b).

Using competencies allows us to avoid reducing the mastering of mathematics to just the mastering of some syllabus, and to avoid inessential trivial comparisons between different mathematics curricula, in which we can only identify the differences between curricula X and Y by listing the syllabus components in $X \cap Y$, $X \setminus Y$, and $Y \setminus X$, respectively. The differences between two kinds of mathematics teaching and learning are typically both more fundamental and more subtle than the differences reflected in the syllabi.

In summary, the competencies provide us with an appropriate platform for addressing key issues of the level of ambition in mathematics education.

4 What are the challenges and pitfalls?

It will come as no surprise that the competency framework present us with many challenges. The most important challenge is to expand the empirical foundation of the competencies. Even though the competencies have been supported empirically by a growing body of research, especially regarding subsets of the competencies, there are still issues to deal with. Would it, for example, be possible to define another set of competencies that are as good as or better than the eight competencies presented here when it comes to theoretically or empirically capturing mathematical competence? Would, for instance a smaller set do? Another issue is the internal relationships amongst the competencies. One might say, perhaps, that aspects of the symbols and formalism competency and of the aids and tools competency are special forms of the representation competency. If so, is it then reasonable to have them placed on a par with the representation competency? When making use of the competencies in empirical investigations do we not run into problems when the competencies are overlapping rather than disjoint? How can we then disentangle the roles and impact of the different competencies? Wouldn't a set of mutually disjoint competencies not provide a theoretically and empirically more satisfactory tool than the ones at hand?

At the moment we do not have definite answers to all these questions. Each of them is a case for research to be conducted to produce answers. I would greatly welcome any undertaking to that end.

References

Hadamard, J. (1945) *An essay on the psychology of invention in the mathematical field.* Princeton: Princeton University Press

Niss, M.; Jensen, T. (2002) *Kompetencer og matematiklæring. Ideer og inspiration til udvikling af matematikundervisning i Danmark.* Uddannelsesstyrelsens temahæfteserie.18. Copenhagen: The Ministry of Education. (English translation of the title: Competencies and the learning of mathematics. Ideas and inspiration for development of mathematics education in Denmark.)

OECD (2003) *The PISA 2003 Assessment Framework. Mathematics, Reading, Science and Problem Solving Knowledge and Skills.* Paris: OECD.