From drag and drop with the mouse to finger manipulations on multi touch devices: how ICT practices can foster mathematical inquiries[1]

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Abstract

Many national curricula at all grades suggest involving students in the manipulation of (real or virtual) materials. The current great diffusion of ICT in all aspects of everyday life pushes towards a massive use of such tools in the school. Their practices introduce an "experimental" dimension into mathematics, as well as a dynamic tension between the empirical nature of activities with them, which encompasses perceptual and operational components, and the deductive nature of the discipline, which entails a rigorous and sophisticated formalization. The talk illustrates the pedagogical possibilities offered by the tension between these two aspects when ICT are introduced into the classroom. Some short video clips from the classroom life make palpable this dynamic tension.

Key words

ICT, experimental mathematics, embodiment, proof.

Resumen[2]

Muchos programas nacionales en todos los niveles sugieren la participación de los estudiantes en la manipulación de materiales (reales o virtuales). La corriente de gran difusión de las TIC en todos los aspectos de la vida cotidiana empuja hacia una utilización masiva de este tipo de herramientas en la escuela. Sus prácticas introducen una dimensión "experimental" en las matemáticas, así como una tensión dinámica entre la naturaleza empírica de actividades con ellos, que abarca componentes de percepción y de funcionamiento, así como la naturaleza deductiva de la disciplina, lo que implica una formalización rigurosa y sofisticada. Este documento ilustra las posibilidades pedagógicas que ofrece la tensión entre estos dos aspectos cuando se introducen las TIC en el aula. Algunos clips cortos de vídeo de la vida en el aula hacen palpable esta tensión dinámica.

Palabras clave

TIC, matemáticas experimentales, materialización, demostración.

[1] Este trabajo corresponde a una conferencia plenaria dictada en la XIV CIAEM, celebrada en Tuxtla Gutiérrez, Chiapas, México el año 2015.

[2] El resumen y las palabras clave en español fueron agregados por los editores.

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1. Proofs in the classroom: the rigorous side of mathematics?

The teaching and learning of proofs has been discussed deeply in literature because of the difficulties showed by the students when approaching them. Research has pointed out the main difficulties when teachers try to make their students aware of the meaning of proof or able to produce proofs by themselves.

As a first result, according to Fischbein (1982) the notion of formal proof is beyond students' capacities, and the way people elaborates everyday knowledge and the scientific knowledge itself contradicts the mathematical idea of absolute certainty, which, according to Fischbein, grounds the epistemic basis of mathematical knowledge. Proving is not an "intuitive" activity, where according to terminology of Fishbein an intuition is "a representation, an explanation or an interpretation directly accepted by us as something natural, self-evident, intrinsically meaningful, like a simple, given fact" (Fischbein, 1982, p.10). If one knows something intuitively, "he will not feel the need to add something which could complete or clarify the notion (for instance an explanation, a definition, etc.)" since it is "something natural, self-evident, intrinsically meaningful, like a simple, given fact" (ibid.). A proof states a "formal extrinsic type of conviction indirectly imposed by a formal (sometimes a purely symbolical) argumentation" (Fischbein, 1982, p.11). According to Fischbein, teaching proof is so difficult since it involves the training of logical capacities and this is a difficult task, since "the main concern [of this training] has to be the conversion of these mental schemas into intuitive efficient tools, that is to say in mechanisms organically incorporated in the mental behavioral abilities of the individual" (Fischbein, 1987, p.81).

A second type of the difficulties met when teaching proofs in the classroom is pointed out by T. Dvora in her PhD dissertation (Dvora, 1982). Dvora (2012) analyses the proofs of 182 Israeli secondary school students in Geometry and finds that many of their proofs are incorrect and 88% of mistakes include not justified assumptions, often very near to the claim to be proved. She concludes stating that most students are not aware of the main aspect of a proof: "These classifications reveal that most of the errors made by high school students are not accidental but rather can be shown to have a rational basis and can be derived by a quasi-logical process that makes sense to the student." (Dvora, 2012, p.134).

A third research by Reiss, Klieme & Heinze (2001) shows that out of the students they examined only 57% acknowledges when a formal proof is correct, 42% when the proof is verbal, while 46% thinks that an empirical argument is incorrect and 33% find a incorrect a circular formal argument.

Similar results were found in Healy & Hoyles (2000), who analysed the conceptions of proof in algebra in about 2500 students of grade 10, who were good mathematics achievers.

In a nutshell, all these results point out a double gap when considering proofs in the classroom. From the one side, there is an epistemological gap between what is empirically perceived as true and what is logically valid within a suitable theoretical framework (Balacheff, 1988). From the other side, there is a cognitive gap between a first arguing phase in students' productions (when they are asked to explore a situation and make conjectures) and the proving phase (when they are asked to prove their conjectures in a more formal way). A possible consequence of these problems is that either one gives up from teaching proofs in secondary schools (the first edition of NCTM Standards went in this direction), or teachers must find suitable ways so that students can acquire a fresh, "non natural" basis for developing their mathematical arguments, as explicitly stated by Fischbein with his distinction between intuitive and non intuitive concepts.

Within this second stream, many approaches are suggested and tried in the literature. Some authors stress the importance to make students experience an explanation, an exploration and a justification of conjecture phase before approaching the formal aspects of proofs (Hanna & De Villers, 2012). Many studies portray the very importance of using Dynamic Geometry Software not only to convince students of the truth of a geometric theorem but also to perceive the difference between the figural and the conceptual aspects of geometric objects (Fischbein, 1982). Dynamic figures, in fact, allow students to deal with all the figures characterized by the same construction properties. Thanks to this variety of the same geometric object, students have the possibility to perceive the geometric invariants, which constitute the core of the discipline.

A crucial observation is that proving a theorem requires not only to enter into a theory and to handle the contents of the discipline but also manage the rules of logic that guarantee the truth preserving. Unfortunately, an approach that gives almost the same importance to the exploration-argumentation phase and to the formal proof writing, it is not widespread in math education: neither it is presented in the textbooks, nor is so alive in teachers' practises (Kosko, Rouge & Herbst, 2014).

A second observation is that most of the approaches that try to bridge the gap between the intuitive arguments and the formal proofs base on the assumption that a formal proof is not intuitive at all and hence an epistemic and cognitive jump is needed to arrive to grasp it.

In our research we are facing the problem also the other way round, namely we are analysing the logical structure of mathematical formal reasoning with the aim of bridging the gap from this side too. We change from the usual approach to logic and base on a new method, the so-called Logic of Inquiry, due to the research of an eminent logician, J. Hintikka, as we will sketch below.

The logic of inquiry, from the one side is a rigorous form of logic that grounds all the standard mathematical logic, but from the other side, it is next to what Fischbein called intuitive concepts, because of its intertwining with the games of game theory and the strategic rules that students develop to win a game.

Following this point of view, we are trying to study the role played by the strategic rules of thinking inside the proving process in a more compact and unified fashion: namely not only the proof writing but all the argumentation and exploration processes, which precede it. For this reason, we are designing tasks in which geometric theorems or properties are presented through strategic games. We believe that, not only the students need to understand which objects and relationships are involved in a theory, but also they have to develop a strategic type of thinking strictly related with the rules that govern geometry. This kind of reasoning, in fact, is the source of abductions which are “the only logical operations which introduce any new ideas” (Pierce, 1960. 5.171).

We think that strategic games could give to students the appropriate tools to become aware of it and use it in the field of mathematics, because of different reasons. Games motivate students’ discussion about the possible moves and the different strategies available in a given situation. The discovering and the selection of good strategies develop abilities that help students in the selection of the right geometric knowledge that allows them to write a proof.

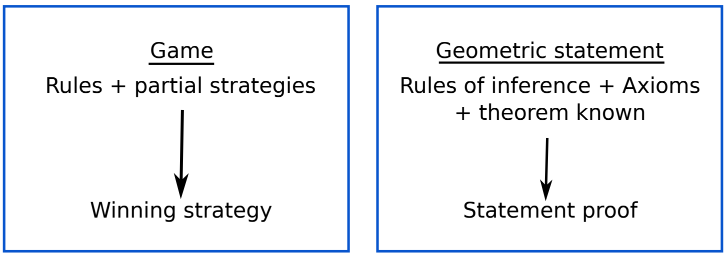

Figure 1: Games vs Theories

Moreover, from the structural point of view, the parallel between a strategic game and the geometric theory is quite evident, and can be exploited in order to make students understand what does “enter in a theory” mean: the games’ rules and the strategies can be compared to the rules of inferences and the known set of axioms/theorems. In order to win a game you are requested to develop the winning strategy in-between a given set of rules and possible moves, as well as in order to prove a statement you have to develop a strategic logic chains in-between the set of axioms and known results.

The game approach

Hintikka (1999) claims that any kind of activity directed to the reach of an aim can be conceptualized as a game between two players. As we mentioned above, developing this idea, he found a new type of logic, the Logic of Inquiry, based on the Game Theory. In this logic, each statement is interpreted as a debate between two players a falsifier F, who tries through his actions to disprove it, and a verifier V, who tries to prove it.

If we consider, as instance, the sentence “$\forall z \exists y | P(x, y)$”, the dialectic between F and V will develop in this way: F starts by showing to V a particular individual a, chosen in the most unfavourable situation. If V finds an individual b, such that P(a,b) is true, V wins the game, otherwise F wins. In this way, the process to describe the truth of a sentence became a dialectic process: each action hides a questioning/answering dynamic, ruled both by definitory rules (the rules of inference) and strategic rules (the rules of well reasoning).

Hintikka sees an analogy between the strategic rules, that in a game tell the player which actions is the best to do, and the abductions that is the process of selecting a theorem’s result, in order to prove a new statement (namely selecting the rule of which the result is the case) or forming what Peirce calls explanatory hypotheses. In a game, the strategic rules have to refer to sequences of moves and not to moves taken one by one:

> [strategic rules] have to refer in the first place to entire strategies, or at least to partial strategies. They cannot normally be formulated by reference to particular moves. They do not tell us what move to make in some particular situation, except insofar as that move is a part of some overall strategy. (Hintikka, J., 1999, p.4)

In a similar way, it is very difficult to have an abduction while writing a proof, without a clear vision of the results you have to reach and on the hypothesis you need, namely both on the previous passages of the proof and the next ones.

In our approach, developing the Logic of Inquiry in the classroom can be a possible way to help students in the proving process, because it allows them not only to become familiar not only with the definitory rules but also with the strategic one.

In the talk I will exemplify the issues above basing on teaching experiments with multi-touch devices made in Italy and Brasil, using DGS software like Geometric Constructor (designed by Yasuyuki Iijima at Aichi University of Education[3]), SketchPad Explorer[4] and Sketchometry[5] (Arzarello et al., 2013, 2014a, 2014b). The new technology, allows having more than one subject simultaneously operating on the screen of a tablet using as many fingers as they wishes: this facility, not possible within the mouse click-and-drag modality of DGS, makes it possible to design tasks where geometrical properties are introduced in a problematic way according to a game theoretical transposition. I will illustrate it with an example: the property "two circles in a plane intersect if and only if the sum of their radiuses lengths is lesser or equal to the distance of their centres" becomes the following (full-information) two-players game on a tablet (Fig. 1), which students must solve.

2. A teaching experiment

The design engineering of the games

Referring to the notion of truth in the Logic of Inquiry, we are developing some games activities based on it. For the moment, the activities are five, but for reasons of space, we present here in detail only the first one relating to the distance between the centres of circles and the sum/difference between the radiuses. The structure of the activities are very similar and all of them refer to theorems in the field of elementary geometry related to circles. Therefore we hope that the description of the first one suffices to understand our approach. The game is plaied in a Dynamic Geometry Environment.

Player Z can move points B and E (see Figure 1), by moving B he changes the length of the radius of circle E, while by moving point E he changes the position of the circle. Player Y can move D and F and the results of his moves are the same as player Z. Z's aim is to intersect or to touch Y, while Y's aim is to inhibit B's goal. The circles

[3] http://souran.aichi-edu.ac.jp/profile/en.7RRZ6p1fkRx0afMM47vMnA==.html
[4] https://itunes.apple.com/en/app/sketchpad-explorer/id452811793?mt=8
[5] http://www.sketchometry.org/

can intersect the sides of the rectangle, but their centres have to remain inside. We ask students to play the games more times, changing the starting position between the two circles.

Moreover, in case the game is played in multitouch devices (e.g. in iPad or Androids: see below) the players can move their objects at the same time, we explicitly tell students that the winning strategy should not depend on the speedy with which they move their points.

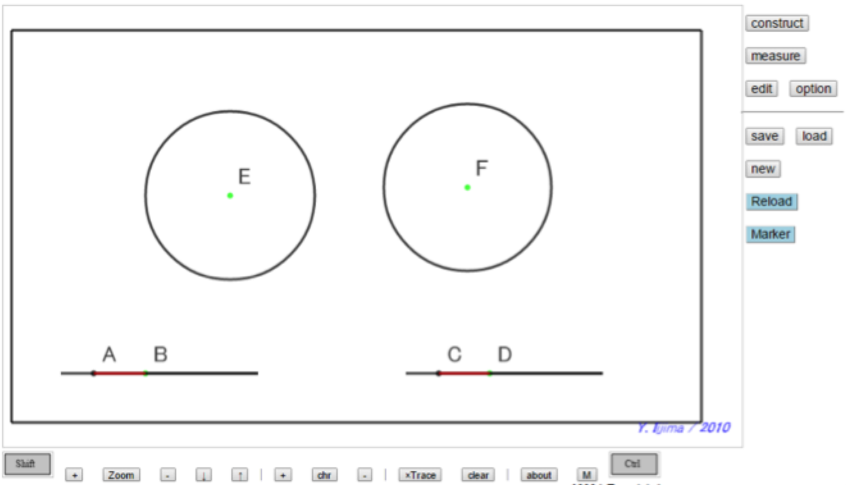

Figure 2: A game situation on the tablet.

The proposition on which the game is based is: "for any move made by F there always exist a move made by V such that the two circles touch or intersect."

The discovering of the strategy to win the game is not taken for granted: it requires students to switch their attentions from the particular games played, to a general game that could be played. This passage is very delicate because students to succeed have to take detach themselves from the concrete situation and think it in an abstract way. In literature, the importance of this shift has been already underlined:

> ...'to see the general through the particular and the particular in the general' and 'to be aware of what is invariant in the midst of change' is how human beings cope with the sense-impressions which form their experience, often implicitly. The aim of scientific thought is to do this explicitly. (Mason, J., 2005)

While students are discovering the strategy to win, they work in pairs and their reasoning is guided by some questions, contained in a worksheet:

Who is the player that can always win, provided she plays well?

Can you write to someone else a way for winning?

How do you know that the method always works?

The first question refers completely to the game. To answer it, students should know what a strategic game is and what the sentence "play well" means. In view of the importance of this knowledge for the development of all the teaching experiment, in the first activity, we ask students to write their own definition on the paper and, during the discussion phase, the teacher institutionalizes the meaning to the entire classroom.

The second question aims to make the students think on the moves made while playing. Generally, to answer the question, students play again the game but in a fictitious way: they speak to each other anticipating the moves; most of times, they do not play simultaneously but they decide to play one after the other or one player plays for both and the other observe what he does. The dynamic of the game offers the possibility to create great contests for meaningful discussions on necessary and sufficient conditions for winning the game. In the subsequent section, we will analyse an extract of a classroom's discussion.

Generally, at this point of the activity, the actions of the students are already concepts in formation; however the theorem is still in action (Vergnaud, 1982): students deal only with particular instances of the theorem, and as such the theorem itself is not yet part of the classroom knowledge. We believe that it is very important that students convince themselves on the truth of the theorem and perceive the logical aspects of its formulation before writing a proof. Therefore, we present them the statement of the theorem in form of conjecture. For instance, in the first activity, the conjecture is "Luisa thinks that if two circles are secant then the distance between their centres is greater than the sum of their radii. Do you think she is right? Justify your answer". After that we ask students to formulate true propositions, by suitably linking with the logical relation of implication (if…then) the statements from two lists, A and B. We specify to students that the statements to produce must be of the following type:

If A then B

If B then A

A if and only if B

For example, in the first game, the statements are the following:

For A:

1. Two circles are externally tangent
2. Two circles are external
3. Two circles are secant
4. Two circles are internal
5. Two circles are internally tangent

For B:

1. The distance between centres in minor of the sum of the radiuses and major of their difference

2. The distance between centres is equal to the difference between radiuses
3. The distance between centres is equal to the sum of the radiuses
4. The distance between centres is minor then the difference between radiuses
5. The distance between centres is major then the sum of the radiuses

Thanks to the conjecture and the request to create true propositions, the geometric theory emerges in a more engaging and reflecting way that by directly enunciate the theorem. In fact, students first explore the situation in order to understand what the conjecture or the sentences A and B mean and reflect over their meaning. Generally, they are not able to prove the conjecture but they support it with sound arguments.

After having answered to these questions, students can justify the strategy for winning the game with the mathematical theory. We prefer to let the third question as homework, so that they have more time to reflect on the game in a mathematical way, while in the classroom we ask students to prove one of the sentence they discovered.

The teaching experiment

The episodes described below is part of two teaching experiments developed in two tenth grade Italian scientifically oriented classes, "Maria Immacolata" high school in Pinerolo and "Giordano Bruno" high school in Albenga. Each classroom is composed by twenty students; during the activity they work in pairs, and have to read the task, to play the game on the tablet and to answer some questions on a worksheet. The software used in high school "Giordano Bruno" was GeoGebra, while in "Maria Immacolata" was GC/htlm5, a newest version of Geometric Constructor (one of the free dynamic geometry software used in Japan since 1989). They are both compatible with iPad and with Android tablet, but while the last software mention is multi-touch, the first one is not, so that the screen manages the input coming from only one finger, as with the mouse practice. We decide to experiment both the types of mediations because we would like to know if the request of playing a perfect game (a game in which players make the moves at the same time) obstacles or not students' formulations of strategies. Playing with simultaneous moves, in fact, brings into the game one more variable: the time, which could lead students to formulate strategies based on the speed of the moves made, bringing them far from the geometric theory. In a complete game (namely in a game in which players play in turn one after the other) the time does not enter into the game, as a consequence the speed does not affect the strategies.

During the work-in-pairs phase, the role of the teacher is to observe students and to help them if they are in troubles, whereas the role of the researcher is to videotape a single group. The first part of the activity is followed by the "Devil's Advocate reflection", namely the moment in which the teacher, starting from the answers given by the pairs of students, triggers a discussion about the activity, to make students reflect on what they have done and to systematize the mathematical content. Generally both the work-in-pairs activity and the Devil's Advocate reflection last one hour each.

Since the aim of this paper is not to compare the game played using multi-touch or single touch, we will present extracts from both the experimentations and we will concentrate our reflections on the awareness of logical aspects that the game-approach can bring into the classroom.

The teaching experiment deals with some themes related to a classical topic included in the National Curriculum 2012 (Indicazioni Nazionali): the circle. The teacher commits almost twelve lessons to the project, developing six themes: the reciprocal position between two circles, the reciprocal position between line and circle, the chords theorem, the angles at the centre and at the circumference, the inscribed quadrilaterals.

Some extracts from the first activity

In this section, we present part of the first activity we proposed to "Maria Immacolata" classroom. The aim is to have insight on the students' strategic reasoning and of how it influences the students' awareness of the logical aspect. In the moment of the dialog reported here, the pair of students video-recorded, has already played the game, and is trying to discuss what is the meaning of the expression "play well".

1. Student Y: "Play well" means... Applying strategies while playing...
2. Student Z: Let's try for a moment, do something (the students move to the DGE and play again)
3. Student Y: You always win
4. Student Z: Won (while intersecting)
5. Student Y: Play well means...
6. Student Z: Yes, but I always win
7. Student Y: I know, but what does "play well" means?
8. Student Z: Without cheating
9. Student Y: Without cheating, ah, yes
10. Student Z: Have a look, it (centre E) is still inside
11. Student Y: One could say without cheating. But you always win, so even if I would cheat you win (Y moves the centre F outside of the rectangle and then Z moves E to make the two circles intersect)
12. Student Z: Because... "play well" in the sense that...
13. Student Y: In the sense that ...
14. Student Z: Respecting the rules of the game and applying strategies... That might be winning
15. Student Y: Let's write this!

Moving from the concrete situation, students are trying to establish the meaning of the expression "play well". They question each other explicitly and the game implicitly. Thanks to this process the definition of "play well" evolves: starting from a meaning more linked with the everyday words "do not cheating" they move towards the mathematical meaning (applying strategies). Although they use the word "strategy", they do not explain what this word means, in particular, they do not relate the development of the strategies to the opponent's moves. Anyway, they are aware that the knowledge of the rules of the games are not sufficient to win.

1. Student Z: So, how do I win?
2. Student Y: Mmm... You always win, there is not much [to say]! (Moves circumferences random)
3. Student Z: Then, try to reproduce the first case. [Circles] are external.
4. Student Y: You must tell me how can I win... (Provocatively)

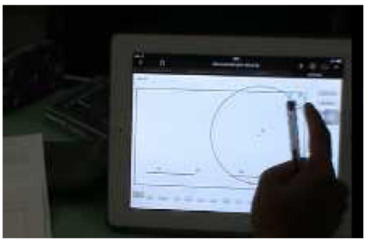

Figure 3

Figure 4

5. Student Z: Look, the point must remain inside the rectangle, even if [point F] is in the corner, point [E] remains inside, and I... Saying in some way, I catch you (pointing to the intersection points, Figure 3). I can also .(makes the circle E tangent to circle F, Figure 4)
6. Student Y: Ok, but you have to explain to me how I can win.
7. Student Z: It is sufficient bringing the radius to the maximum value and then moving towards F
8. Student Y: Ok, but the circle can exit, isn't it?
9. Student Z: Yes, the circle can, but not the point (enlarging the circle). You should be small, although I catch you immediately

The students are reflecting upon the moves made during the game in order to find out the strategy to win. Z proposes to play a fictitious game, in fact he uses the verb "reproduce" not "play" (line 3), starting from external circles. Y seems not engaged in the task, he declares that "there is not much [to say]" because Z always wins and asks provocatively Z how it is possible to win (line 2 and 4). Z starts moving the circles for both the players in order to search and to make explicit the moves that allow him to win. Z puts the circle F in the corner, that is the farthest position possible and then drags circle E until intersecting circle F (line 5). Finally, he drags circle E a little bit backwards until making the two circles tangent. The actions made by Z reveals that he has a clear idea of what a strategy is, in fact he chooses immediately to put circle F in the most unfavourable position to be reached and then makes the move that guarantee him to win even in this situation. It seems that he takes the decision to bring the circle backward because it is not necessary such a long movement to win, while it is possible to stop the move in the moment in which the two circle touch each other in the tangent point. Y observes Z movements but he seems not to understand the strategy just showed, in fact he ask for an explanation of the winning strategy (line 6). Then Z explicates with worlds what he has just done on the tablet: "It is sufficient bringing the radius to the maximum value and then moving toward F" (line 7). The discovering of the winning strategy is the result of an abduction: between the all-possible moves he is selecting the best ones which allow him to win in any situation. Finally, Z explicates also the fact that Y should be as small as possible in order to have some possibilities to save himself.

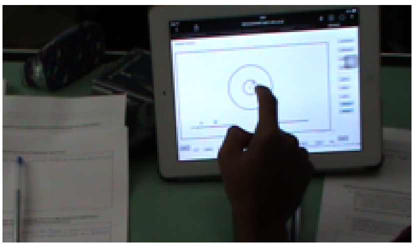

Figure 5

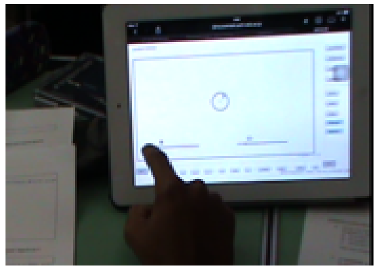

Figure 6

1. Student Z: [...]But wait a moment, try to put it [circle F] here, inside. (Making the two circles concentric and F with minimum radius, Figure 5)

2. Student Y: I at the minimum

3. Student Z: It's true, I always win. (making the radius of circle E at minimum, Figure 6)

4. Student Y: But in this case...

5. Student Z: Because I thought that if you were inside me and you reduce yourself to the minimum value, I must bring myself to the minimum value, too. So that the circles overlap. But if instead it is the opposite...

6. Student Y: Wait! Bring myself to the maximum value

7. Student Z: What do I do? What would you do?

8. Student Y: I cannot care about it, you catch me anyway! (enlarging the radius of circle E)

Z is not already totally convinced about the possibility of winning in any situation, in fact he wants to do more checks. He is now considering the case in which, at the beginning of the play, the two circles are concentric and the one, which has to catch, is external. First of all, Z brings the radius of circle E to the minimum values to verify that they overlap and after that (line 5) he justifies his actions. In this situation, the possibility to win the game is not so evident: if the minimum value of radius of circle E was major then the minimum value of circle F and the two players reduce both to the minimum, the two circles do not intersect. Y understands what Z is doing and decides to verify also the opposite situation: circles E is inside circle F and F has the maximum value of the radius. If the maximum value of the radius of circle E is minor then the maximum value of the radius of circle F, E would not catch F.

It is interesting to underline that, while discussing the strategies, students identify themselves with the circles: they never speak about circle E and F but only about "myself" and "yourself". They are engaged in the discovery of the strategy to win in any situation. Even Y, who is not so motivated at the beginning, after having understood the type of reasoning provide by Z, starts checking his hypothesis (line 6).

The use of questions as "What do I do? What would you do?" underline the fact that students are playing in a reflecting way and they are cooperating in the search of a strategy. The use of a strategic kind of reasoning is evident: they are thinking about the possible move in order to select the best one. These questions are the same ones a students should pose himself in order to find the suitable result's theorem that allow him to discover the proof.

After having make explicit the strategy to win in any situation, the students move to the following question "Luisa thinks that if two circles are secant then the distance between their centres is greater than the sum of their radii. Do you think she is right? Justify your answer". We report here the dialog between Z and Y.

1. Student Z: Wait, let's try it here (on the tablet)
2. Student Y: Secant, distance greater than the two radiuses ... do you think is right?
3. Student Z: This is the distance between E and F (pointing the imaginary segment EF)
4. Student Y: Exact! Yes, it is grater, it is grater!
5. Student Z: No!
6. Student Y: Look, take this part here... (pointing the radius of a circle F)
7. Student Z: It is smaller, not greater!
8. Student Y: No, no greater! Look, look, look, that is, from here to here, from E to here (pointing a generic point on circle E) is less than that from E to F and EF is greater, unless it is like this ... (puts the point F on the circle with centre E)

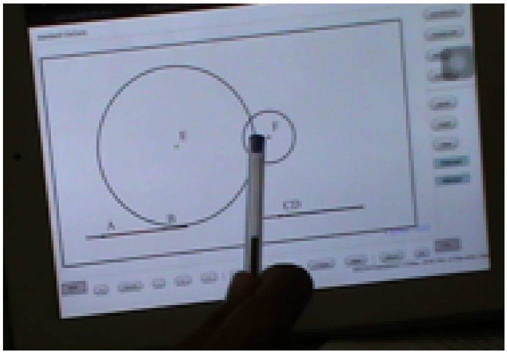

Figure 7

9. Student Z: If they were like this, namely secant, they share this piece, then they take off (pointing the common segment of the two radius, Figure 7), but if you add up their radiuses, suppose this is 2 and this is 3...

10. Student Y: No, but he intends to...

11. Student Z: The sum of the radii is always greater ... of course because they have a length in common here!

12. Student Y: Eh, you're right! Ah, yes, true, true, true!

The students decide to use the Tablet to visualize the meaning of Luisa's statement. From the dialog emerges that Y has misunderstood it, because he does not compare the distance between centres with the sum of the radiuses but with the single radius of the circle. Both Z and Y try to convince each other on the correctness of their own reasoning. Z argues that Luisa is wrong by showing that, if you use the radius to measure the distance between the two centres you have to subtract one of the shared part, the part in which the radiuses overlap, in order not to count it twice. Y, instead, argues that Luisa is right and to justify it, shows on the Tablet that the distance between the centres is major than the two radiuses of the circles or could be equal to one radius if the centre of one circle lays on the circumference of the circle. Only when Z repeated the worlds "the sum of the radiuses" (line 11) he realizes his mistake and gets convinced by Z. After this discussion, they write on the paper "Luisa is wrong because the distance between E and F includes some common measures and so it is always major". Y and Z are not happy with their answer, they believe to have the concept in their mind, but not being able to write it and so they call the teacher who shows them how to prove it.

For reasons of time this pair of students do not finish the worksheet and so we do not have evidence about their ability to solve the exercise in which they are asked to produce a true sentence from the given statements. Anyway, we have evidences from other pairs of students that shows some difficulties to understand what is the request of the task. The pairs who rescue in approaching it, use the Tablet to discover to which statement B is linked a given statement A, then make a table that shows the link without specifying if A implies B or vice versa or both. After the teacher's discussion their ideas on the task become clear.

Some extract from the first discussion

We report here an extract of Devil's Advocate discussion developed after the work-in-pairs activity in Albenga. The teacher is at the blackboard and is talking about the winning strategy.

1. Teacher: F has moved here now (drawing at the blackboard circle F in the corner of the rectangle up on the left). Suppose we only act on the slider, does the strategy work? Does Z win?
2. Students: No. Not always. It depends on the position of the two circles. If the slider was really large
3. Teacher: If I act only on the slider what can happen? As the slider is limited then it is clear that, if F is a circle big enough, I can take it, or if E has not moved very far. But it can also happen that F is very small and has moved very far so that in this way I cannot take it. So it is not enough act on the slider…
4. Student: Then it is better just moving…
5. Teacher: Then what is more convenient to do for winning?
6. Student: Moving the centre of the circle!
7. Teacher: Is not a sufficient condition working on the slider to win, right? And work on the centre is it a sufficient condition?
8. Students: Yes!
9. Teacher: Do you agree?
10. Students: Yes!
11. Teacher: Is it necessary?
12. Student: In which sense?
13. Teacher: I said, is it necessary working on the centre to win?
14. Student: In this case yes
15. Student: It depends on the radiuses of the circles
16. Teacher: Can you make us an example in which you show us that under certain conditions it is not necessary to work on the centre to win?
17. Student: It depends on the distance between the two circles… if they are close enough it is not necessary. If I increase the sliders the two circles touch each other

The teacher decides to vary the rules of the games, inhibiting the movement of the centre of the circles. In this way, all the students start reasoning on a different case and he can orchestrate their discussion, showing implicitly the way in which they have to work in pars. First of all, the teacher draws the new situation at the blackboard: circle F is in the corner of the rectangle and circle E is very far from it. Then he asks students if Z can win by moving only the slider (line 2). Students understand that the answer depends on its length, the teacher captures the answers and institutionalize them to the classroom with a quite long comment (line 3). Gradually, the discussion becomes more dynamic and the teacher gives to the students the responsibility of the institutionalization of the answers: at the end, he calls a student to the blackboard to do it, in his stead.

The discussion on the necessary and sufficient condition to win creates concrete situation to refer in the next part of the lesson, when students are talking about the logical dependence between geometric statements.

3. Discussion

One remarkable result of the teaching experiments regards the effects of using a Tablet during the class lessons. We notice that students tend to flip the question they have to answer to the used technology. From the extracts reported above it is clear that the two students do not hesitate one minute in doing it, even if the question does not refer to the game, they use it to get some suggestions. I am referring in particular to the searching of the answer to the question "What does the expression 'play well' mean?", Z before saying anything proposes: "Let's try for a moment, do something". The same attitude is repeated when they are searching the answer to the question: "Can you write someone else a way for winning?". In fact Z says "Let's try to reproduce the first case". They use the technology "to draw" sketches: the Tablet substituted completely the piece of paper. This is also evident, in particular, when they are discussing about Luisa's conjecture.

Even though the game is very simple, students have some difficulties to find the answers to the questions, because the formulation of them requires taking distance from the concrete situation. In particular Y's words, "You always win, there is not much", reveals that, according to him, the method for winning is: "to be player Z, because he always wins". Students are not used to take distance from the empiric aspects and to reflect over them. The majority of students, when are requested to explain why something happens, are able only to provide empirical evidences which convince the interlocutors but not justify. The same behaviour can be seen when students are proving something: if they see the result in a DGE they have not the necessity to proving it, they are sufficiently convinced about it. This attitude provides evidences on the fact that to convince and to prove is the same thing to them. When the students are trying to justify Luisa's conjecture, they always use empirical and visual arguments.

However, when the task pushes them to reflect in a detached way on their activities during the played game, they can become aware of the strategies they used in the game. In fact, at each stage of a deductive argument, there are normally several propositions that can be used as premises of valid deductive inferences. The so-called rules of inference will tell you which of these alternative applications of the rules of inference are admissible. They do not say anything about which of these rules' applications someone ought to make or which ones are better than others. For that purpose one needs strategic ideas. The detached reflection on the game can provide them: of course it is necessary the careful coaching of the teacher, who poses the right questions at the right moment during the classroom discussion.

The answers provided to the questions "What does the expression 'play well' mean?" reveals that not all the students have a clear idea of what a strategy is and this fact should alarm teachers about students' ability to write a proof alone. Students should be aware of the existence of the strategic reasoning and should become familiar with the use of it. I wonder how it is possible for students having ad abduction during the proving process and feel comfortable with proofs if they have not clear ideas of what a strategic reasoning is. At each stage of a deductive argument, there are normally several propositions that can be used as premises of valid deductive inferences. The so-called rules of inference will tell you which of these alternative applications of the rules of inference are admissible. They do not say anything about which of these rules' applications someone ought to make or which ones are better than others. For that purpose you need strategic rules.

The design engineering of the game is a very powerful instrument for students' understanding of not only strategic thinking, but also the mathematical content. The videos' analyses reveals that students really get convinced on the validity of the theorem, because before approaching it they have played with it and explored all the possible cases.

We think that it is important doing deeper research on this kind of approach to proof investigating how playing games not only convinces students of the truth of a theorem but also gives a structure to the proving process. In fact, while working in a DGE, people naturally wonder why a situation is like this, how could be different, etc. Thanks to the exploration, the ascendant and descended control over the geometric objects (Arzarello & all.,2002), students could find some answers to these questions, make conjectures or local deductive steps. It is difficult that just by working into a DGE students wonder what is possible to do in this situation and what is better to do. These questions arise typically during games. For this reason, putting games into DGE could prompt students' reflection on the local deductive step and on the conjecture made during the exploration phase. Thanks to the game, the local component of reasoning emerged in the exploration phase are not forget but could be reorganized in global deductive chain. "A game can provide a structure for the learning that takes place in the environment" (Devlin, 2011, p.32).

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