Engaging with Mathematics in the Digital Age[1]

Professor Dame Celia Hoyles
UCL Institute of Education, University College London
United Kingdom
C.Hoyles@ioe.ac.uk

Abstract

There is widespread acceptance that mathematics is important, for an individual and for society. However there are still those who disagree arguing that the subject is boring and irrelevant. It is therefore crucial that mathematics teaching strives to engage all learners at all levels, without of course sacrificing the rigour and 'essence' of the subject. In this talk, I will argue that one way to achieve both rigour and broader access to mathematics lies with using appropriately designed digital technology. I will illustrate my argument with examples from research and practice.

Key words

Mathematics Education, digital technology.

Resumen[2]

Existe una aceptación generalizada de que las matemáticas son importantes para un individuo y para la sociedad. Sin embargo, todavía hay quienes están en desacuerdo y argumentan que la disciplina es aburrida e irrelevante. Por tanto, es crucial que la enseñanza de las matemáticas se esfuerce en involucrar a todos los estudiantes en todos los niveles; por supuesto, sin sacrificar el rigor y la "esencia" de la disciplina. En este trabajo voy a argumentar que una manera de lograr tanto el rigor como un acceso más amplio a las matemáticas se encuentra con el uso de la tecnología digital diseñada adecuadamente. Voy a ilustrar mi argumento con ejemplos de la investigación y la práctica.

Palabras clave

Educación matemática, tecnología digital.

1. Introduction

The importance of mathematics for an individual and for society is widely recognized. In the UK there has even been a report that has quantified the ways in which Mathematical

[1] Este trabajo corresponde a una conferencia plenaria dictada en la XIV CIAEM, celebrada en Tuxtla Gutiérrez, Chiapas, México el año 2015.

[2] El resumen y las palabras clave en español fueron agregados por los editores.

Recibido por los editores el 10 de noviembre de 2015 y aceptado el 15 de enero de 2016.

Sciences Research influences economic performance and its economic value in terms of direct employment and Gross Value Added:

> Working in partnership with the Council for the Mathematical Sciences (CMS), the Engineering and Physical Sciences Research Council (EPSRC) commissioned an independent study which has shown that 10 per cent of jobs and 16 per cent of Gross Value Added (GVA) to the UK economy stems from mathematical sciences research (EPSRC, 2012).
>
> The report goes on to argue that:
>
> The fruits of mathematical research affect the daily lives of everyone in the UK, for example:
>
> Smart-phones use mathematical techniques to maximise the amount of information that can be transmitted
>
> Weather forecasting is based on complex mathematical models
>
> The latest Hollywood blockbusters take advantage of the mathematics behind software for 3D modelling to showcase cutting-edge special effects
>
> Elite athletes at the 2012 Olympic Games used tools based on sophisticated maths to maximise their performance.
>
> It is not just contemporary mathematics research that can have an impact. Research from the last century has paved the way for technology used in a range of activities, goods and services, such as mobile telecommunications and medical devices. (EPSRC, 2012).

As a result of reports such as this one and the growing awareness of the importance of mathematics, alongside I must add the intrinsic interest of the subject, there is a relentless drive to improve access and engagement with the subject at every level of education, including increasing recruitment to mathematics courses for students post-16 years, an age when students at the moment in England can drop the subject.

Increasing awareness of the significance and standing of mathematics has proved to be a success story in England. However there are still challenges to be faced, not least that the most commonly-held view of mathematics still tends to be that it serves little or no purpose. This position is aptly summarised in a recent piece by a well-known journalist in a national newspaper in the UK, Simon Jenkins:

> I learned maths. I found it tough and enjoyable. Algebra, trigonometry, differential calculus, logarithms and primes held no mystery, but they were even more pointless than Latin and Greek. Only a handful of my contemporaries went on to use maths afterwards. *(emphasis added, Jenkins, Guardian 18 February 2014)*

Mathematics, as many have noted, is plagued by a culture of 'speed', 'getting an answer quickly with apparently little effort, of 'winning the race'; a 'genius' culture that all too often leads to many giving up the subject, as aptly summarised in the following from Alex Bellos:

> Athletes don't quit *their sport just because one* their teammates outshines them. And yet I see promising young mathematicians *quit every year, even though* they love mathematics, because someone in their range of vision was 'ahead' of them(Bellos' review of Ellenberg's book 'How not to be wrong: the Hidden Maths of Everyday Life' 2014.)

This 'genius culture' along with learned helplessness "I simply can't do maths" (for an early discussion of learned helplessness, see Diener & Dweck, 1980) is further aggravated by the fact that most people of all ages regard mathematics simply as a set of procedures and rules. They fail to glimpse in their efforts to master all the 'machinery of the subject', the key mathematical concepts, structures and relationships of the subject. It is only by providing a mathematical lens on the world and on school mathematics – making it more visible - that learners might come to see the point of all their efforts.

My claim is that a major challenge for increasing such engagement with mathematics is *to* address its current *invisibility*, and one way to do this is to harness digital technology, and, crucially, to do this in ways that are systematically tested to be effective in design research, a point I will return to later. All too often mathematics is a black box that is kept closed, either as there is no reason to try to open it, or it is deemed as too complicated to even try. Strings of symbols tend to be meaningless to most people. But is it not possible for mathematics educators to work together to open the 'black box' *just enough* to convey the mathematical concepts behind in ways that are comprehensible to the audience at hand? The Mathematics Matters series of the Institute of Mathematics and its Applications provides an example of one possible approach. (http://ima.org.uk/i_love_maths/mathematics_matters.cfm.htm). In these case studies, mathematics research has been described in language that attempts to be meaningful to a diverse audience.

> The industry and technology that surrounds us owes a great debt to modern mathematics research, yet this fact is perfectly hidden in its physical manifestation. The concern with this state of affairs is that what is unknown cannot be appreciated or valued. This is not a simple matter to resolve since, although current mathematics significantly influences the familiar, the mathematics itself may seem impenetrable to the very people whose views we seek to influence. The Mathematics Matters case studies have been written to resolve this problem. Examples of contemporary research and its applications have been presented in a series of papers which describe the mathematics, without resort to technical detail but also without patronising over-simplification. In this way, policy makers can understand how mathematics research influences so many areas of modern life. However, in order to provide a satisfying level of detail to those with a more scientific training, each paper also includes a technical supplement, which describes the work in more detail, and may include references to published work which confirm the credentials of the research. From its original concept, the work has now progressed over four phases and includes such case studies as:
>
> On Your Bike: Accurately Measuring Cycling Numbers
>
> Official estimates suggest the number of cycle journeys in the UK could be declining. With a leading transport charity arguing otherwise, mathematics is being used to paint the true picture of cycling in the UK in order to secure important government funding.


Advancing the Digital Arts

"New mathematics is pushing forward the boundaries of the animation industry."

On the Radar Calibrating Air-traffic Control Antennae

Air-traffic control is a vital part of the aviation system that contributes billions of pounds annually to the UK economy. Without mathematics, however, the radar antennas that underpin the network of primary and secondary surveillance radars operating to ensure the safety of air vehicles operating in and beyond UK airspace would take longer and require more effort to calibrate.

A Smarter Future for Next Generation Local Electricity Networks

The vision of a low carbon future brings its own challenges when it comes to maintaining an effective electricity supply system. Mathematicians are working to give decision makers richer understanding, greater flexibility and a more solid evidence base on which to inform their important choices.

(http://ima.org.uk/i_love_maths/mathematics_matters.cfm.htm).

One early case study concerns the computer animation industry that relies on a steady stream of mathematicians to produce the images found on our cinema and television screens (case study Advancing the Digital Arts), which I will illustrate briefly in my presentation.

Becoming aware of the potential power of digital technology and how it is framed by mathematics points to a possible way that we might engage more learners with mathematics in the digital age, which I will elaborate a below.

2. The potential and challenges for research in mathematics education

In my keynote to the ICME 11 congress in Mexico in 2008, I drew on the mass of evidence from research and practice, to set out what I saw as a vision for the potential of digital technologies to transform the teaching and learning of mathematics and to reinvigorate engagement with mathematics. I suggested that digital technologies could offer:

- *dynamic & visual tools* that allow mathematics to be explored in a shared space;
- tools that *outsource processing power* that previously could only be undertaken by humans;
- *new representational infrastructures* for mathematics;
- *an infrastructure for supporting connectivity* to support mathematics collaboration;
- *connections between school mathematics and learners' agendas and culture*;
- *intelligent support* for learners while engaged in exploratory environments.

Time has moved on since 2008. Nonetheless I adhere to these six headings as a framework for future research. There are other potential areas for study that might now be added to my list: for example, the potential of digital technology to build into student activity 'invisible' formative assessment with data collected as students work on their solutions (individually or collectively) resulting in assessments that are more genuinely personal and adaptive; or, functionalities embedded in activities that are particularly tuned to students' previously identified learning needs or goals.

In one area at least there has been a dramatic change, and that is massive increase in infrastructure to support *connectivity* and access to the web, which was in its infancy in 2008, at least in schools. But the question remains as to how far this connectivity is effectively exploited in the interests of mathematics education: should it be and if so how? Can this new functionality that is so widely available and taken for granted in the daily lives of many be harnessed for the purpose of helping learners and teachers share, discuss and take ownership of the mathematics, and better appreciate the point of the subject beyond its calculational side?

3. Theoretical background[3]

There has been much discussion and writing about relevant theories that have been developed, to underpin research into using digital technologies in mathematics education (for an overview see for example the ICMI Study Technology Revisited, Hoyles & Lagrange, 2011). It is also noteworthy that for this ICMI study we were unable to collect papers on this subject except from those that were specifically invited. My own research has taken inspiration from the work of Seymour Papert and I remain committed to constructionism as a way of thinking about using computers for mathematics learning. So what is constructionism? Seymour Papert launched the notion of *constructionism* in the mid-nineteen eighties, with the central idea that a powerful way for learners to build knowledge structures in their mind, is to build with external representations, to construct physical or virtual entities that can be reflected on, edited and shared:

[3] Some of the following text builds on Hoyles, C. The proceedings of the Vth SIPEM (2012) Petrópolis, Rio de Janeiro Sociedade Brasileira de Educação Matemática – SBEM pp 1-12

> Constructionism [...] shares constructivism's connotation of learning as "building knowledge structures" irrespective of the circumstances of the learning. It then adds the idea that this happens especially felicitously in a context where the learner is consciously engaged in constructing a public entity, whether it's a sand castle on the beach or a theory of the universe. (Papert & Harel, 1991, p.1).

Thus, the constructionist environment must first represent a *compelling* medium in which to explore and learn, much as one can master a foreign language by living in the country where it is widely spoken. Second, in the environment, the learner *is able to* adopt a construction-based approach to learning in which there is some ownership by learners of the construction process, and which, potentially at least, leads to their engagement, confidence and empowerment. Third, exploration through building enables the learner to encounter 'powerful ideas' or intellectual nuggets, while ostensibly constructing something else. This has led to the design of microworlds, where a successful microworld is both an epistemological and an emotional universe, a place where powerful (mathematical, but also scientific, musical or artistic) ideas can be explored; but explored 'in safety', acting as an incubator both in the sense of fostering conceptual growth, and a place where it is safe to make mistakes and show ignorance: And, of course, centrally these days, a place where ideas can be effortlessly shared, remixed and improved (for an earlier discussion of these twin aspects of engaging through building and sharing, see Noss and Hoyles, 2006).

It is important to emphasise that, as Papert was at pains to point out, constructionism seeks to develop knowledge structures in the mind alongside physical or virtual structures external to the mind, and as such is as much a theory of epistemology as of pedagogy, (see Harel & Papert, 1991). In fact in the ICMI study mentioned above (Hoyles & Lagrange, 2011), we tried to insist all participants in the study conference should think about 'Papert's 10%, the 10% of knowledge that would need to be rethought given the use of new tools.

Over the years, constructionism has provided the framework for a fertile strand of research and development and continues to attract innovative ways of designing tools and working with learners from across the world and with different age groups (reference the recent Constructionism conference, 2014, and to the Special Edition of *Mathematics Today* that will be published in Dec 2015, Windows on Advanced Mathematics, Hoyles and Noss, eds).

One prevailing challenge is that although microworlds designed with a constructionist agenda are intended to orient students towards a way of thinking carefully structured by the designers, learners must at the same time have some autonomy. This means, of course, that learning will not occur precisely as planned. Thus, one has to ask how is it possible to balance self-motivated activity while maximising the opportunity to encounter the planned powerful ideas (see the 'Play Paradox', Noss & Hoyles, 1996).

There is also the complex issue of the role the tools play in shaping the mathematical knowledge and mathematical learning, and at the same time being shaped by the interactions of the students, called the process of instrumental genesis by some researchers. Drijvers et al (2010) put it thus when talking about the instrumental approach:

> According to this approach [The instrumental approach], the use of a technological tool involves a process of instrumental genesis, during which the object or artefact is turned into an instrument. This instrument is a psychological construct, which combines the artefact and the schemes (in the sense of Vergnaud, (1996)) the user develops to use it for specific types of tasks. In such instrumentation schemes, technical knowledge about the artefact and domain-specific knowledge (in this case, mathematical knowledge) are intertwined. Instrumental genesis, therefore, is essentially the co-emergence of schemes and techniques for using the artefact.

Much attention has been paid by researchers to the issue of instrumentation but rather less to instrumentalisation - the reciprocal relationship whereby the medium or the tools are changed during interaction and along with this the knowledge developed (we argued in Noss and Hoyles 1996 that the medium shapes that mathematical meanings through its use and at the same time is shaped by use).

4. Dynamic and visual tools

Digital technology can provide tools that are dynamic, graphical and interactive. Using these tools, learners can explore mathematical objects from different but interlinked perspectives, where the relationships that are key for mathematical understanding are highlighted, made tangible and manipulable. The crucial point is that the semiotic mediation of the tools can support the process of mathematising by focusing the learner's attention on the things that matter: as Weir (1987) put it, "the things that matter are the things you have commands to change." (p. 65). The computer screen affords the opportunity for teachers and students to make explicit that which is implicit, and draw attention to that which is often left unnoticed (Noss & Hoyles, 1996).

Central to this research endeavour is to identify which out of all the aspects that might change are judged by students to be important and which not? My conjecture is that to engage with the dynamic microworld in the ways anticipated by the designer, it is important that *some aspect of the constructionist agenda is intact* – the black box is opened 'just a little', the microworld not quite complete and students actually able to build something for themselves. When these openings are on offer, I would argue that the tool is more likely to open a window on the mathematical ideas. In fact evidence for this can be gleaned from research undertaken around, what are called half –baked microworlds, microworlds that are intentionally designed as malleable and improvable with students challenged to find faults and fix them, (see for example Healy & Kynigos, 2010).

Other research from outside the school mathematics community that focussed on workplaces is relevant here (see Hoyles, Noss, Kent, & Bakker, 2010). In a later summary, Hoyles, Noss, Kent, & Bakker, 2013 argued that there are contrary views regarding the mathematical needs of employees in workplaces, a problem exacerbated by the ubiquity of information technologies and the widespread automation of routine procedures, which leave little if any trace of the mathematical processes going on. We pointed to a particular difficulty, that of widespread *pseudo-mathematical interpretation* of symbolic output in workplaces. This certainly impedes communication, but can be challenged by designing what we termed *'technological enhanced boundary objects'* (TEBOs) with appropriate dynamic digital technology that focussed on moving 'beyond' the calculational side. I take just one example. We identified a ubiquitous requirement in workplaces to understand and reduce variation, summarized in two process capability indices, Cp, a measure of spread, and Cpk, a measure combining spread and central tendency in relation to specification limits (Hoyles, Bakker, Kent, & Noss, 2007; Bakker, Kent, Noss, & Hoyles, 2009). Let us look at Cp, which summarises the spread of a distribution in relation to the required specification for the process:

$$C_p = \frac{USL - LSL}{6s} \quad \text{where:} \quad \begin{array}{ll} USL & = \text{upper specification limit} \\ LSL & = \text{lower specification limit} \\ s & = \text{standard deviation} \end{array}$$

We found that employees were shown these formulae and plugged in values for the variables but interpreted the results pseudo-mathematically, making little if any connections to data or underlying mathematical relationships (Hoyles et al., 2010). The capability indices were supposed to illustrate that the spread of the data was within specification limits (or not). But in fact for most employees did not 'see' this and simply knew that their manager would complain or '*they would be beaten up' for low Cp's"*. Thus Cp and Cpk were viewed simply as management devices unrelated to the data from the production line: the models were just too baffling. We sought to address the problem by designing a TEBOs where employees could manipulate the key variables: the one for Cp is illustrated in Fig 1 below.

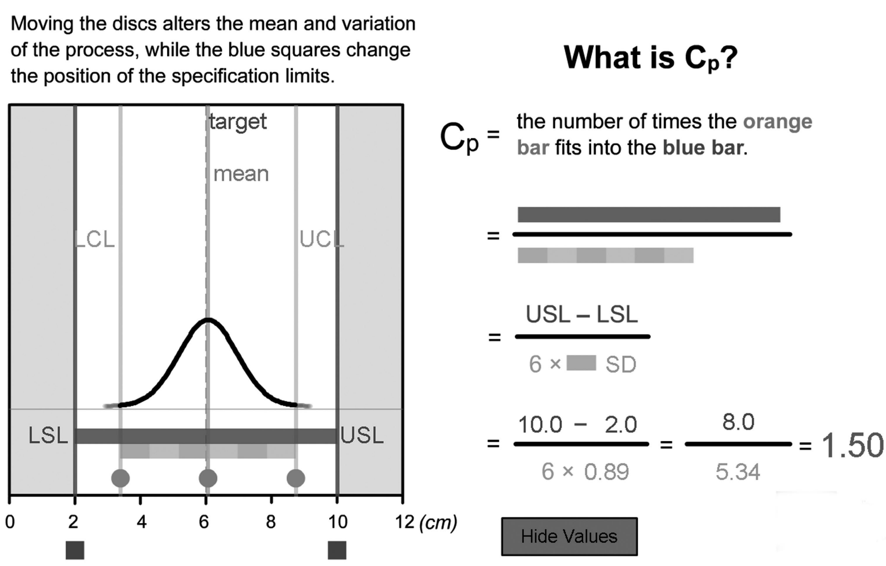

Figure 1: Screen capture image of the Cp tool.

Figure 1 illustrates the TEBO for Cp that aims to reveal the fundamental nature of Cp without the need for engagement with the algebraic definition or manual calculation. The TEBO allowed employees to manipulate the key variables, the mean and the spread, in relation to the specification limits and to see that what looks like a complicated formula is just "the number of times the 'bottom line goes into the top line" – and the bottom line is 6 standards deviations long.

Although our samples were small, there was remarkable take-up in the use of the tools, not only with the shop-floor workers but also with supervisors and engineers; and beyond the original factories, spreading to SPC courses worldwide (Bakker et al., 2009).

5. From design experiments to innovation at scale

So let us return to my main agenda, which is to enhance engagement with mathematics in this digital age. It is clear from research evidence and from practice that for this to happen teachers must be a central part of the process. But how can they best be supported so as to fulfil this role? I suggest the evidence points to the following set of prerequisite activities:

i) Teachers tackle the mathematics for themselves with the digital tools (before and alongside thinking about pedagogy and embedding in practice), thus allowing them, regardless of experience, the time and space to take on the role of learner,

ii) Teachers co-design activity sequences that embed the digital tools and make explicit appropriate didactic strategies,

iii) Teachers try out the activities iteratively in classrooms as a collective effort and debug together.

This design process is time-consuming and challenging, not least because at every phase the dialectical influence of tools on mathematical expression and communication must explored. I will give just one example of this design process, that of Cornerstone Mathematics (CM). CM set out to exploit the dynamic and multi-representational potential of digital technology to enhance learners' engagement and understanding of some key mathematical ideas that most (or all?) students aged 11-14 years will face in school. In brief, the CM approach is to design interventions that integrate professional development, curriculum materials, and software in a *unified* curricular activity system (see Vahey, 2013), where the activities and in particular the use of digital technology focused on core, deep and challenging mathematics. Thus CM comprises three inter-dependent elements, each of which are critical for any innovation and each of which have been extensively researched and developed over many years, that is: digital technology designed and tuned for specific mathematics learning, iteratively designed student curriculum to replace current practice along with a teachers' guide, and professional development for teachers. CM to date comprises three curriculum units, on linear functions, geometrical similarity, and patterns and expressions. What we call *'landmark activities'* are designed so that students through their explorations with the software are bound to come up against inevitable epistemological obstacles.

A major challenge – arguably, *the* major challenge – is then to design support for the student that provides enough freedom so they can actively engage in *their* task, yet with adequate constraints so as to be able to generate feedback that assists them to achieve *our* goals. We are confronted in exactly this same dilemma when working with teachers. We are exploiting the 'landmark' activities in our research into teachers' developing mathematical knowledge for teaching as a way to expose problematic issues of around mathematical understanding and representation (Clark-Wilson et al, 2015).

Clearly there is complexity and variability in implementing any activities in classrooms and huge issues of ensuring alignment to school ethos and schemes of work, national curriculum and assessment – and, last but not least, ensuring that the schools have access to all the necessary artifacts: materials, hardware, software, texts and evaluations. All too frequently, the costs and challenges of using digital technologies in mathematics are noted as the reason why in so many cases, impact has not reached expectations. But with ever increasing knowledge, a more robust theoretical basis, along with systematic evidence from the research community, we should be able together to move forward and support students in trajectories of learning *with* digital tools.

This will of course mean that we have to study how to build evidence-based, sustained, and scalable professional development for teachers: a growing area of research. It is undoubtedly complex as it requires systematic investigation at school, regional and national levels. For example, a fundamental challenge for CM was not whether the nature of the innovation 'changed' in use – this is inevitable – but how far are these changes or 'mutations' were "legitimate" or lethal" using the terms adopted by (Hung et al 2010): that is in our context aligned or not with the vision and aims of CM. (Clark-Wilson et al 2015)

The issue of evidence-based CPD and scaling sustainable interventions is explored in a recent special issue of ZDM and in the survey paper for this Special issue, Roesken-Winter, Hoyles & Blömeke (2015) pointed to challenges of scaling CPD from four perspectives: "First, ….crucial aspects of teacher learning and what taking the learning of these crucial aspects entails. Second, ...different CPD frameworks to showcase developments in CPD research and practice over the last 40 years and the influences of different views of CDP. Third, …what developing CPD in an evidence- based way means, before we finally discuss crucial issues of spreading CPD on a large scale". In this last perspective, we drew on Coburn's four dimensions characterizing the process of scaling CPD interventions, depth, sustainability, spread, and shift in reform ownership (Coburn, 2003).

I end by returning to this notion of 'shift in ownership', which resonates exactly with my overriding concern to open up the mathematical way of thinking to more learners. I mention one potentially exciting potential shift at least in the constructionist agenda: the massive popularity of the Scratch programming language (https://scratch.mit.edu/), where young people can be put in the role of designing and creating with digital media rather than simply playing and searching online (Resnick, 2012). This phenomenon is global and massive; at the time of writing there are 8,705,136 projects being shared via the Scratch website. It is largely an initiative outside of formal education, (see also the Hour of Code movement http://hourofcode.com/us). However in England we have a compulsory Computing curriculum - alongside an ongoing compulsory mathematics curriculum. In our project Scratchmaths (EEF, 2014), we are seeking to align these two curricula at points where we can design to enhance engagement with mathematics and mathematical reasoning in the ways explored in this paper. Thus plan is to support the building of mathematical knowledge with programming, thus harnessing the enthusiasm and energy for programming for mathematics learning as well as providing a glimpse of the underlying structures. We do not underestimate the challenges – so many were documented in the 1980s - but we intend to learn from this past experience and at the very least plan and design from the start in partnership with teachers and detailed curricula. Is this a way we might just be able to:

> let the students learn mathematics as applied mathematics ... in the sense that mathematical knowledge is an instrument of power, making it possible to do things of independent worth that one could not otherwise do ... (Papert, ICME 1972)

References

Bakker, A., Kent, P., Noss, R. & Hoyles, C. (2009). Alternative representations of statistical measures in computer tools to promote communication between employees in automotive manufacturing. *Technology Innovations in Statistics Education,* 3(2).

Bellos, A. (2014) How Not to Be Wrong: The Hidden Maths of Everyday Life by Jordan Ellenberg – review. *The Guardian.* http://www.theguardian.com/books/2014/jun/13/how-not-to-be-wrong-hidden-maths-jordan-ellenberg-review

Clark-Wilson, A., Hoyles, C., Noss, R., Vahey, P. & Roschelle, J. (2015) Scaling a technology-based innovation: windows on the evolution of mathematics teachers' practices. *ZDM Mathematics Education:* 47

Clark-Wilson, A., Hoyles, C., & Noss, R. (2015). Conceptualising the scaling of mathematics teachers' professional development concerning technology. *9th Congress of European Research on Mathematics Education, Prague, Czech Republic. 4th - 8th February 2015.*

Coburn, C. (2003). Rethinking Scale: Moving Beyond Numbers to Deep and Lasting Change. *Educational Researcher, 32*(6), 3-12.

Diener, C.I, Dweck, C.S. (1980) An analysis of learned helplessness: II. The processing of success Journal of Personality and Social Psychology, 39 (1980), pp. 940–952)

Drijvers, P, Kieran, C & Mariotti M et al (2011) Integrating Technology into Mathematics Education: Theoretical Perspectives in Hoyles. C & Lagrange J-B (eds) (2009) *Mathematics Education and Technology- Rethinking the terrain* Springer Chapter 7 pp 89- 132.

Education Endowment Foundation (2014) 'Scratchmaths' research project, http://educationendowmentfoundation.org.uk/projects/scratch-programming/ accessed 27th March 2015

EPSRC (2012) 'Deloitte Report' - Measuring the Economic Benefits of Mathematical Science Research in the UK - http://www.epsrc.ac.uk/newsevents/news/mathsciresearch/

Healy, L., & Kynigos, C. (2010). Charting the microworld territory over time: Design and construction in mathematics education. ZDM: *The International Journal on Mathematics Education*, 42(1), 63–76.

Harel & S. Papert (eds.), (1991) *Constructionism*, pp. 269-394. New Jersey: Ablex Publishing Corporation.

Hour of Code website http://hourofcode.com/us accessed 27th March 2015.

Hoyles, C. Noss, R., Kent, P. & Bakker, A. (2010) *Improving Mathematics at Work: The need for techno-mathematical literacies* Routledge

Hoyles, C. *The proceedings of the Vth SIPEM* (2012) Petrópolis, Rio de Janeiro *Sociedade Brasileira de Educação Matemática – SBEM* pp 1-12

Hoyles, C., & Lagrange, J.B. (Eds.), (2009) *Mathematics education and technology: Rethinking the terrain* (pp. 89-132). New York: Springer.

Hoyles, C., Noss, R., Kent, P., & Bakker, A. (2013) Mathematics in the Workplace: Issues and Challenges in Damlamian, A., Rodrigues, J.F., Strässer, R. (eds.), *Educational interfaces between mathematics and industry: report on an ICMI-ICIAM-study*, pp. 43 – 51 New ICMI study series, pp. 43 - 51

Hoyles, C., Noss, R., Roschelle, J. & Vahey, P. (2013) Cornerstone Mathematics: Designing Digital Technology for Teacher Adaptation and Scaling. *International Journal on Mathematics Education (ZDM),* 45 (7). p. 1057. ISSN 1863-9690.

Hung, D., Lim, K., & Huang, D. (2010). Extending and scaling technology-based innovations through research: The case of Singapore. In Organisation for Economic Co-operation and Development (Ed.), *Inspired by Technology, Driven by Pedagogy [electronic resource]: A Systemic Approach to Technology-Based School Innovations* (pp. 89-102): OECD Publishing.

Institute of Mathematics and its applications (2011) Advancing the Digital Arts (report). http://www.ima.org.uk/viewItem.cfm-cit_id=383289.html

Institute of Mathematics and its applications. Mathematics Matters. http://www.ima.org.uk/i_love_maths/mathematics_matters.cfm.html

Jenkins, S. (2014) For Britain's pupils, maths is even more pointless than Latin. *Guardian*. http://www.theguardian.com/commentisfree/2014/feb/18/maths-more-pointless-than-latin-british-pupils-china

Noss, R. & Hoyles, C. (1996) *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht: Kluwer Academic Publishers.

Papert, S & Harel I. (1991) Preface, Situating Constructionism, in Harel & S. Papert (Eds), Constructionism, Research reports and essays, 1985-1990 (p. 1), Norwood NJ.

Resnick, M . (2012) Mother's Day, Warrior Cats, and Digital Fluency, Stories for the Scratch online Community, *Proceedings of Constructionism 2012* Educational Technology Lab pp 52-58

Roesken-Winter, B. Hoyles C & Blömeke, S. Evidence-based CPD: Scaling up sustainable interventions, *ZDM Mathematics Education* (2015) 47:1–12

Scratch website https://scratch.mit.edu/ accessed 27th March 2015.

Vahey, P., Knudsen, J., Rafanan, K. & Lara-Meloy, T. (2013). Curricular Activity Systems Supporting the Use of Dynamic Representations to Foster Students' Deep Understanding of Mathematics. In C. Mouza and N. Lavigne (Eds.). *Emerging Technologies for the Classroom,* pp. 15 – 30. Springer, NY.

Weir, S. (1987). *Cultivating Minds: A Logo Casebook.* London: Harper