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Artículos de investigación

A NARRATIVE LITERATURE REVIEW OF CONTEMPORARY HISTORIC-EPISTEMOLOGICAL STUDIES ON ALGEBRA: SOME IMPLICATIONS FOR MATHEMATICS EDUCATION

UNA REVISIÓN BIBLIOGRÁFICA NARRATIVA DE ESTUDIOS HISTÓRICO-EPISTEMOLÓGICOS CONTEMPORÁNEOS SOBRE EL ÁLGEBRA: ALGUNAS IMPLICACIONES PARA LA EDUCACIÓN MATEMÁTICA

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ABSTRACT

Various approaches to teaching and learning of Algebra within Mathematics Education come from classical studies in the History of Mathematics. Consequently, in this second discipline, a narrative literature review of contemporary sources, between 2000 and 2018, in prominent data bases and journals was carried out to identify new elements that could contribute to the strengthening of these approaches. We focused this review on the contributions of recent renowned mathematics historians that have had immersed in new findings related to the development of Algebra. In this paper, we present at least six considerations that can be problematized from the perspective of Mathematics Education, which generate new routes of investigation that could contribute significantly to a more robust and profound understanding of algebraic activity in general, and positively impact on the understanding of development of algebraic activity in mathematical education.

Keywords: history of algebra, history-epistemological studies, algebra development, symbolic algebra.

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RESUMEN

Diversos enfoques para la enseñanza y aprendizaje del Álgebra dentro de la Educación Matemática provienen de estudios clásicos de la Historia de la Matemática. En consecuencia, en esta segunda disciplina, se realizó una revisión bibliográfica narrativa de fuentes contemporáneas, entre los años 2000 y 2018, en bases de datos y revistas destacadas para identificar nuevos elementos que pudieran contribuir al fortalecimiento de estos enfoques. Enfocamos esta revisión en los aportes de reconocidos historiadores de las matemáticas recientes que han estado inmersos en nuevos hallazgos relacionados con el desarrollo del Álgebra. En este trabajo presentamos al menos seis consideraciones que pueden ser problematizadas desde la perspectiva de la Educación Matemática, las cuales generan nuevas rutas de investigación que podrían contribuir significativamente a una comprensión más robusta y profunda de la actividad algebraica en general, e impactar positivamente en la comprensión del desarrollo de la actividad algebraica en la educación matemática.

Palabras clave: historia de las matemáticas, estudio histórico-epistemológico, desarrollo del álgebra, álgebra simbólica.

1. INTRODUCTION

Historic-epistemological studies (HES) in Mathematics Education (ME) represent a key element in didactic research to improve the educational practice in mathematics (Artigue, 1990) since they humanize the mathematical activity and provide elements to enrich the knowledge currently taught in schools (Buendía and Montiel, 2011; Furinguetti, 2004; Panasuk & Horton, 2012). There is a vast number of compilations and works regarding the use of the historical dimension in the practice of mathematics education (see Barbin, et. al., 2018; Barbin, Guillemette, & Tzanakis, 2020; Clark, Kjeldsen, Schorcht & Tzanakis, 2018; De vittori, 2023; Díaz-Chang & Arredondo, 2023; Fauvel & Van Maanen, 2002; Fried, 2001, 2007, 2008; Haverhals & Roscoe, 2010; Katz & Tzanakis, 2011; Panasuk & Horton, 2012). For instance, Tzanakis et. al. (2000) established that it contributes to the learning of mathematics, in terms of making visible the progress of ideas, techniques, processes, problems, and questions that are often overshadowed in teaching and, that can be regarded as teaching contents.

Furthermore, other aspects lead to understand the nature of mathematics and its activity. Some of these aspects make the didactic frameworks more robust for teachers, or become affective considerations regarding mathematical activity, such as perseverance, the appreciation of misunderstandings, mistakes, and persisting ideas as part of the mathematical doing; and even a broader perspective of mathematics as a cultural effort (Tzanakis et. al., 2000).

Among the HES objectives stands the understanding of the formation of mathematical thought processes to ground didactic intervention. The processes mentioned refer to the genesis of mathematical ideas, the conditions of their emergence, their evolution, and the persistence of certain problems in specific cultures and periods (Bartolini & Sierpinska, 2000; Gallardo, 2002; Radford, 1997, 2000). These insights can lead us to: i) understand the structure and nature of mathematics as scientific knowledge and its complex development; and ii) have more comprehensive and less simplistic views of the relevance, adaptation, and incorporation of both curricular content and mathematics activity in the classroom. Furthermore, the insights can inform the mathematics educator to devise relevant questions that become decisions on what elements of the mathematical culture should be put into play while teaching mathematics (Kidron, 2016). Moreover, Radford (1997, 2000) states that it is important to have robust theoretical frameworks and methodologies to adequately explain the construction of mathematical knowledge as well as to inform the articulation between the historical and psychological domains for the instructional design.

Particularly, research in the Algebra domain of ME is far more extensive than that in any other topics (Charalambous & Pitta-Pantazi, 2016). However, the foundational HES in this domain came from classical studies in the History of Mathematics, to name a few: Boyer (1986), Høyrup (1986), Klein (1968), Mahoney (1981), Nesselmann (1842) and, Piaget and Garcia (1982). These studies provided a more thorough understanding of the development of this area, and they set the foundation for important approaches to the teaching and learning of algebra. However, for a particular study (López-Acosta, 2023) we found that in the last decades, research in the History of Mathematics (HM) has brought new advances regarding the understanding of the algebraic activity. This led us to consider the implications of these insights in the field of the HM compared with previous models and approaches within ME.

In this paper, we present some examples of contemporary HES in the field of HM, regarding the development of symbolic algebra. Furthermore, the examples provide elements and conceptualizations that, as far as this review showed, have not been addressed in ME yet. Therefore, our aim in this work is to provide just a few accounts for new paths for didactic research within ME. This accounts, we believe, are important implications for the research in ME in terms of the potential for new understandings about the algebraic activity not previously considered in the learning and teaching of Algebra.

In the first part of the paper, we present some methodological aspects that guided the narrative literature review. The second part consist in presenting two main results based on HES in algebra within ME that we found more relevant to discuss, considering the new explorations and results derived of later historical studies in HM. We briefly focus on the tripartite model of Nesselman (1842) for the development of algebra and the typically reported contributions of Viète and Descartes to the development of algebra. The third part is oriented to address those recent research findings in the HM, regarding the alternative models for the development of algebra and, other relevant aspects of algebraic reasoning or contributions of Viète and Descartes not previously reported in ME. Based on the second and third parts, we will discuss six relevant considerations providing new insights, further explanations, or comparisons for the development of symbolic algebra and algebraic activity. All these could orient us to ask new questions and find new elements to incorporate in the research of the development of algebraic activity in ME.

2. THE NARRATIVE LITERATURE REVIEW

Narrative reviews, also known as *unsystematic narrative reviews,* are narrative syntheses of previous information (Green, johnson & Adams, 2006). They also can be characterized as an iterative, non-structured and multi-layered process without rigid steps and rules (Green, johnson & Adams, 2006; Juntunen & Lehenkari, 2021). However, a narrative review should "be well structured, synthesize the available evidence pertaining to the topic, and convey a clear message" (Green, johnson & Adams, 2006, p. 106). Typically, as reported by Juntunen & Lehenkari (2021), the literature review process contemplates *the definition of the objective and research questions*, *developing and validating a review protocol*, *searching the literature*, *selecting the literature*, *analysing*, *synthesising*, *concluding* and, *reporting*. We followed this process to do our review.

This narrative literature review was part of a bigger study related to the emergence of the parametric equations in the works of Viète and Descartes (López-Acosta, 2023;

López-Acosta & Montiel, 2021, 2022). In these studies, it was very important to compare the works of these two mathematicians with the previous algebraic tradition to better understand the innovations of both. As a result of the literature review, we found in resent studies in the HM important considerations that we believed were worthy to be problematize in ME, even when those were not related to the foci of the studies cited before.

Thus, in this work we want to convey relevant considerations, to provide new insights into previous models of the development of symbolic algebra and algebraic activity, that are needed to be explored in further studies in Mathematics Education. We were interested in new findings of these topics in the History of Mathematics field and compare them to those of the classical HES within Mathematics Education. The research questions that guide the literature review were: *What new explanations of the development of symbolic algebra have been developed in the history of mathematics? What new types of symbolic operations have been present throughout history?* and *how these insights can bring new paths for research in ME regarding the learning of symbolic algebra?*

The protocol to collect sources different from those in the ME field, as suggested by Siebert (2019), were consulting handbooks, chapters, and prominent compilations; searching highly regarded journals and reference sections of the papers located; and browsing the web. The search began with relevant key terms in databases[3], such as 'algebra', 'symbolic algebra', 'history of algebra', 'symbolic operability', 'algebra development', 'history of symbolic algebra', 'Viète', and 'Descartes'. The last two categories were included eventually on account of the role both mathematicians played in the development of symbolic algebra. We consulted the period between 2000 and 2018.

After collecting the sources, we selected those which could help us respond the research questions and we particularly locate the authors who had studied specific elements related to the development of symbolic algebra, seeking insights concerning the ones established in ME. Finally, we went deeper into the *primary sources* (mainly books, chapters, and papers)—in the sense of Fraenkel, Wallen, & Hyun (2012)—of those authors and their insights to assess how they could provide new questions, research paths, and new elements to advance in the understanding of the algebraic activity. In this phase, we reviewed some original treatises by algebraists considered in the primary sources to fully understand and complement the findings. We did this by solving some of the problems commented in these papers to compare the different types of reasoning and methods addressed. Some of these original treatises are cited in the fourth section.

We found that scholars, such as Albrecht Heeffer, Chikara Sasaki, Giovanna Cifoletti, Henk Bos, Jaqueline Stedall, Jeffrey Oaks, Michel Serfatti, Maria Massa Esteve, Roy Wagner, to mention a few, had made, recently, important contributions concerning the development of algebra in specific periods (Middle Ages, Early Renaissance, Renaissance, and post-Renaissance), cultures, and algebraic practices not fully explored or revisited in ME.

3. SOME HISTORICO-EPISTEMOLOGICAL RESULTS IN ALGEBRA WITHIN MATHEMATICS EDUCATION

It is important to recognize that works like those of Freudenthal (1977), Kieran (1992), Sfard (1995), Gascón (1989, 1994-1995), Charbonneau (1996), Radford (1995, 1996, 1997, 2001), Rojano (1996), Puig (1998), Filloy (1999), Malisani (1999), Gallardo (2002), Puig &

3 Some of the data bases and prominent journals consulted were: Jstor, Web of Science, Science Direct, Springer, Historia Mathematica, Archive for History of Exact Sciences, Philosophica, Foundations of Science.

Rojano (2004), Katz & Barton (2007) and Filloy, Puig & Rojano (2008) had established the fundamentals that led to didactic approaches to algebraic thinking and language development in our field. For the purpose of argumentation in this paper, we only present two main relevant considerations deriving from some of these studies: 'phases in the development of algebra' and 'some elements of the symbolic algebra and the relevance of Viète and Descartes'.

3.1. PHASES IN THE DEVELOPMENT OF ALGEBRA

It is well known that the work of Nesselman (1842) has had an important influence in the research of algebra because of his tripartite model of algebra development. Based on the three phases, *rhetorical*, *syncopated*, and *symbolic,* it was possible to understand the complex and long process it took humanity to develop algebraic symbolism (Malisani, 1999; Kieran, 1992), and formulate reflections and explanations regarding the difficulties students encountered in developing this symbolism. Kieran (1992, p. 391) stated that "some of the cognitive processes involved in learning school algebra find their roots in the historical development of algebra as a system of symbols". This approach has led to questions about the parallelism between phylogenesis and ontogenesis in the case of algebra (see Harper, 1987; Kieran, 1992; Sfard, 1995). Another argument derived from this is that in cognitive terms, the transition between each of the three phases implies a change from procedural to structural thinking (Kieran, 1992; Sfard, 1995).

Nevertheless, this characterization of the development of algebra has been criticized by some researchers pointing out a positivist vision of historical events (see Radford, 1997; Chorlay & de Hosson, 2016) showing that these phases reflect in a limited way the true innovations that took place in each one of them. In addition, not only was the distinction of the phases criticized, but also were the *recapitulationist* approaches to parallelism and the relation of phylogenesis and ontogenesis in general. These critics argued that this relationship was more complex than was thought, and that the incorporation of the socio-cultural dimension in HES came to contradict the recapitulationist approaches (Radford, 1997; Radford and Puig, 2007; Schubring, 2011).

> From a sociocultural perspective, this division of algebra seems to be completely different: syncopated algebra was not an intermediate stage of maturation in which knowledge took a kind of rest in its tiring race towards symbolism. Instead, it was merely a technical strategy that the limitations of writing and the lack of printing in past times imposed on the diligent scribes that had to copy manuscripts by hand (Radford, 1997, p. 27).

As we are going to show later it is precisely the syncopated phase of algebra that has been strongly challenged for some historians.

3.2. SOME ELEMENTS OF SYMBOLIC ALGEBRA AND THE RELEVANCE OF VIÈTE AND DESCARTES

Some works have particularly studied more in depth the production of Viète's *Analytical art* and Descartes' *Cartesian method* during the symbolic phase of algebra (see Harper, 1987; Charbonneau, 1996; Rojano, 1996; Gascón, 1989, 1994-1995; Puig and Rojano, 2004; Filloy, Puig and Rojano, 2008). These works emphasize the degree of generality that algebra reached thanks to the use of symbolism that distinguished between known and unknown numbers, highlighting how algebra emancipated itself from geometry to become an autonomous mathematical field (Charbonneau, 1996; Rojano, 1996). In this stage, the construction of the *algebraic formula* was achieved since the equations did not contain specific coefficients, but coefficients in terms of parameters. Thus, some authors (e.g., Chevallard, 1989; Gascon, 1989, 1994-1995), building on Jacob Klein's (1968) ideas, pointed out the fact that authentic algebraic activity is that which makes a systematic use of parameters and unknowns for the modeling of different kinds of problems, whether arithmetic or geometric. However, one of the most common characterizations of the notion of the algebraic parameter, first found in Viète, refers to its function "to represent *givens* in expressing general solutions and as a tool for proving rules governing numerical relations" (Kieran, 1992, p. 391, original emphasis). In this regard, some researchers (e.g., Chevallard, 1989; Gascon, 1989, 1994-1995) have argued that it was a partial way of understanding the algebraic activity.

Based on this brief description, we must say that the above mentioned and other research works not reported here have produced a vast number of results that have had an impact—to a greater or lesser extent—on the decisions made for mathematics education today (Socas, 2011). Nevertheless, as we will show next, the review of more recent studies in the field of the History of Mathematics shows some progress in aspects we had previously referred to, generating new explanations and models for the algebraic activity, on the functionality of the algebraic symbolism, and about new insights on Viète and Descartes' algebraic analysis.

4. CONTEMPORARY STUDIES IN THE HISTORY OF MATHEMATICS CONCERNING ALGEBRA

As we specified in section 2, by contemporary studies we consider historical studies regarding the development of algebra after the 2000 and, consequently, insights that were not previously incorporated on the approaches for the learning of algebra in the fiel of ME.

One of the most important aspects of contemporary HES in the development of algebra is that the methods and views are based on more contextual accounts of history, trying to avoid anachronisms (Heeffer, 2014). Discussed in this section, studies by Heeffer (2008a, 2008b, 2009, 2010a, 2010b, 2014), Massa Esteve (2008, 2012), Oaks (2018), Sasaki (2003), and Stedall (2000, 2003, 2007, 2008, 2011), provide a more robust understanding of the cultural and social characteristics that had an impact on the development of algebra, considering that unlike classical HES in this topic, more sources are available today.

4.1. NEW EXPLANATIONS REGARDING ALGEBRA DEVELOPMENT

Heeffer (2009, 2010a) emphasizes the need to generate alternative models to Nesselmann's, arguing how obsolete it is for current scientific practice to continue considering such model. He finds three problems with the tripartite distinction centered on '*the myth of syncopated algebra*'. The first problem is the chronology of the division since he points out that the rhetorical and syncopated phases overlap with each other. Nesselman considers the period of rhetorical algebra from Iamblichus to Arabic algebra, the Italian algebra of the abacus and Regiomontano, a period that spans approximately from 250 to 1470. The period of syncopated algebra spans from the *Arithmetica* of Diophantus to the European algebra of the mid-17th century, which includes even Viète and Descartes. Finally, the period of symbolic algebra is modern algebra with the symbolism we know today. However, scholars of Diophantus place the *Arithmetica* between 250 and 350. Thus, the rhetorical and syncopated stages overlap almost completely, which leaves the question whether the two systems did not influence each other. The second problem refers to the role of scribes in the translation of manuscripts, he indicates that the first Arabic translations of the *Arithmetica* were obtained around the 12th century, which separates by a long time the original manuscript from its first translation and that, in that period of time innovations were made by the scribes in the transcription of the manuscripts, including abbreviations of the words to save time, effort and money, for which Heeffer points out that the syncopation could not be an invention of Diophantus but of the scribes. Furthermore, he mentions that the first translations of *Arithmetica* into Arabic did not show signs of syncopated structures. Finally, since the symbolism in *Arithmetica* is not close to the algebraic symbol (in the sense of Klein, 1968) but to a ligature, the syncopated category is deprived of the element that distinguishes it from the rhetorical phase.

Based on these arguments and in his own studies, Heeffer (2008a, 2009, 2010a) proposes an alternative model:

1. *Non-symbolic algebra:* this is an *algorithmic type of algebra* dealing with numerical values only or with a non-symbolic model. Typical examples are Greek geometric algebra or the Chinese method of solving linear problems with multiple unknowns (Fāng chéng).

2. *Proto-symbolic algebra*: *algebra which uses words or abbreviations* for the unknown but is not symbolic in character. This would include Diophantus, Arabic algebra, the early Abbacus algebra, and the early German cossic algebra.

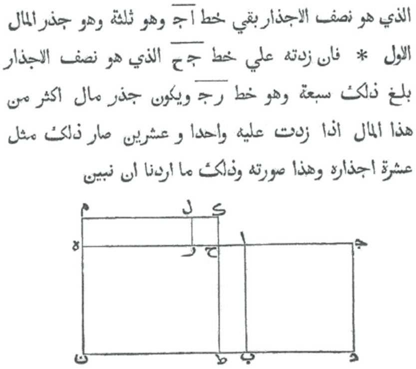
3. *Symbolic algebra: algebra using a symbolic model*, which allows for manipulations on the level of symbols only. It was established around 1560, and prepared by later abbacus and cossic algebra, Michael Stifel, Girolamo Cardano, and the French algebraic tradition. (Heeffer, 2009, p. 9, emphasis added)

Heeffer shows that before the algebraists usually cited for having constructed the algebraic symbolism closest to the modern one—such as Viète and Descartes'—, the way of thinking already possessed a symbolic rationale despite the lack of a symbolic semiotic resource like the modern one has.

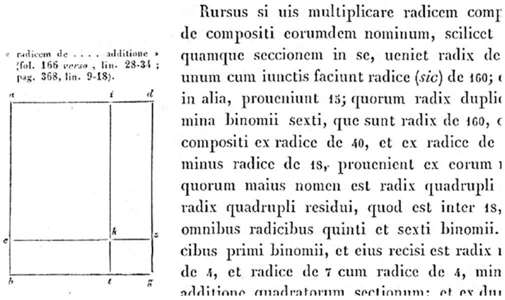
> In our opinion, the road to symbolic algebra was paved by several previous stepping-stones that were functional in developing the symbolic mode of reasoning. The major obstacle in recognizing the importance of previous developments has been the *confusion between the use of symbols and symbolic reasoning*. […] [S]everal instances of symbolic reasoning in algebraic problem solving can be identified while no symbols are being used (Heeffer, 2008a, p. 153, emphasis added).

Heeffer (2010a) argues that during the sixteenth century there was a transition from reasoning based on geometric models (Figure 1) to symbolic reasoning, and characterizes it as based on arithmetic rules that can be applied to non-arithmetic objects, assigning to the symbolism the ability not only to represent but also to create new objects (Heeffer, 2008a, 2010a, 2014). Furthermore, he identified that this transition came from the generalization of arithmetic rules to the creation of explanations of 'not understood or accepted objects' in ontological terms, such as the case of negative, irrational, and imaginary numbers (Heeffer, 2008a, 2009, 2010a).

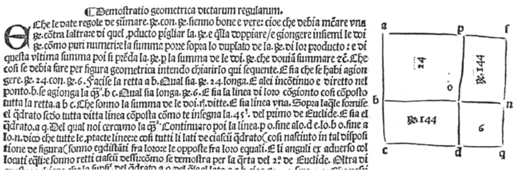
Figure 1 – Epistemic justification based on geometric reasoning.



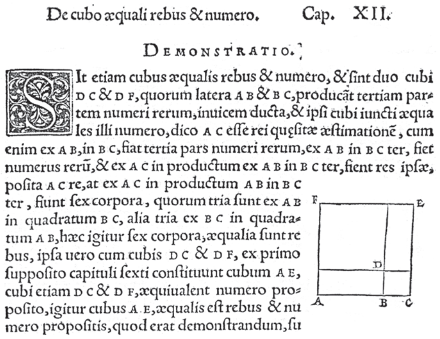
Justification of the rule "*a treasure plus twenty-one dírhems equals ten roots*" (al-Khwârizmî, in Rosen, 1831, p. ١٣).



Justification for the extraction of some root by Fibonacci (Boncompagni, 1857, p. 368).



Demonstration for the extraction of some root (Pacioli, 1523, fol. 117).

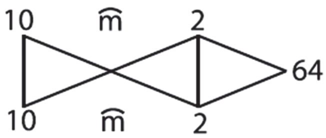


Demonstration of the case *cube equals square plus number* (Cardano, 1545, fol. 31).

Source: Self elaboration.

Heeffer (2014) further shows that, just as those geometric models had the function of *epistemic justifications* for procedures that defined rules for certain cases, arithmetic diagrams also had the function of justifying arithmetic rules and that these were used as a means to validate unaccepted results (Figure 2). In particular, he shows the case of Dardi, cited in Høyrup (2010), who justifies the product of two negative numbers and the case of how Cardano justifies the product of √-15 • √-15, both under the cross-multiplication algorithm.

Figure 2 – Dardi's justification based on cross-multiplication.



Source: From Høyrup (2010, p. 23).

Dardi justifies that the product of two negative numbers is positive using the following reasoning: since 8 • 8 = 64, which is the same as (10-2) • (10-2) = 64, then, based on the cross-multiplication rule (10-2) • (10-2) = 100 - 20 - 20 + (-2) (-2) = 64, and so (10 - 2) • (10-2) = 60 + (- 2) (- 2) = 64, implying that to ensure the relation between the quantities it must be true that (- 2) (- 2) = 4.

Thus, cross-multiplication is an epistemic justification that validates other objects whose nature is unclear, allowing the construction of new knowledge.

Another of Heeffer's contributions is the characterization of the six phases that led to the emergence of what he calls the *symbolic equation* (Heeffer, 2008a, 2009, 2010a, 2010b) which was completed by Buteo (1559) and, which happened in the period between Cardano's and Gousselin's treatises (1539-1577). These phases—see full description in Heeffer (2008a)—describe a progressive process of objectification of algebraic numbers (negative and imaginary), polynomials, equations, and systems of equations. Except for the third phase, all the others are attributed to Cardano (1539) which could be understood as a sign of a paradigm shift. The phases are as follows:

1. *The expansion of arithmetic operations to polynomials*. The arithmetic operations are applied to objects that are not necessarily natural numbers, such as polynomials and whole numbers, fractions, and irrationals —even those not completely accepted, such as negative and imaginary numbers—.

2. *Equating polynomial expressions*. There is a shift from the classical practice of operating with *co-polynomials* to the operations with polynomials by making it explicit that the affectation in operations occurs on both sides of the equality (Figure 3-I).

   The term *coaequare* denotes *the act of keeping related polynomials equal*. The whole rhetoric of abacus texts is based on the reformulation of a problem using the unknown and the manipulation of coequal polynomials to arrive at a reducible expression in the unknown. One looks in vain for equations in abacus texts (Heeffer, 2008b, p. 119, original emphasis).

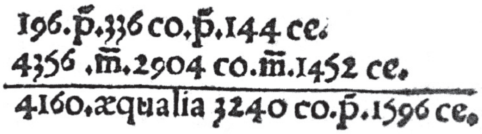
3. *Introduction of the second unknown*. It implies a different treatment compared with equations with only one unknown. Heeffer (2010b) explains that before 1560 it was unusual to use more than one unknown in the solution of problems (Figure 3-II).

4. *Expansion of arithmetical operators to equations*. The multiplication or division by a scalar number on equations is applied. This was first found in Cardano (1539) (Figure 3-III).

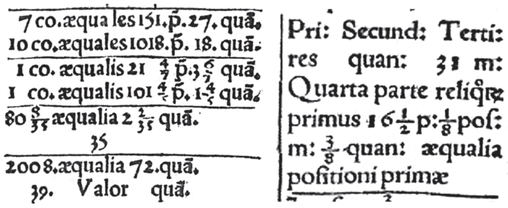
5. *Introduction of letters for multiple unknowns*. The use of several letters is explicitly used to represent each of the unknowns of the problem, a practice that essentially did not exist until before Stifel's work (1544).

6. *Systematic manipulation of linear equations to eliminate unknowns*. Acknowledging the distinction of several unknowns led to the expansion of arithmetic rules applied to the manipulation of equations. Therefore, it was possible to add or subtract for the systematic elimination of unknowns. It was Buteo (1559) who concluded about the construction of the symbolic equation since he used arithmetic operations to eliminate unknowns not only in one equation but also in sets of equations (Figure 3-IV).

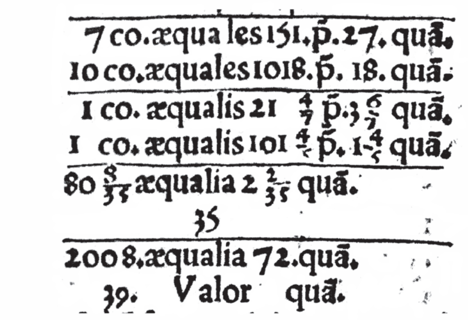
Figure 3 – Transition to the symbolic equation according to Heefer.



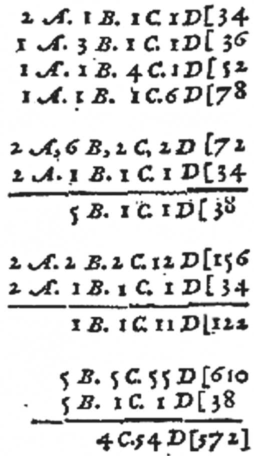
I. Cardano's first equality of polynomials (1539, p. 424).



II. Introduction to Cardano's second unknown (1539, p. 435 & 1545, p. 21).



III. Cardano's operations in equations (1539, p. 435).



IV. Buteo's systematic manipulation of linear equations (1559, p. 194).

Source: Self-elaboration.

4.2. THE RELEVANCE OF GEOMETRIC WORK IN THE DEVELOPMENT OF THE ALGEBRAIC ANALYSIS OF VIÈTE AND DESCARTES

Stedall (2007, 2008, 2011) argues the importance of the geometric work that Viète developed in his mathematical production. She states the following:

> Viète gave algebra a startling new priority as a tool for investigating and analyzing the problems and theorems of classical geometry. Even the hitherto intractable difficulties of doubling the cube or trisecting an angle were now, in his opinion, amenable to algebraic treatment (Stedall, 2011, p. 28).

This interpretation of the relevance of geometry in Viète's work has also been recently argued by Oaks (2018). This author emphasizes that basic geometric knowledge for astronomy was of great interest to Viète before 1570, which led to the development and improvement of geometric models for astronomical calculations (Oaks, 2018).

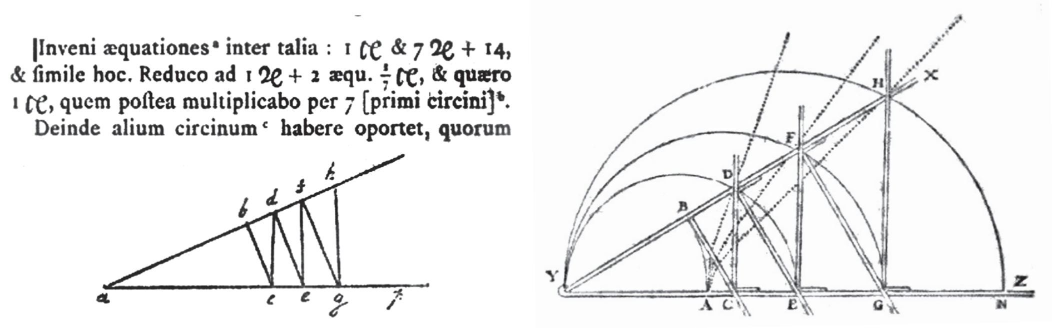
For Oaks (2018) it is sufficient to say that the notion of number in Viète is that of a geometric magnitude. Oaks reinterprets Klein's (1968) dual sense of the number in Viète, implying that Viète was building an *algebra for geometry*, about which Oaks circumvents three problems with non-arithmetic geometric magnitudes, and mentions that this mathematician explicitly solved in his *analytical art:*

> How can equations be formed if the magnitudes participate in the category of "quantity" through ratio and proportion? If magnitudes of different dimension are heterogeneous, how can they be added and subtracted? How can meaning be given to magnitudes of dimension greater than three? (Oaks, 2018, pp. 275-276).

According to Oaks (2018), the first problem was solved using the theory of proportions. Since each proportion establishes an equality, this allows a natural transition from *a:b :: c:d* to *ab=cd*. For the second, the homogeneity law establishes that to compare or operate with the species it is necessary to compare them with magnitudes of the same dimension, which allows the operation as is the case of the equation $AC^3\text{-}3(AC \times AB^2\ ) = (CE \times CD^2)$—see Viète (1646, pp. 248-249)—, which is carefully constructed from the comparison of quadratic and cubic expressions with planes and solids, respectively. Finally, Viète solved the third problem by mentioning that magnitudes of dimension greater than three are useful to calculate and solve problems of angular sections, so it can be justified and necessary to work with this kind of dimensions. This is an aspect that Oaks considers Klein (1968) overlooked; and thus, Viète's astronomical and cosmological program makes sense. Furthermore, Oaks (2018) argues that this consideration is where it is possible to see the function of algebraic symbolism because although Viète does not find meaning or significance in these types of dimensions, they are useful. It means that he works through symbolism with objects that are not well understood.

According to Bos (2001) and Sasaki (2003), Descartes had also expressed the importance of geometry in his project of new science, which he called *Mathesis Univesralis* in the *Regulae ad directionem ingenii* (1628, published posthumously in 1701). In a letter to Beeckman in March 1619, Descartes stressed how important his compasses were to him since he used them to demonstrate the solution of equations. Specifically, he showed how to solve a cubic equation ($x^3 = 7x + 14$, in anachronical notation) using the *mesolab*, which appeared in *La Géométrie*, with which the geometric progression $1, x, x^2, x^3, x^4, x^5,\ldots$ could be constructed (Figure 4).

Figure 4 – Descartes's mesolab.



I. *Cogitationes Privatae* (Adam & Tannery, 1908, p. 234)

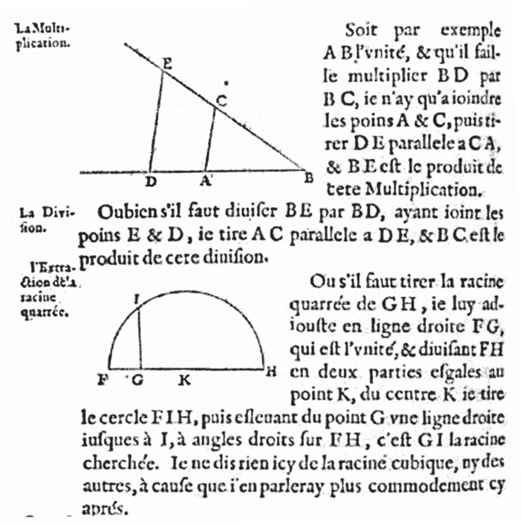
II. *La Géométrie* (Descartes, 1637, p. 318)

Source: Self-elaboration.

The use of compasses and geometric instruments shows Descartes' interest in finding the solution to equations geometrically (Sasaki, 2003).

Considering the classical geometric analytical method, the synthesis implied the construction of the figure. Therefore, if his project used algebra as a new tool, he had to ensure the construction of the equations and their solutions using geometry. Consequently, algebra was only a part of the cartesian method (Bos, 2001). Under this premise, it makes sense why from the beginning of *La Géométrie* he establishes geometric constructions of the arithmetic operations and the solution to equations (Figure 5).

Figure 5 – Descartes' arithmetization of geometry.



La Multiplication.

Soit par exemple A B l'vnité, & qu'il faille multiplier B D par B C, ie n'ay qu'a ioindre les poins A & C, puis tirer D E parallele a C A, & B E est le produit de cete Multiplication.

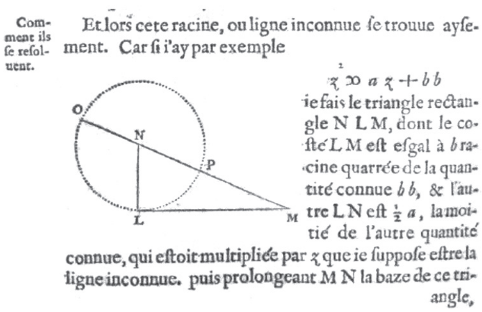
La Diuision.

Oubien s'il faut diuiser B E par B D, ayant ioint les poins E & D, ie tire A C parallele a D E, & B C est le produit de cete diuision.

l'Extraction de la racine quarrée.

Ou s'il faut tirer la racine quarrée de G H, ie luy adiouste en ligne droite F G, qui est l'vnité, & diuisant F H en deux parties esgales au point K, du centre K ie tire le cercle F I H, puis esleuant du point G vne ligne droite iusques à I, à angles droits sur F H, c'est G I la racine cherchée. Ie ne dis rien icy de la racine cubique, ny des autres, à cause que i'en parleray plus commodement cy aprés.

Constructions for multiplication, division, and square root (Descartes, 1637, p. 298)



Comment ils se resoluent.

Et lors cete racine, ou ligne inconnue se trouue aysement. Car si i'ay par exemple

$z^2 = az + bb$

ie fais le triangle rectangle N L M, dont le costé L M est esgal à b racine quarrée de la quantité connue b b, & l'autre L N est ½ a, la moitié de l'autre quantité connue, qui estoit multipliée par z que ie suppose estre la ligne inconnue. puis prolongeant M N la baze de ce triangle,

Construction of the solution to the equation $z^2 = az + b^2$ (Descartes, 1637, p. 302)

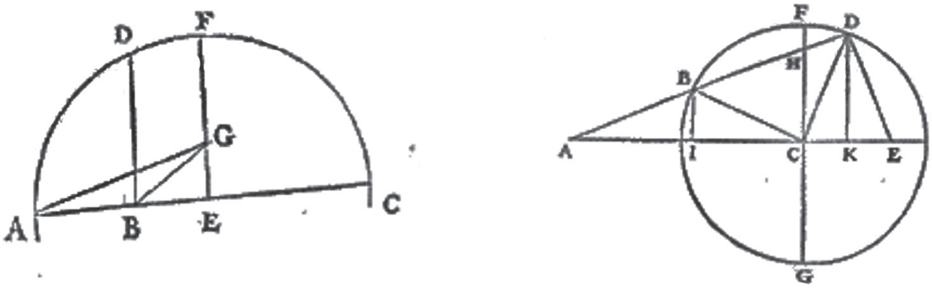
Source: Self-elaboration.

As these studies show, the affirmations regarding the emancipation of algebra from geometry to become an autonomous mathematical field (Charbonneau, 1996; Rojano, 1996), indicates that it was not the case for these mathematicians. Furthermore, in the studies of López-Acosta (2023), López-Acosta & Montiel (2021, 2022), it is discussed that the invention of the parametric quantities was influenced by the geometrical nature of the problems that were solved by both.

It is known that by the time of the *Regulae ad directionem ingenii* Descartes already possessed a large part of the scheme of thought that he definitively embodied in *La Géométrie*—where geometric analysis and algebra played a central role—, however, he still had to overcome the obstacle of dimension. This step was fundamental for the construction of the algebra of segments, first embodied in *La Géométrie* and where parametric quantities appear systematically and explicitly. This fact led to López-Acosta (2023), López-Acosta & Montiel (2021, 2022) to question what happened between 1628 and 1637 that allowed Descartes to make this leap.

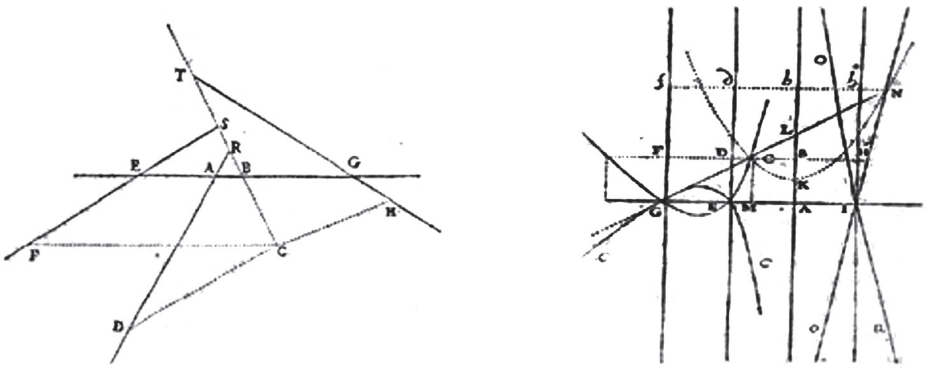
In 1631 it was proposed to Descartes to solve the Pappus problem. This *locus* problem, like many of the problems solved by Viète, involves many relations that are not perceptible from the geometrical diagram present in the text, which requires not only a precise system to characterize the known and unknown quantities (see Figure 6 and 7), but a system to approach the geometrical magnitudes. Therefore, it is conjectured that it was the geometrical nature of the problems that allowed Descartes to refine his analytical method (Bos, 2001; Sasaki, 2003).

Figure 6 – Problems concerning the trisection of angles in Viète's *Supplementum Geometriae.*



Source: Viète (1646, p. 248 and p. 249 respectively).

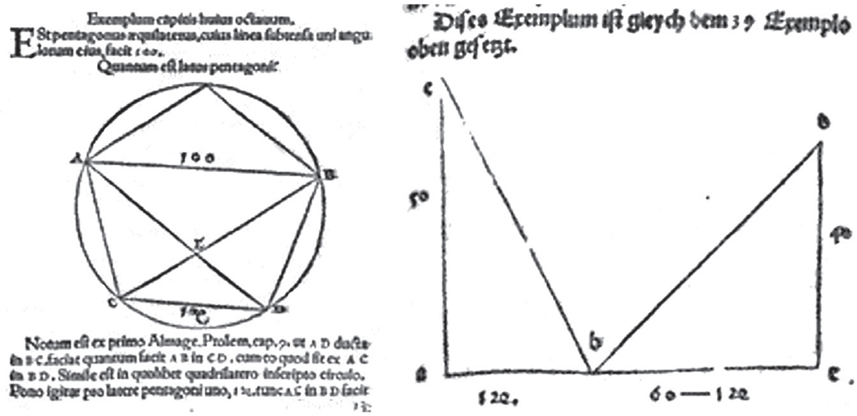
Figure 7 – The Pappus problem for four and five lines in *La Géométrie*.



Source: Descartes (1637, p. 309 and p. 336 respectively).

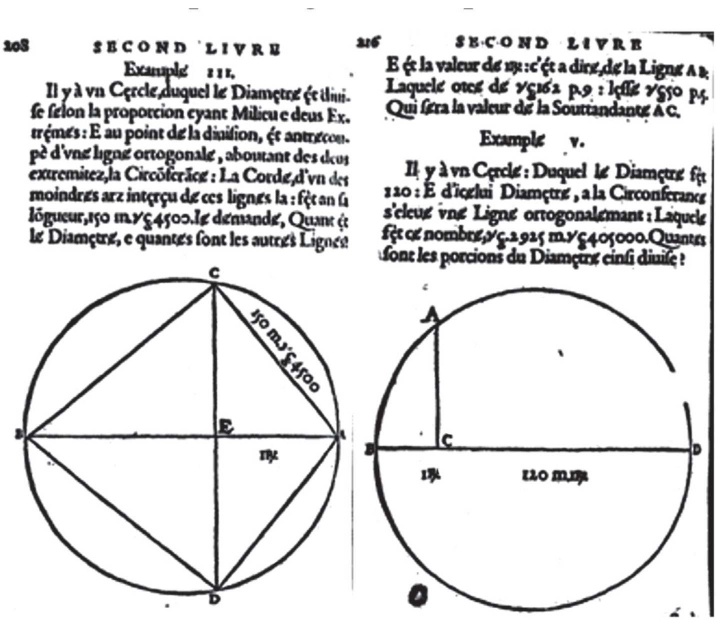
A review of algebraic treatises from between 1494 and 1585—belonging to Luca Pacioli, Girolamo Cardano, Nicola Tartaglia, Jaques Peletier, Ioannes Buteo, Petrus Ramus, Pedro Nunez, Rafael Bombelli, Guillaume Gosselin and Simon Stevin—shows that in the algebraic tradition prior to Viète and Descartes, examples of geometric problem solving did not possess this 'complexity' that was recognized in these later two. For example, in the problems solved by Stifel (1544, 1553) and Peletier (1554)—Figure 8 and 9 respectively—, two things can be identified: the first is that the algebraic expressions involved do not present parameters, but specific coefficients; the second is that the resolution of these problems—unlike the approaches by Viète and Descartes—, was not associated to the construction of formulas or general expressions, but to the determination of the unknown that satisfied the geometric relation established by the problem.

Figure 8 – Examples of geometric problems in Stifel.



Source: Stifel (1544, p. 286 and 1553, fol. 305 respectively).

Figure 9 – Examples of geometric problems in Peletier.



Source: Peletier (1554, p. 208 y p. 216 respectively).

Based on these considerations López-Acosta (2023), López-Acosta & Montiel (2021, 2022) identify that the parametric equation came from the need, both of Viète and Descartes to build an algebra for geometry (Oaks, 2018). For this, as Klein (1968) emphasized, it was needed an extension of the object of study to which the algebra of his predecessors referred, considering not only numbers, but also geometric magnitudes. However, this last consideration makes more sense when analyzing the mathematical activity immersed in the type of geometric problems that both mathematicians solved (see López-Acosta, 2023; López-Acosta & Montiel, 2021, 2022), something that, due to the philosophical and ontological nature of Klein's work, is not possible to see clearly.

4.3. ROOT APPROXIMATION BASED ON ALGEBRAIC REASONING

Another contribution of Stedall (2011) is revealing Viète's work *De Numerosa Potestatum* about the numerical solutions to equations. Such work has not been studied in ME regarding the underlying algebraic thinking. Stedall shows an example of Problem II of this treatise and explains the rationale of the method with which Viète numerically approximates the roots of equations with a degree greater than three. The problem she takes up from Viète's (1646, pp. 166-168) consists in analytically extracting the root of a given cubic number: 157,464.

Viète establishes that an approximation to the root is 50. The reasoning involved considers the expansion of the cube of 50 plus a number k, as shown below:

$$(50 + k)^3 = 50^3 + 3(50)^2k + 3(50)k^2 + k^3 = 157{,}464$$

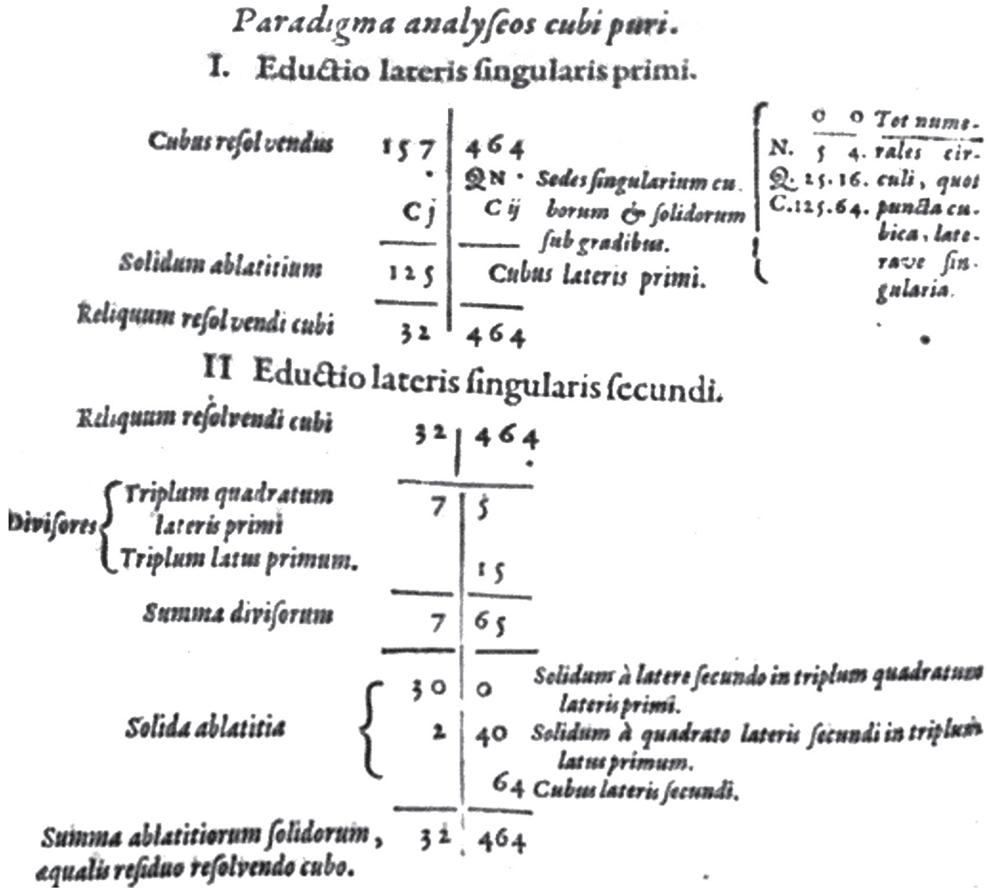
$$125{,}000 + 3(50)^2k + 3(50)k^2 + k^3 = 157{,}464$$

On this relationship, which is described in rhetorical terms in the original text, it begins by subtracting the value of the cube from , obtaining:

$$3\,(50)^2k + 3(50)k^2 + k^3 = 32{,}464$$

Then, he approximates k dividing 32,464 by the coefficient of k (7500), obtaining 4 as a result. This was a strategy previously defined as part of the method with another example (see Viète, 1646, p. 165). Once he gets this approximate value of k, he calculates the values of $3(50)^2\,(4) + 3(50)\,(4)^2 + (4)^3$ and subtracts them from 32,464 getting the rest zero, which is why he determines that the root is 54. Otherwise, Viète states, the root would be irrational. This reasoning is represented schematically using tables in the treatise (Figure 10).

Figure 10 – Viète's schematization of the method of root approximation.



Source: From Viète (1646, pp. 167-168).

This example shows a method for the approximation of roots based on algebraic reasoning that can be further studied.

4.4. THE USE OF ALGEBRAIC SYMBOLISM AS A TOOL TO INVESTIGATE THE STRUCTURE OF EQUATIONS

Further contributions by Stedall (2000, 2007, 2008) are her works related to Thomas Harriot, one of Viète's followers. Stedall shows how Harriot took Viete's results beyond regarding the structures of the equations, managing to investigate and obtain relations between the roots and the coefficients of the polynomials thanks to a more convenient symbolism (Figure 11). "Symbolism became for him not just a more concise way of writing, a kind of mathematical shorthand, but also an investigative tool" (Stedall, 2007, p. 390). Just as Viète, Harriot used vowels for the unknowns and consonants for the known numbers, but he eliminated the words to describe the powers and replaced them with the repetition of the unknown as many times as the power indicated. He also substituted the word equality with the equal sign as Recorde (1557) did.

Figure 11 – Harriot's rewriting of Viète's expressions.

Isagoge
To add (Z square)/G to (A plane)/B
the sum will be (G in A plane) + (B in Z square)/B in G

Praxis

$$\frac{ac}{b} + \frac{dd}{g} = \frac{acg + bdd}{bg}$$

Source: From Stedall (2008, p. 465).

Figure 12 shows that Harriot was investigating the structure of polynomials by multiplying linear factors, which according to Stedall (2011) is one of his greatest contributions to the theory of equations, and that with it, the relationship between the roots and the coefficients of the polynomial could be seen in a "transparent" way. Stedall (2007, p. 383, original emphasis) mentions:

> Harriot's mathematics is almost wordless because he expects (and he is almost always right) that his reader will be able to *see* what he is doing either by following a symbolic argument or from the layout of his material on the page.

Figure 12 – Harriot's investigation of the structure of polynomials.

$$\begin{array}{l|l} a+b & = aa+ba \\ a-c & \quad -ca-bc \end{array}$$

$$\begin{array}{l|l} a+b & = aaa+baa+bca \\ a+c & \quad +caa-bda \\ a-d & \quad -daa-cda-bcd \end{array}$$

$$\begin{array}{l|l} a+b & = aaaa+baaa+bcaa \\ a+c & \quad +caaa+bdaa \\ a+d & \quad +daaa+cdaa+bcda \\ a-f & \quad -faaa-bfaa-bcfa \\ & \quad -cfaa-bdfa \\ & \quad -dfaa-cdfa-bcdf \end{array}$$

Source: From Harriot (1631, p. 4).

To show more clearly this visual aspect, let us consider Theorem I of Chapter XV of the treatise *Æqvationvm Recognitione Et Emendatione Tractatvs Dvo*, where Viète (1615) constructs a quadratic equation by considering the following expressions:

$$B - A = S \text{ with greater than } A$$

$$A - B = S \text{ with greater } B,$$

Where B is known, and S is the difference between A and B. By squaring both sides of the equation (for the first case), it is obtained that

$$B^2 - 2BA + A^2 = S^2$$

$$2BA - A^2 = B^2 - S^2$$

Similarly, the same expression is obtained for the second case

$$2BA - A^2 = B^2 - S^2$$

Viète states that if you have and , then you get the equation:

$$12x - x^2 = 20$$

And therefore, it can be determined that $x = 2$ and $x = 10$.

In this example, we can see a direct relationship between the coefficients and the root of an equation, and this relationship can be obtained from the exploration of the symbolic expression. If we start from the general expression $2BA - A^2 = B^2 - S^2$, it is possible to establish that $12 = 2B$, while $B^2 - S^2 = 20$. From these expressions we can obtain the values of B and S and, therefore A, considering that A is the sum or difference of the values of B and S.

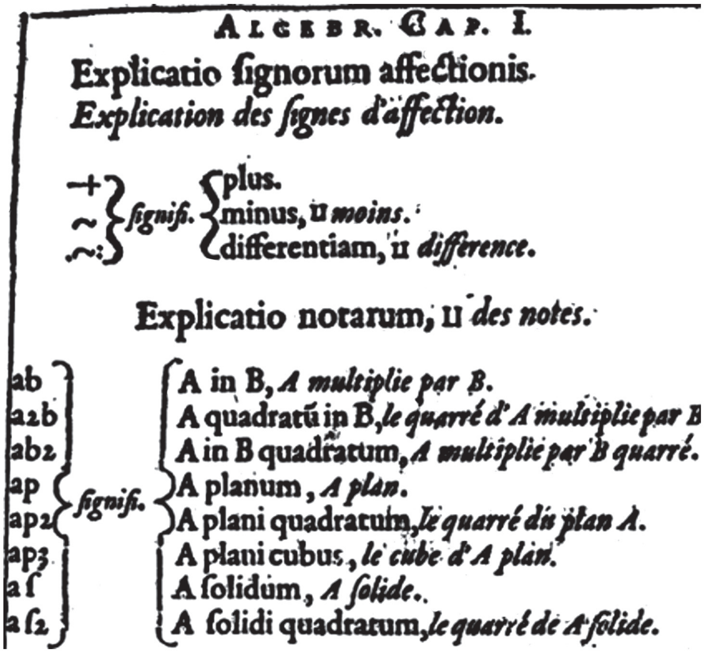
4.5. THE PRACTICES OF INCORPORATION AND REWRITING OF PREVIOUS TREATISES

The work of some mathematicians after Viète (see Massa Esteve, 2008, 2012; Stedall, 2007) had addressed the development of the algebraic symbolism and how they produced symbolism increasingly independent of the rhetorical text. Those works argue that this was possible through the rewriting of Viète's original texts to make the reading clearer and to expand on previous results.

For instance, Massa Esteve (2008, 2012) highlights that Viète's work was an inspiration for Pierre Hérigone, who built a full and clearer symbolic writing for mathematical demonstrations. According to her, Hérigone had in mind a didactic plan in which simplicity, clarity, and structure of the writing were fundamental. His aim was "to introduce a symbolic language as a universal language to deal with both pure and mixed mathematics using new symbols, margin notes (which he called "citations"), and abbreviations." (Massa Esteve, 2008, p. 286).

Hérigone not only modified Viète's notations as Harriot had, but he also elaborated a system of abbreviations for recurrent rhetorical expressions. This is derived in highly symbolical texts almost without words. For instance, the powers of the unknown were associated with a numeral at the end of the algebraic term (Figure 13).

Figure 13 – Hérigone's notational symbolic system.


ALGEBR. CAP. I.

Explicatio ſignorum affectionis.
Explication des ſignes d'affection.

+ } *ſignif.* { plus.
~ } *ſignif.* { minus, ıı *moins.*
~: } *ſignif.* { differentiam, ıı *difference.*

Explicatio notarum, ıı *des notes.*

ab } *ſignif.* { A in B, *A multiplie par B.*
a2b A quadratũ in B, *le quarré d'A multiplie par B*
ab2 A in B quadratum, *A multiplie par B quarré.*
ap A planum, *A plan.*
ap2 A plani quadratum, *le quarré du plan A.*
ap3 A plani cubus, *le cube d'A plan.*
aſ A ſolidum, *A ſolide.*
aſ2 A ſolidi quadratum, *le quarré de A ſolide.*

Source: From Hérigone (1634, pp. 5-6).

5. DISCUSSION AND CONCLUSIONS

Based on this brief presentation of some findings in the history of mathematics, related to symbolic algebra, which we have called contemporaries, we have identified new paths, reflections, and questions worthy to *accommodation* (Fried, 2001) for the expansion and development of algebraic thinking within ME; since they have not been explored and studied explicitly yet. This is what we have considered as implications to ME, in the sense that provide new accounts to rethink how algebra was developed in the history and, how these insights could bring new paths to research concepts, heuristics and mathematical practices in order to enhance the understanding of the construction and development of algebraic activity in our field. We recapture six elements from these findings that, as Barbin, et. al. (2020) propose, can be considered as *epistemological contributions* to the teaching and learning of mathematics if incorporated:

1. New explanations for the development of algebra and symbolic algebra

Heeffer's studies (2008b, 2009, 2010a) show an alternative view of the typical division of algebra development as *rhetorical*, *syncopated,* and *symbolic* in ME. It supports the disagreements of other ME researchers about this tripartite model (see Radford, 1997). The division as *non-symbolic algebra*, *proto-symbolic algebra,* and *symbolic algebra*—based on a particular conception of *symbolic reasoning*—vindicates the relevance of the innovations of medieval algebraists, both in their symbolism and functionality. This insight is helpful because it is not based on the semiotic nature of algebraic writing but on a type of mathematical reasoning. Nevertheless, we consider pertinent to consider what Chorlay and de Hosson (2016) say about the division of phases. This division usually implies a positivist vision that tends to implicitly promote that subsequent stages are better than the previous ones, demeaning these in the sense that a subsequent phase replaces the former one. However, during the historical development, this was not the case.

Developed in the period between Cardano (1539) and Buteo (1559), this model acknowledges the emergence of the *symbolic equation,* as a mathematical object. This incorporates arbitrary symbols for both arithmetic operations and unknowns and the equal sign, which in turn implies the systematic operation on itself. In this sense, it would be worthy to deepen, re-contextualize, and/or adapt the results of the development of the symbolic equation in empirical studies related to the development of Heeffer's six moments in the students' algebraic thinking.

2. New characterizations related to algebraic thinking: symbolic reasoning and epistemic justification

The theoretical constructs of *symbolic reasoning* and *epistemic justification* that Heeffer discusses in his research could be incorporated into the ME research in algebra. The first one presents innovative elements that could expand and be articulated with other approaches to characterize algebraic thinking. Kaput (2008, p. 10), for instance, states that the two core aspects of algebraic reasoning rest in the "generalization and the expression of generalization in increasingly systematic, conventional symbol systems" and in the "syntactically guided action on symbols within organized systems of symbols". In this line of thought, Radford (2006, p. 3, original emphasis) proposes that the algebraic activity be characterized by three elements: (a) a sense of *indetermination*, (b) indeterminate objects handled *analytically*, and (c) a peculiar *symbolic* mode to *designate* its objects.

In both characterizations, we can notice the weight that is assigned to symbolization, making it an enhanced characteristic. Although both authors consider that symbolic systems do not necessarily have to correspond to the formal ones of current algebraic symbolism, we believe that these explanations can be strengthened by considering the property of the creation of new knowledge that Heeffer highlights in his research. This would lead to incorporate a pragmatic dimension that transcends from its efficient capacity to store and transmit information (Pimm, 1987; Drouhard and Teppo, 2004), to a creative potential based on a visual function.

The notion of *epistemic justification* also contributes to the discussion of characterization of algebraic activity since it provides a framework to distinguish types of reasoning underlying such activity. To illustrate, consider the *figural* (related to the reconfigurations of geometric forms to demonstrate equation-solving techniques) (Hoyrup, 2002; Radford, 1995, 1996, 2001), the *arithmetic operability* (related to the use of arithmetic operation schemes to demonstrate the existence of unacceptable numbers such as negative, irrational or imaginary), and the *visual symbolic* (related to the use of symbolism as an argument to detect visual patterns in the structures of the equations), among others, which have recently been used by López-Acosta (2023). Under the adoption of this construct, the author distinguishes the algebraic activity of Viète and Descartes from some representative cases of the previous algebraic tradition.

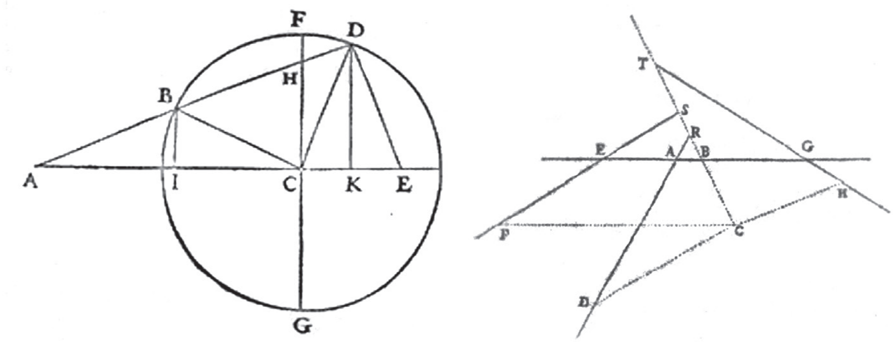
The distinction of these epistemic justifications can also generate approaches for the development of algebraic thinking in students, for which it is equally relevant to carry out empirical studies in our field.

3. The relevance of geometric problems in the development of algebraic analysis

Other works within ME (see Gascon, 1989, 1994-1995, 1999; Ruiz-Munzón, 2010, Ruiz-Munzón, et. al., 2011) describe and justify a model for algebraic activity, based on the epistemological considerations of Piaget and García (1982) and Klein (1968). These works addressed the essential role played by the use of parameters and unknowns in algebraic activity, giving rise to the emergence of mathematical formulas, a great step for the subsequent development of mathematical activity in general, as well as reformulations of the method of analysis-synthesis which Viète used to build his new algebra. Nevertheless, we identified that little has been mentioned about the characteristics of the geometrical activity that Viète and Descartes carried out in their time and how these implied the emergence of the *parametric equation*, i.e., the equations that use parameters and unknowns.

From the works that had addressed the importance of geometric activity in mathematics projects in which Viète and Descartes were involved, López-Acosta (2023), López-Acosta & Montiel (2021, 2022) have had recently conjectured about the emergence of the parametric equation which did not exist in the previous algebraic tradition. The authors ascertains that more than the algebraic work *per se*, it was specifically the complex geometric problems which both mathematicians solved in their attempt to renew the method of geometric analysis through algebra (Figure 14) that gave birth to the parametric equation.

Figure 14 – Complex geometrical problems in Viète and Descartes.



Source: The first image, taken from Viète (1646, p. 249), is related to an angle trisection problem. The second one, taken from Descartes (1637, p. 309), corresponds to the Pappus problem. Both kind of problems were considered as complex by the ancient Greek mathematicians.

With such findings, this research is carrying out didactic explorations with students and mathematics teachers to determine the scope of this conjecture in the creation and use of parametric equations (see López-Acosta et. al, 2024; López-Acosta & Romero-Fonseca, 2023). However, more research in this area is needed.

4. The use of algebraic symbolism as a tool to investigate the structure of equations

Another aspect that has not been significantly addressed in ME is the exploration of the visual character of symbolism, derived from the algebraic analysis first investigated by Viète and which other mathematicians further developed more prolifically, as mentioned by Stedall (2000, 2007, 2008). The visual character of the symbolism is one of the most relevant functions of the modern algebraic symbolism; however, it is hardly ever addressed in the teaching/learning process because algebra school disregards the importance of the formalism of scientific writing and its role as an instrument of thought (Bolea, 2003).

Unlike Viète, Descartes and Harriot, among others dedicated to the *construction of polynomials* to detect visual patterns and regularities between coefficients and roots in equations, in the current school practice, this visual argument is not used. The broadly used practice in which schools approach products such as $(x + a)$ $(x - b)$ is based on pre-established rules, yet to be explained to students. Thus, the algebraic activity at school focuses on rule memorization: 'the square of the common term, plus the product of the sum of the uncommon terms by the common term, plus the product of the uncommon terms. Consequently, these approaches distort the visual potential of symbolism, using the construction of polynomials allowed in its genesis. This aspect highlights and supports the importance of structural approaches to the learning and use of algebraic language (see Kirshner, 1989, 2001; Kirshner and Awtry, 2004).

5. The practice of incorporating and rewriting previous treaties by algebraists

One consideration that may significantly contribute to address the refinement of symbolism in school activities may come from the progressive rewriting of basic algebraic texts, as suggested in the works by Massa Esteve (2008, 2012) and Stedall (2007). They recognized the innovations, simplifications, and prolific ways to improve algebraic symbolism by algebraists based on Viète's texts.

This insight could provide didactic elements to work with students since it would be plausible to set environments dedicated to improving algebraic writing based on initial texts, something not yet addressed in algebra research.

6. New characters and algebraic treatises to be studied

Overall, the review of these contemporary sources can lay foundations to determine new research objects for HES in ME in algebra, since it allows the identification of both algebraists and algebraic treatises that have not been analyzed yet. For instance, the analysis of the work *De Numerosa Potestatum* by Viète could provide new insights that might have an impact on the development of algebraic thinking related to root approximation. Algebraists such as Stevin, Stifel, Peletier, Gousellin, Harriot, Herrigone, and their respective works, among many others analyzed in these studies, could provide new techniques, symbolism, and reasoning that may have been overlooked so far.

In conclusion, these studies can contribute significantly to a more robust and profound understanding of algebraic activity in general and have a positive impact on mathematics education. This should be feasible if relevant theoretical and methodological frameworks for empirical research are constructed, as pointed out by Radford (2000). In this way, with these few examples, we have presented the relevance of contemporary HES in the history of mathematics, showing the possibility of posing new objects of study at different levels. In short, we refer to those related to (i) *thinking*: symbolic reasoning, epistemic justification, the constitution of the symbolic equation, geometrical activity in the emergence of algebraic analysis, the visual character of symbolism to detect patterns between roots and coefficients in the equations; (ii) *historical development*: the non-symbolic, pre-symbolic and symbolic algebra; (iii) *algebraists and treatises not studied before*; and (iv) *theoretical constructs* that could strengthen the methods to analyze algebraic activity.

Thus, we propose that these signaled paths are worthy to be incorporated in the ME research of algebra to study more in depth and to identify their scope in the mathematics education of young students.

STATEMENTS OF AUTHORS' CONTRIBUTIONS

LLA conceived the idea presented along with the respective literature review, the analysis of sources, its narration, as well as the design of the structure of the manuscript. GME actively participated in the review of the methodological structuring of the work and the argumentative structure of the manuscript.

DATA AVAILABILITY STATEMENT

Data supporting the results of this study will be made available by the corresponding author, LALA upon reasonable request.

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