# The Danish KOM project and possible consequences for teacher education[1]

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**Resumen**[3]

Se define el concepto de competencia matemática y se describen las ocho competencias matemáticas identificadas por el proyecto KOM. También se analizan sus implicaciones en la formación de profesores de matemáticas y la propuesta del proyecto KOM al respecto.

**Palabras clave**

Competencia matemática, formación de profesores, competencias del profesor de matemáticas.

**Abstract**

It defines the concept of mathematical competence and describes the eight mathematical competencies identified by the KOM project. It also discusses their implications for the training of teachers of mathematics and KOM project proposal in this regard.

**Key words**

Mathematical Competence, Teacher Training, Teacher's Math skills.

## 1. Introduction

This paper is guided by an attempt to answer two main questions: *What does it mean to master mathematics? and What does it mean to be a good mathe-*

[1] Este artículo fue aportado generosamente por M. Niss en el contexto de su participación como conferencista magistral inaugural en la XIII CIAEM que tendrá lugar en Recife, Brasil, 26 - 30 de junio del 2011.

[2] Mogens Niss fue durante muchos años secretario general de la *International Commission on Mathematical Instruction*.

[3] El resumen, las palabras clave, el abstract y las key words fueron agregados por los editores.

*matics teacher?* In order for the discussion to have a somewhat concrete point of departure, let us begin by considering an example of a possible school task at the upper secondary level. It should be kept in mind that the point is not to recommend (or the opposite) this task as a valuable activity. It serves illustrative purposes only, as fuel for the subsequent general discussion, and has been chosen for its relative simplicity and because it makes sense at the borderlines of various educational levels (depending on country).

## 2. Illustrative example

*Assume a rectangular piece of cardboard, of dimensions $b$ and $2b$. We want to produce an open box by folding up sides from the cardboard. How large a volume can we obtain by this process? If we want to obtain at least a certain volume, what dimensions should the cardboard have?*

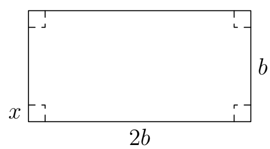


Figure 1: Cardboard of dimensions $b$ and $2b$.

This is a somewhat stylised applied mathematics problem. To answer the questions implied by it, mathematical modelling is needed, e.g. as follows:

1. Mathematising the problem:

Mark off stripes of the same (variable) width, $x$, along all sides of the cardboard, cut the corners and fold the sides to produce the open box. The resulting box will have the volume

$$V(x) = x(2b - 2x)(b - 2x) = 2x(b - x)(b - 2x).$$

The questions in the problem can be translated into mathematics: Is there a (feasible) value of $x$ for which $V(x)$ attains a maximal value? If so, is there more than one such value of $x$? What should $b$ be so as to ensure a certain minimum volume of the open box?

2. Specification in mathematical terms:

Based on the nature of the situation, the feasible values of $x$ satisfy $0 < x < \frac{b}{2}$, even though $V$ as a polynomial function is defined for all $x$ in $\mathbb{R}$. We are now ready to specify the mathematical questions entailed by the problem:

Q 1: Is there a value of $x$ in the feasible interval $I = ]0, b/2[$ for which $V$ is maximal on $I$?

Q 2: If so, what is this value?

Q 3: Is there more than one such value of $x$ in $I$?

Q 4: If so, is one value 'better' than the others if additional criteria are introduced?

Q 5: What should $b$ be in order to obtain at least a given volume?

3. Representing $V$ by its graph:

We observe that $V(0) = V(b/2) = V(b) = 0$ and that the polynomial has no other roots. In the feasible interval $I = ]0, b/2[$, we have $x > 0$, $b - x > 0$ and $b - 2x > 0$, hence $V(x) > 0$ on $I$. For negative $x$, $V(x)$ is negative. For $b/2 < x < b$, $V(x)$ is negative as $b - x > 0$ and $b - 2x < 0$. For $x > b$, $V(x)$ is positive as both $b - x$ and $b - 2x$ are negative.

This allows us to qualitatively sketch the graph of $V$ (see Figure 2 which also contains the graph of the derivative, $V'$, of $V$ for a subsequent purpose).

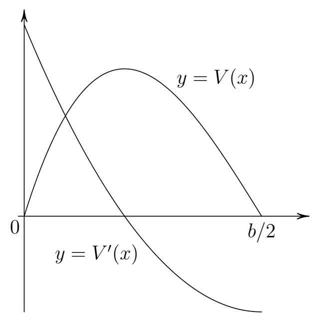


Figure 2: Graph of $V$ and $V'$.

4. Answering Q1:

As $V$ is a polynomial function it is continuous on the closed interval $[0, b/2]$. Hence it attains a maximal value on that interval.

As $V(0) = V(b/2) = 0$, and as $V$ assumes positive values in $I$, the maximal value is attained in the interior, i.e. in $I$. So, the answer to Q1 is "yes!". This answer is visually supported by the reading of the graph of $V$.

5. Answering Q2:

Here we use a little calculus. As we can re-write

$$V(x) = 2x(2x^2 - (2b + b)x + b^2) = 4x^3 - 6bx^2 + 2b^2x,$$

we have that the derivative of $V$ is given by

$$V'(x) = 12x^2 - 12bx + 2b^2 = 2(6x^2 - 6bx + b^2).$$

It follows that

$$V'(x) = 0 \text{ for } x = b(3 \pm \sqrt{3})/6.$$

As $b(3+\sqrt{3})/6$ is non-feasible ($> b/2$), whereas $b(3-\sqrt{3})/6$ is feasible ($\sqrt{3}/6 < \frac{1}{2}$), there is *only one possible, feasible solution*, i.e. $x_0 = b(3 - \sqrt{3})/6$.

The value of $V$ in $x_0$ is

$$V(x_0) = (2b/6)(3-\sqrt{3})[b - b(3-\sqrt{3})/6][b-(3-\sqrt{3})/3] = b^3\sqrt{3}/9 > 0.$$

What we have found is an extreme point for V on I. Can we be sure that it is a maximum point? Yes, for in 4. we argued that there is a maximum point for $V$ in $I$, and since we have identified only one extreme point, $x_0$, in $I$, $x_0$ has to be a maximum point.

(We could also argue that

$$V''(x) = 12(2x - b),$$

so that

$$V''(x_0) = 12[b(3-\sqrt{3})/3 - b] = -4b\sqrt{3} < 0,$$

which implies that $x_0$ is a maximum point for $V$.)

The value $x_0$ is an exact value. If we want an estimate which is useful for practical purposes, e.g. $x_0 \approx 0.211b$, we typically have to make use of tools or aids such as calculators, computers or tables to obtain it.

In summary, we can conclude that there is *one and only one feasible solution* to the problem, namely $x_0 = b(3-\sqrt{3})/6 \approx 0.211b$.

This answers Q2.

6. Answering Q3 and Q4:

There is no other maximum point for $V$ on $I$. Hence there is no alternative to consider.

7. Answering Q5:

We found that the maximal volume that can be obtained on $I$ is $b^3\sqrt{3}/9$. So, if we want to obtain a certain given minimum volume, $V_0$, this can be obtained by choosing

$$b = (3\sqrt{3}V_0)^{1/3}.$$

Again, a numerical estimate for a given $V_0$ can be obtained by means of computers, calculators or tables.

This answers Q5 and the entire problem has now been solved.

It is now time to analyse what it takes to do what we have just done. In order to successfully complete this task one has

- to be able to perform *mathematical modelling* of a pseudo-practical situation as a means to *ask questions* and *solve problems* pertinent to that situation.
- to possess of a feel for *the kinds of questions* we ask in mathematics and the *kinds of answers* we can expect, which is a key element in *thinking mathematically.*
- to be able to pose, specify and solve *mathematical problems.*
- to formulate and *justify* statements, solutions, and conclusions, i.e. to *reason mathematically.*
- to make use of different *mathematical representations*, and to translate between them. In this case we have encountered verbal, graphical, and symbolic representations.
- to be able to handle mathematical *symbolism* and *formalism.*
- to be able to *communicate* about mathematical matters
- to make use of mathematical *tools and aids*

These are exactly the main components in the Danish KOM project which we shall describe in the next section.

## 3. The Danish KOM project (Competencies and the Learning of Mathematics)

Without entering into details of the genesis of the Danish KOM project 2000-2002, which had a complex brief from the Ministry of Education and the then National Council for Science and Mathematics Education, a main task for the project (directed by the author of this paper) was to consider – and possibly answer – the question “What does it mean to master mathematics?”. For a variety of reasons we immediately decided to adopt a competency based approach. This requires a definition.

*Definition*: Possessing mathematical *competence* means having knowledge of, understanding, doing and using mathematics and having a well-founded opinion about it, in a variety of situations and contexts where mathematics plays or can play a role.

A mathematical *competency* is a distinct major constituent in mathematical competence.

### Mathematical competencies

In the KOM project we have identified eight such competencies, which may well be seen as forming two clusters, each containing four competencies.

| The ability to ask and answer questions in and with mathematics: |
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*Mathematical thinking competency - mastering mathematical modes of thought*, includes

- understanding and dealing with the roots, scopes, and limitations of given *concepts*;
- *abstracting* concepts, *generalising* results;
- distinguishing between different *types of mathematical statements*, e.g. definitions, theorems, conjectures, statements concerning single objects and particular cases;
- possessing awareness of *the kinds of questions* that are typical of maths, and insight into the *kinds of answers* to be expected;
- possessing an ability to *pose* such questions.

*Problem handling competency - formulating and solving mathematical problems*, includes

- detecting, formulating, delimitating, and *specifying mathematical problems*, pure or applied, open or closed;
- possessing an ability to *solve problems*, posed by oneself or by others, if desirable in different ways.

*Modelling competency - being able to analyse and build mathematical models concerning other areas*, includes

- *analysing* the foundations and properties of existing models, and assessing their range and validity;
- *performing active modelling* in given contexts i.e. structuring and mathematising situations, handling the resulting model, drawing mathematical conclusions from it, validating the model, analysing it critically, communicating about it, monitoring and controlling the entire process.

*Reasoning competency - being able to reason mathematically*, includes

- *following* and *assessing* others' mathematical reasoning;
- *understanding* what a *proof* is (not) and how it differs from other kinds of reasoning;
- understanding the logic behind a *counter example*;
- uncovering the *main ideas in a proof*;
- *devising* and *carrying out* informal and formal *arguments*, including transforming heuristic reasoning to valid proof.

The competencies in the second cluster focus on

| The ability to deal with mathematical language and tools: |
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*Representation competency - being able to handle different representations of mathematical entities*, includes

- *understanding* (decode, interpret, distinguish) and utilising *different kinds of representations* of mathematical entities;
- *understanding* the *relations* between different representations of the same entity;
- choosing, *making use of*, and switching between different representations.

*Symbols and formalism competency - being able to handle symbolic language and formal mathematical systems*, includes

- *decoding* symbolic and formal language;
- *translating* back and forth between symbolic language and natural language;
- *handling* and *utilising* symbolic statements and *expressions*, including formulae
- understanding the *nature of formal mathematical systems*.

*Communication competency - being able to communicate, in, with, and about mathematics*, includes

- *understanding*, examining, and interpreting different kinds of written, oral or visual mathematical expressions or texts;

- *expressing oneself* in different ways, and at different levels of precision, on mathematical matters to different sorts of audiences.

*Tools and aids competency - being able to make use of and relate to the tools and aids of mathematics*, includes

- having knowledge of the *existence* and *properties* of different relevant tools and aids for mathematical activity (e.g. rulers, compasses, protractors, tables, centicubes, abaci, calculators, computers, the internet);
- having insight into the *possibilities* and *limitations* of such tools;
- reflectively *using* tools and aids.

It appears that these competencies provide a systematisation of the elements involved in dealing with the initial example.

The competencies are *closely related*, yet they are *distinct*. One may think of and depict each competency as constituting a blurredly delineated web of increasing density towards a centre of gravity. The different webs overlap, but as each has its own centre of gravity the webs are clearly discernible from one another. Moreover, if we focus is on one of the competencies, the others can be called upon as auxiliary means to pursue the ends of the one in question. Assume, for instance, that we have the *problem handling* competency in mind. Then it is clear that, say, the *representation* competency, the *symbols and formalism* competency and the *reasoning* competency, and perhaps the *tools and aids* competency as well, all come in very handy to assist in the specification and above all in the solution of mathematical problems.

The competencies all have a *dual nature*, in that each of them contains two sides. One side emphasising the individual's ability to understand, follow, relate to, analyse, and judge others' exercise of the activities encompassed by that competency, and one emphasising the individual's own independent pursuit and performance of these activities.

The competencies also comprise what some may want see is independent competencies, such as *intuition* and *creativity*. Intuition is on the agenda in most of the competencies, for instance when we speak of the kinds of questions and the kinds of answers that are characteristic of mathematics, of developing heuristic reasoning, and of making use of different representations. Creativity can be seen as the amalgamation of all the performance sides of the eight competencies.

Finally, the competencies are *specific to mathematics*, even if their titles may make sense in other fields and subjects as well. And, most importantly, they are *overarching* across educational levels, from primary school to university, and across topic areas, from arithmetic to topology.

### Overview and judgement regarding mathematics as a discipline

The mathematical competencies are manifested in dealing with *situations* and contexts in which a task or a challenge related to mathematics is directly or indirectly present. In the KOM project, we further wanted, however, to insist that mathematics education should also serve to establish a thoughtful, sound and well-balanced image, with the leaerners, of mathematics as a discipline. We therefore identified three forms of overview and judgement regarding mathematics as a discipline, which we saw as cornerstones of such an image:

- The **actual application** of mathematics in other subjects or practice areas. Here we emphasise that mathematics is actually being applied and employed outside of mathematics in other subjects and fields of scientific or societal practice, which should be reflected in mathematics education.
- The **historical development** of mathematics, both internally and from a societal point of view. Here we emphasise that mathematics has been and is being developed in time and space, in society and culture, and by human beings with different kinds of backgrounds and roles and for a variety of different purposes. Again, this should be reflected in mathematics education.
- The **nature of mathematics** as a discipline. Here we emphasise that mathematics has important features in common with other scientific fields, and equally important features that distinguishes it from other such fields. This should not only be reflected but also be reflected *upon* in mathematics education.

In summary, here the focus is on important aspects of mathematics as a whole, not on mathematical situations.

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How can all this be employed and utilised in mathematics education? Well, basically in three different ways.

Firstly, in a **normative** way, when we decide upon goals and aims of teaching and learning, design curricula, set priorities, produce teaching materials, and so forth, and when we monitor coherence and progression in mathematics education.

Secondly, in a **descriptive** way, when we want to know and understand what actually happens (or does not happen) in mathematics education. Using the competencies and the three forms of overview and judgement as lenses through which the reality of mathematics teaching and learning can be depicted and analysed, does also allow us to compare, say, teaching, curricula, and institutions, and it may help us identify causes of transition problems from one segment of the educational system to another.

Finally, it may serve as **metacognitive support** for teachers and students, when they struggle with questions concerning the path that teaching or learning is currently taking, the emphases adopted, the problems that have occurred, etc.

## 4. Possible consequences for the education of mathematics teachers

We are now prepared to address our second guiding question "What does it mean to be a good mathematics teacher?".

It may come as no major surprise that, in the KOM project, the a first approximation to an answer is this:

*A good mathematics teacher is one who can effectively foster the development of mathematical competencies with her/his students.*

Although this answer does suggest a focus and some priorities which are not necessarily commonplace amongst mathematics educators, it has to be admitted that it is not overly informative either. We have to be more specific to obtain a useful answer.

Needless to say, if a mathematics teacher is to foster the development of the mathematical competencies with the students, that teacher must possess these competencies her/himself. The same holds for the three forms of overview and judgement.This may be slightly more controversial, in particular in education sectors where teacher training focuses on educating the generalist teacher with an emphasis on general education and general pedagogy, while substantive issues concerning the teaching subject(s) are seen as being of secondary importance. Nonetheless, most *mathematics* educators tend to agree that since teachers teach what they know, mathematics teachers must know and be able to do what they (should) teach. So, in many places teacher education programmes are constructed as some kind of Cartesian product of general education and pedagogy, on the one hand, and subject matter knowledge on the other. In still other places, one has chosen to integrate these two components for prospecive teachers, so as to result in a specific, didactically modified teaching of mathematics as a *school subject* rather than as a discipline.

The KOM project has taken another route. It proposes three main components of the education (and professional development) of a mathematics teacher: A mathematical education based on the competencies approach. A component of general education and pedagogy, and – and this is probably the more novel part in this context - *didactical and pedagogical competencies with specific regard to mathematics* .

We have identified six such competencies, as follows:

***Curriculum*** *competency*

This includes the ability to understand, analyse, assess, relate to, and implement existing mathematics curricula and syllabi, as well as the ability to construct new ones should it be desirable or necessary.

***Teaching*** *competency*

This includes the ability to devise, plan, organise, orchestrate and carry out mathematics teaching, including: Creating a rich spectrum of teaching/learning situations; finding, assessing, selecting and creating teaching materials; inspiring and motivating students; discussing curricula and justifying teaching/learning activities in discussions with students.

***Uncovering of learning*** *competency*

This includes the ability to uncover, interpret and analyse students' learning of mathematics, as well as their notions, beliefs and attitudes towards mathematics. It further includes identifying development, including progression, with the individual student

***Assessment*** *competency*

This includes the ability to identify, assess, characterise, and communicate students' learning outcomes and competencies, so as to inform and assist the individual student and other relevant parties. It includes knowing, selecting, modifying, constructing, critically analysing, and implementing a varied set of assessment forms and instruments to serve different formative and summative purposes.

All of these are supposed to concern and address different categories of students, of different background, in different situations, and at different levels, while paying attention to the individual student's needs and opportunities. Next come two competencies that focus on the position of the mathematics teacher in professional and institutional environments.

***Collaboration*** *competency*

This includes the ability to collaborate with different sorts of colleagues in and outside mathematics, as well as with others (parents, superiors, authorities, employers) concerning mathematics teaching, its boundary conditions and circumstances.

***Professional development*** *competency*

This includes the ability to develop one's own competency as a mathematics teacher (thus it is a meta-competency), including participating in and relating to activities of professional development, such as in-service courses, research and development projects, conferences; reflecting upon one's own teaching and needs for development; keeping oneself up-dated about new developments and trends in research and practice.

In Denmark, this way approach gives rise to major intellectual, structural, cultural, political and financial challenges to our systems of pre-service education and in-service development for teachers of mathematics for whichever level. In fact, implementing it in any serious way requires fundamental reforms of our systems. I would not be surprised if the same were true in other countries, also amongst the neighbours of Denmark.

There are indications in the Danish debate that elements of reforms along those lines may be initiated in the years to come. The degree to which this will happen is largely dependent on political and economic conditions, most of which are quite controversial. However, fortunately we are not faced with a "0-1 problem", leaving us with the choice between "doing nothing" and "profound and total reform" . The ideas and intentions sketched here can be implemented within a continuum of possibilities, ranging from local in-service initiatives on an experimental basis with a group of teachers in a single institution, to large scale profound changes of the entire system.

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