

A MATHEMATICAL STUDY OF THE SRAFFA MODEL

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RESUMEN

Este documento tiene dos objetivos principales. En primer lugar, se presenta una rigurosa derivación del modelo de Sraffa utilizando algunos teoremas descubiertos por Frobenius y Perron al principio del siglo veinte. Esta presentación es utilizada posteriormente para analizar el concepto de economías productivas o de excedente, la solución al problema de existencia del sistema de precios, y también mostrar claramente la derivación del concepto de mercancía standard.

El segundo objetivo es delinear el modelo de crecimiento balanceado de von Neumann y mostrar como se relaciona al modelo de Sraffa. Mostrando el rol de los supuestos en la derivación de estos modelos, se puede establecer explícitamente la relación entre ambos modelos, particularmente el rol del supuesto de irreducibilidad para derivar la unicidad y unificar el estudio de ambos modelos.

PALABRAS CLAVES: SRAFFA, ECONOMÍAS DE PRODUCCIÓN, ECONOMÍA CLÁSICA, TEORÍA DE DISTRIBUCIÓN, PRECIOS.

ABSTRACT

This paper has two primary objects. On the one hand, it presents a rigorous derivation of the Sraffa model recurring to some theorems discovered by Frobenius and Perron at the beginning of the twentieth century. This presentation is used to study the concept of productive economies or surplus economies, the solution to the price system and also to provide a clear derivation of the concept of the Standard Commodity.

The second goal of the paper is to sketch von Neumann's balanced growth model and show how Sraffa's model is related to Neumann's model. Showing the role of the assumptions, one can establish the relationship between both models, particularly the role of irreducibility to obtain uniqueness and unify the study of both models.

KEY WORDS: SRAFFA, ECONOMIES OF PRODUCTION, CLASSIC ECONOMICS, THEORY OF DISTRIBUTION, PRICES.

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INTRODUCTION

It is unknown whether Piero Sraffa was aware of the Perron-Frobenius's work (1907 and 1912) when he started to work on his book "Production of Commodities by Means of Commodities [11]" in the late forties. Nevertheless, his book reflects a clear demonstration of the existence of a maximum positive eigenvalue associated with a unique, up to a scalar, positive eigenvector when working with an irreducible non-negative matrix.

It is through this mathematical result that the concept of the standard commodity developed by Sraffa acquires full generality; the Sraffian-Price system $PA(1+r)+a_n l=P$ can use as its numeraire $r=\rho(A)(1-w)$, a numeraire developed by using the standard product. Sraffa employed this result to solve the problem posed by Ricardo [10] of finding a commodity whose capital and labor proportions were such that those price fluctuations created by changes in distribution, say changes in the rate of profit or wages, did not affect this price. Employing the price of this composite commodity as numeraire, one could generalize Ricardo's result of the corn model and establish a linear relation between the real wage and the rate of profit.

One must add from the start that this standard commodity does not imply that fluctuations of prices are not obtained from changes in distribution as Ricardo had already recognized; rather, the standard commodity is a clever constructed numeraire which does not fluctuate to changes in distribution so that one can understand and measure fluctuations in prices obtained from changes in distribution.

Furthermore, it brings enough criticism by itself on any definition of the term "capital". It is immediate that any fluctuation in distribution brings about movements in relative prices. Then, defining capital as a bundle of commodities whose value is determined by supply and demand becomes circular and indeterminate. The very idea of a demand curve for a factor of production implies that changes in distributional rates occurs, the vertical axis, and hence fluctuations in the value of those defined "capital goods", the

horizontal axis, changes in an indeterminate manner.

This paper has two primary objects. On the one hand, it presents a rigorous derivation of the Sraffa model recurring to some very abstract mathematical theorems discovered by Frobenius and Perron at the beginning of the century. This presentation is used to exploit the concept of productive economies or surplus economies, and also to provide a clear derivation of the standard commodity. The second goal of the paper is to sketch von Neumann's balanced growth model and show how Sraffa's model is related to Neumann's model. By studying the set of assumptions underlying both models, one can study the relationship between both models and its implications.

The paper is organized as follows. First it introduces the theorems of Frobenius and Perron and analyze subsistence economies. Next it introduces productive economies and study the solution of these economies and the structure of these systems proposed by Sraffa. Next, the concept of the Standard Commodity is derived and analyzed. The paper follows with the sketch of the model of von Neumann and its properties. Then the relation of Neumann's model to the model of Sraffa is studied. Conclusion remarks follows.

2. THE STANDARD COMMODITY IN THE MODEL OF SRAFFA

First, consider the fundamental theorems of Frobenius on nonnegative matrices which extend the results obtained by Perron for positive matrices

Theorem 1

1. (Frobenius non-negative) Let $A \geq 0$ be an $n \times n$ matrix then $\rho(A) \geq 0$ is an eigenvalue

2 Particularly important is the fact that the maximum root is not unique and hence the eigenvector although nonnegative are not unique.

- of A and there is a nonnegative vector $x \geq 0$; $x \neq 0$, such that $Ax = \rho(A)x$.²
2. A irreducible iff $(I + A)^{n-1} > 0$
 3. (Frobenius nonnegative and irreducible) If $A \geq 0$ and irreducible then $\rho(A)$ is a real, multiplicity one eigenvalue. The left and right eigenvectors are positive and unique up to a scalar. Moreover $\rho > 0$ (Frobenius) and for any other $\lambda \leq \rho(A), p_{\lambda, i} < 0$ for at least one $i = 1, \dots, n$ (Pasinetti [9]).

For a proof see Horn [3] and Pasinetti [9].³

Let represent the technical coefficient of the economy. We avoid any discussion on how these coefficients are determined a fact that has actually determined different schools in economic thought. For instance, one can have some so-called Neo-Ricardians who tend to find on a kind of technology as such. In a line close to some thinkers of the “second international socialist”, these authors deliberately emphasize on the technical side and reduce the social configuration of such structure to its minimum. Moreover, it is a logical implication from this attitude their recognized neglecting of the Labor Theory of Value. For our present purpose, is an non-negative matrix for a single product and denominated “an economy.”

2.0.1 Subsistence system and prices

Consider first the case denominated by Sraffa “Production for Subsistence.” Sraffa proposes to find out a set of exchange values that allow economic agents to exchange all quantities in such a way that production can take place next period and there is no surplus created after production. This exchange values, he adds, spring directly from the methods of production. In matrix notation he argues that $pA=p$.

For $pA=p=\lambda p$ to hold, one argues that $\lambda(A)=1$ where $\lambda(A)$ is an eigenvalue of A . There

3 I omit certain proofs to focus on the main argument of the paper. Sources including these proofs are always included.

are some attributes associated to this type of economy that deserves to be emphasized and will lead us into the role of basic and non-basic commodities.

We just assume that A is non-negative. One interesting issue here is to see what other restrictions are required to generate economic meaning for this type of model. From theorem (1), if $A \geq 0$ then one knows that $\rho(A) \geq 0$ and the price vector associated is $p \geq 0$. The spectral radius is an eigenvalue not unique and hence the eigenvector associated is not unique either.

The economy is denominated a reducible economy in the sense that not all its sectors are somehow connected. Mathematically speaking, this means that there is a permutation matrix P which allows one to re-write A as

$$P^T \cdot A \cdot P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}$$

Matrix B and D are square and irreducible matrices⁴. Furthermore, this normal form of matrix A is unique up to a permutation of the diagonal (Gantmacher [2]).

Let us spell out these properties⁵. First, as claimed before, $\rho(A)=p$ has a positive solution for the eigenvalue $\lambda=\rho(A)$. Note that we work with economies which are only nonnegative but we restrict this set of economies to only those with $\rho(A) \leq 1$.⁶

The case for $\rho(A)=0$ is equivalent, not with a $A=0$, but rather with $A^m=0$ for some m integer. Then, although the economy could have initially some technical positive values, they will eventually become null. These eventually null economies shall be disregarded as having no economic meaning.

Second, the vector of prices can have zero elements. I return to this point below.

4 This new matrix is named as the *normal form* of the reducible matrix A .

5 We actually work out the surplus and subsistence economy simultaneously. This issue becomes clear in our discussion below.

6 It will be shown below that this requirement makes possible to have surplus economies.

Third, one can write the A matrix on its normal form. In economic terms, this means that we can partition the economy in groups. Matrix B and D would be square matrix that are actually irreducible ones and represent different subsectors of the economy. Moreover, if C contains some nonzero elements, then one has that D is related to B but not viceversa. Sraffa denominated subsector B of A as a sector composed of *basic commodities* and the rest sectors as nonbasic commodities. It is direct that nonbasic commodities depend upon basic ones for its production and since this partition is unique, one has that it is not possible to modify this partition of commodities.

Dividing the vector of prices for each group, one gets

$$[P_I, P_{II}] \cdot \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \lambda(A) \cdot [P_I, P_{II}].$$

One has that prices for sector B do not depend upon the rest of the economy-prices while viceversa is not the case. Additionally, it can easily be shown that the rate of expansion of the economy or in mathematical terms the maximum eigenvalue of economy A is the same that the maximum eigenvalue for B . Thus, permutations within the diagonal are always possible, the partition is unique and $\rho(A) = \rho(B)$. Sraffa employed these important properties to construct the standard commodity as shown below. This last property let Sraffa take subsystem B instead of A which has a positive Frobenius root and a positive left and right eigenvector associated to work out the Standard Commodity.

Fourth, there is the fundamental issue on prices which must be closely studied when working with a nonnegative matrix. The nonnegative economy allows to have a decomposable system that can be written as the equation above and hence allows for the very important notion of basics and nonbasics. This main notion lets one distinguish between some sectors of the economy that are totally independent⁷ from those dependent ones. This distinction is a very important one for

theories of economic development and industrial organization for instance. In this model, it is even more important in the sense that it lets one to have more general economies where the only requirement is a nonnegative economy with spectral radius less or equal to one⁸. This explains why it is so important for Sraffa to demonstrate that these type of economies do not restrict the application of the standard commodity system which is derived from $\rho(A) = \rho(B)$.

However, there is still a concern of clarification on prices for the economy A that must be done if such generalization is to be possible. Theorem (1) is very general and abstract in the sense that little economic insight is obtained from it. The theorem basically asserts that the maximum eigenvalue is nonnegative, not a simple root and its associated eigenvector is nonnegative and hence not necessarily unique. We have already disregarded zero spectral radius. Now we must clarify the implication on zero prices and the fact that $\rho(A)$ might not be a simple root, that is, one may have the maximum eigenvalue having multiplicity larger than one. This result will produce many different nonnegative eigenvectors associated to these roots which can actually become infinite by linear combinations.

Let us first consider the zero price case. Turning to the notion of prices, one has that if the economy is just nonnegative then some prices could be zero. Let us write

$$[P_I, P_{II}] \cdot \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} = \lambda(A) \cdot [P_I, P_{II}],$$

which can be explicitly written as

$$\begin{cases} P_I B = \lambda(A) \cdot P_I \\ P_I C + P_{II} D = \lambda \cdot P_{II} \end{cases}$$

From the fact that B represents a basic sub-system⁹ and since it is an irreducible

7 This statement should be reviewed nevertheless for the joint production case.

8 Preferably a strict inequality for surplus economies, see below.

9 I denominate this submatrix as subsectors. This is not in line with Sraffa's context where he uses the concept to refer to a different issue.

nonnegative matrix one knows from the Frobenius-Perron theorem (see theorem 1) that its prices are positive. Thus, if a zero price exist for economy A then it is a nonbasic commodity price, $P_i > 0$. Consider the above equation for P_{II} and hold $P_i > 0$ then if there is a positive requirement for any basic good in the production of the nonbasic commodity, it follows that such nonbasic good price is positive. Hence, the nonbasic commodity has no requirement of basic goods if its price is to be zero, $C_i = 0$. Finally, the weight is on the combination $P_{II} \cdot D$. Leaving aside the trivial null case, one has that since $d_i \geq 0$ with $d_i \in D, \forall i = r + 1, \dots, n$. Hence, if for some $i; d_i > 0$ then $P_i = 0$. That is, for any positive requirement its price must be zero so that all combination cancelled out and the price for such commodity is zero.

For the same nonbasic commodity, the answer is trivial. Nevertheless, if this combination is possible, then it must be the case that there exist another zero price nonbasic good, say j , which is associated with a positive requirement, say d_{ij} . Moreover, if such j nonbasic good has a zero price, the last argument for its price must repeat itself. The case is that this cannot continue forever and at some moment the system must revert itself and take the commodities initially defined with zero prices as i . In this form, the system of zero-price nonbasic goods will be canceled out and it is not possible to have zero prices.

This last discussion of zero prices is important as far as it can bring to the model some confusion about the economic meaning of a zero price and especially if it is related to the zero-price neoclassical good or nonscarce goods (see Gale [1]). We will come back to this issue in our discussion of the Neumann model below.

The next issue in studying nonnegative matrix is the one related to $\rho(A)$ which is a simple root. As shown in theorem (1), if irreducibility is assumed then $\rho(A)$ is unique. That is, there is only one eigenvalue which generates the maximum eigenvalue or spectral radius. In the reducible case, we can have that even all eigenvalues are equal to this radius, for

example the identity matrix case. Thus for each eigenvalue, one has an eigenvector. Thus one can have two different price-vector associated to A . Prices spring out from the methods of production as they are not unique. Even worse, by a linear combination one can generate an infinite number of price-vectors.

This issue is very important and complex. One can have an economic system whose prices are totally different depending on what eigenvector one takes. Still more striking can result the though experiment of taking a price system where a nonbasic commodity has a zero price and canceled out with some other nonbasic goods; then one can choose a second price-vector where the opposite occurs.

Of course these considerations have important implications in the system. In general, assuming only nonnegativity for the economy can reduce the generality of the results. A good generalization is obtained for the standard commodity system and an important distinction gained in relation to basics and nonbasics. The zero price case can be studied and ruled out as the zero economic expansion case. However, there are serious limitations when interpreting prices for the A matrix as a whole since not much is derived for the uniqueness of the eigenvalues and its eigenvectors (see Horn [3,503-5].)

If one works with an irreducible nonnegative matrix then the solutions are straightforward from theorem (1) but in this case there is no notion of nonbasic goods at all. If this is the case, A is irreducible, then all commodities are basics and hence their prices are positive. If we rewrite $Ap = p$ as $pA(1+r) = p$ then by our assumption of $\lambda(A) = 1$ we deduce that our economy is not productive in the sense of Morishima [7], that is,

$$\frac{1}{1+r} = 1 \Rightarrow r = 0.$$

The maximum rate of expansion is zero and one calls this a subsistence economy according to Sraffa.

One affirms that the exchange-value vector springs from the methods of production

in the sense that it is the left positive eigenvector associated to A . All that is needed to obtain p is to define A .

2.1 Productive system and prices

One can move one step forward to a productive economy and let the economy create a surplus. That is, the economic system is such that the production process generates a quantity of goods which are above the requirements. Instead of having $Ay = y$ as assumed before, one now has $Ay \leq y$. Applying the same argument for the right eigenvector, one ends up with $pA \leq p$. If an equality is intended between these variables, one needs an extra variable to convert this inequality into an equality. Following Sraffa’s definition of a uniform rate of profit for all sectors, one ends up with a price model,

$$pA(1 + R) = p$$

Putting this Sraffa’s presentation back into the equation to emphasize the characteristic equation framework one has $pA = \frac{1}{(1+R)p} = \rho(A)p$. The surplus-economy is a nonnegative economy A characterized for the existence of an associated spectral radius $\rho(A) < 1$. In general one has $pA < p \equiv Ap = \rho(A)p$, with $\rho(A) < 1$. Note that in both cases A is nonnegative.

Notice further that it is not the existence of a surplus that allows for the possibility of luxury goods or, alternatively, basics and non-basic goods. On the one hand, surplus is obtained from the particular value attained to the maximum eigenvalue. On the other hand, basic and non basic commodities are obtained from reducible economies. Therefore, it is possible to create basic and nonbasic goods either from a subsistence system or a surplus one inasmuch as a sub sector of the economy is not dependent on the other sectors. All previous considerations apply if the economy is considered as nonnegative alone and allows for basics and non basics. Stronger results are produced if irreducible nonnegativity is

assumed but in this case basic and nonbasic commodities are ruled out.

A straightforward result is that the Sraffa price system has a unique positive solution for prices p from the Frobenius theorem given $A \geq 0$. Moreover,

$$\rho(A) = \lambda_{max}(A) = \frac{1}{1 + R} \Rightarrow R = \frac{1 - \lambda}{\lambda}$$

One says again that exchange-values spring directly from the methods of production in the sense that this is coming from the left positive eigenvector associated to A . All that is needed to obtain p is to define A . This price-vector allows then for reproduction.

Another interesting way to study this productive economy and its surplus is to split the surplus rate R between ‘capitalists’ and ‘workers’. “We shall also hereafter assume that the wage is paid post factum as a share of the annual product.” (Sraffa [11, 10]). Variable w is the wage per unit of labor and L_p, \dots, L_K the total quantities of labor employed in each sector as a ratio to total labor employed. Our price model is now written as,

$$Ap(1 + r) + Lw = p \Rightarrow p = Lw \left[\frac{1}{1 + r} - A \right]^{-1}$$

One can expand this last term as,

$$p = w \left[\sum_{n=0}^{\infty} (1 + r)^n LA^n \right]$$

This series converges as $(1+r) < (1+R)$ far as where as was shown above $\frac{1}{1+R} = \rho(A)$ and either of the formulations $\rho(A) < 1$ or $R > 0$. It is important to notice that now there is a degree of freedom for the system and one variable must be taken as given. It is usually taken w as given though it might be necessary sometimes to employ r . One can show that $w < 1$ is a sufficient and necessary condition for r to be positive and converge¹⁰.

10 Lipietz [6] has shown that this condition is more general than Morishima’s Fundamental Marxian Theorem of requiring a rate of surplus positive if and only if the rate of profit is positive (see, Morishima [7, 8]) This a tricky problem as can be seen from our previous analysis since it seems to depend upon productiveness of A which means that

2.2 The Standard Commodity

Turning to the main concern of study the relationship between r and w , one has that insofar r behaves within its bounds, then for each w , we can find a set of positive prices p . Actually, any change $w > 0$ implies a change in $r < 0$ as Ricardo has predicted. Moreover, it is not clear the changes in the vector of prices as continuous changes of.

Nevertheless, one needs to pay attention to the numeraire being used. In terms of what are we expressing wages? One can take w as the numeraire and everything is then expressed in labor terms. A second possibility is to choose a nonzero price as the numeraire and let w varies as a rate from 1 to 0. In this case, prices will change as w changes and then the real wage changes in an indeterminate way.

This problem drives one to the general problem of finding the Standard Commodity. As Ricardo had understood, positive changes in real wages produce negative changes in the rate of profit but also produce changes in relative prices¹¹. Thus one would like to have a commodity, Standard Commodity whose proportions were such that any change in distribution would not affect its price. If the wage bill then increases, one would like to have a reduction in the rate of profit, followed by changes in prices of commodities in such a way that these effects balanced out. The increase in one side of its price formation for this commodity produces a reduction in the other side in exactly the same proportion such that its price is not affected by changes in distribution rates.

If this is the case then one can employ such commodity as numeraire to express wages and generalize Ricardo's distribution theory. One can then show that there is a linear relationship between the rate of profit and the rate of real wages since one can employ this standard commodity as numeraire to express wages. Moreover one is also able to distinguish

11 Perhaps, this is the basic principle that can drive a general critique toward any notion of capital derived from neoclassical economics. It was already presented in Ricardo's claim.

those values that increases, decreases or stays the same with respect to this standard commodity as the wage and profit rates are changed. This is the main concern with the Standard Commodity.

Our concern is to show what this standard commodity is and how it satisfies our object¹². Let follow directly from Frobenius's theorems to show straightforward Sraffa's construction of the standard commodity by paying special attention to the equalization of proportions between the net product and the total product created by each sector or industry in this economy.

Let $A \geq 0$ be an irreducible matrix¹³ with p and x positive left and right eigenvectors respectively and $\rho(A) = \lambda$ (A) maximum eigenvalue positive. A is transposed here with respect to the previous analysis to facilitate presentation. y is reserved for the net product.

Since the vectors are both of simple multiplicity then one obtains only relative prices and quantities for p and x respectively. One can normalize the total quantity of labor to unity as an extra equation for x to determine absolute levels of production,

$$\begin{cases} Ax^* = \lambda^* x^* \\ a_n x^* = 1 \end{cases}$$

Following, one defines a net product y^* associated to x^* with the particularity of y^* to be a proportion to x^* , that is, we assume that all sectors in the economy holds the same proportion between the total means of production employed and the total production of commodities, "[sic] this complete basic miniature system is endowed with the property that the various commodities are represented

12 We will not come back here to our previous discussion about the role of reducing a decomposable system into an indecomposable one. Basically speaking a decomposable system can be written in such a way that an indecomposable submatrix is obtained with the same maximum eigenvalue than the original. This has an strict positive and unique maximum eigenvalue and unique up to a scalar positive left and right eigenvectors. We assume A to be irreducible to simplify the exposition.

13 See comments on previous footnote, 9.

among its aggregate means of production in the same proportions as they are among its products.” (Sraffa [11, 19]). That is,

$$y^* = (1 - \lambda^*)x^*, \text{ with } \lambda^* < 1 \Rightarrow Ay^* = \lambda y^*$$

Sraffa identifies a system that satisfies this requirement as a Standard System. The vector y^* is the standard commodity and if one employs this standard composite commodity with the total labor employed of the actual system then the standard net product is obtained.

Let introduce the set of Sraffa’s price equations as previously defined and introduce a particular numeraire defined as the value of the standard net product. Note that the total quantity of labor has been introduced already to define absolute amounts on total production above. One has,

$$\begin{cases} pA(1 + r) + a_n w = p \\ py^* = 1 \end{cases}$$

My object is to show from this how the inverse linear relation is established between the real wage and the rate of profit. All prices and the real wage are expressed in the value of the standard commodity which happens to be the commodity desired.

Multiplying by in this equation and substituting the following already defined equations, (a) $a_n x^* = 1 \rightarrow a_n y^* = (1 - \lambda)$, , (b) $py^* = 1$ and thus (c) $p \cdot A \cdot y^* = \lambda \cdot p \cdot y^*$, one has the desired relation,

$$\begin{cases} pAy^*(1 + r) + a_n y^* w = p^* \\ \lambda(1 + r) + w(1 - \lambda) = 1 \end{cases}$$

Since one can define and substitute

$$\lambda = \frac{1}{1 + R}$$

one has

$$r = R(1 - w)$$

Thus r and w are linearly and inversely related as Ricardo had claimed and Sraffa has showed.

The last step is to see that our last result can be used as numeraire instead of $y^*=1$. That is,

$$\begin{cases} pA(1 + r) + a_n w = p \\ r = R(1 - w) \end{cases} \Rightarrow py^*=1$$

This is possible from the properties of A and Theorem (1).

To conclude this section, it is interesting to notice that once allows to define the linear inverse relationship one can reverse and allow the linear relation to determine . It is only required to figure R from .This step is possible because of the uniqueness up to a scalar of the eigenvectors of A deduced from irreducibility. This obviously simplifies the calculation.

3. THE VON NEUMANN’S BALANCED GROWTH MODEL AND THE SRAFFA’S MODEL

This section studies the relationship between the Sraffian model presented above and the von Neumann model of balanced growth with particular attention to maximal growth. It also shows the role of some underlying assumptions leading both models and the implications regarding the set of prices obtained particularly the zero-price case.

We follow Gale’s presentation of the von Neumann’s model (Gale [1]) and make reference to Neumann [12] original model when required. There are basically speaking two issues that have been modified from the original paper. First, Neumann assumed in the original paper that $a_{ij} + b_{ij} > 0$ (see, Neumann [13]). This basically means that all inputs and outputs are employed in the economy. This assumption plays an important role in Neumann model as far as it allows him to obtain uniqueness in the rates of expansion for prices and outputs. Second, the original paper allows for the possibility of having $pBx=0$ where B is the output matrix. This means that the value of output can turn out to be zero. Both issues are discussed below. The presentation due to Gale employs the linear programming approach. Strictly speaking, the model of Neumann is not linear since the set of restrictions present nonlinearities.

The basic problem for the planner is to obtain the largest rate of expansion for the economy, and the vector of prices and output

levels in a dynamic setting. On the one hand, one must obtain the largest rate of expansion λ for the technology of the economy given the conditions that total output in period cannot exceed total availability of the economy. In dynamical terms, this means that total output in period t is at least equal to total inputs next period $t+1$. We want to find out a rate of expansion such that each activity expands at the same rate. Formally defined,

Definition 1. The model (B, A) is said to be expanding at a technological expansion factor at least equal to λ , if there exists a number $\lambda > 0$ and an intensity vector $y \geq 0$ such that $By \geq \lambda Ay$.

We assume for the technology that (1) $\alpha_i \geq 0, \forall i=1, \dots, n$. (2) $B_i \geq 0, \forall i=1, \dots, n$. where matrix A and B are respectively the technical coefficient matrix for input and output. Notice that $\lambda > 0$ turns meaningful in economic analysis for the case

We can now define the maximization problem in a linear programming set up,

Problem 1. To maximize λ subject to the following constraints $(B - \lambda A)y \geq 0$ and $y \geq 0$.

Theorem 2. Gale [1]¹⁴, (Existence) For a model (B, A) , there exists a positive technological expansion factor λ^* .

On the other hand, there is a second concern associated to the expansion of the economy. This is related to the economic specification of the system. The previous problem is concerned with finding quantities while now one is also interested in finding values. In this sense, one must find simultaneously a rate of economic expansion such that the value of production is equal to the cost value times the rate of economic expansion. In dynamical terms this means that the total value of production at time t is at most equal to the cost of production in time $t + 1$ when prices are non-negative.

Definition 2. The model (B, A) is said to be expanding at an economic expansion factor at most equal to μ , if there exists a number $\mu > 0$ and a price vector $p \geq 0$ such that $pB \leq \mu pA$.

The problem is the following.

Problem 2. To minimize μ subject to the following constraints $p(B - \mu A) \leq 0$ and $p \geq 0$.

Theorem 3. (Existence) For a model (B, A) , there exists a positive economic expansion factor μ^* .

If one takes a technology (B, A) that satisfies assumption 1 and 2 then it has been shown that there exist solutions for both problems. Nevertheless, one finds out that $\lambda^* \neq \mu^*$. That is, in general the associated problems have solutions whose value is not the same when associated to y^* and p^* . The following result shows the relation between both rates λ^* and μ^* .

Theorem 4. For a model (B, A) it always holds that $\mu^* \leq \lambda^*$.

The problem presented so far has a linear programming approach, i.e., a maximum problem with a dual problem associated. It has been shown that both problems have a solution and there is an inequality relation obtained from the optimal solutions, $\mu^* \leq \lambda^*$. The fundamental dual theorem is not quite complete yet.

Neumann asserts that given a nonnegative economy one can find an expansion rate and nonnegative vectors for prices and outputs (rays) such that the program of production and the program of "prices" are both satisfied and additionally, the value of total output is equal to total cost times the rate of expansion. Nevertheless, this is exactly what our previous maximum and minimum problems do. One can establish then Neumann theorem.

Theorem 5. (Neumann): For a model (A, B) , there exists a number $\gamma \geq 0$ and semipositive vectors $y, p \geq 0$ such that (1) $(B - \gamma A)y \geq 0$, (2) $p(B - \gamma A) \leq 0$, (3) $p(B - \gamma A)y = 0$

14 The main theorems in these sections are due to Gale. If this is not the case, then it will be noticed.

For a proof see Gale [1].

From this result one has that $\gamma \in [\mu^*, \lambda^*]$ and the triplet $(\gamma; p, y)$ is not unique. The fact is that once Neumann's assumption¹⁵ that $\alpha_{ij} + b_{ij} > 0$ is dropped this implies that the rate solutions cannot be unique¹⁶. In this case, there will be pairs of intensity and price vectors, other than the "optimal pair" y^*, p^* , compatible with the expansion of the model (see Klein [5].) It is important to remark that the solution means that any activity that creates economic losses has zero level of activity and any activity that faces oversupply in this situation has zero price. All equalities imply positive output intensities and positive prices.

In order to have $\gamma = \mu^* = \lambda^*$, Gale adds the assumption that technology is irreducible to obtain the following result¹⁷.

Theorem 6. (Duality): For a model (B, A) with an irreducible¹⁸ technology, $\gamma = \mu^* = \lambda^*$.

If the model is irreducible then one has $\gamma = \mu^* = \lambda^*$. For our interest this assumption implies that our model can be build up from a linear programming approach and the dual

theorem holds. This implies that p and y are unique now since $\gamma = \mu^* = \lambda^*$.

The most important issue and perhaps more obvious in this relation between the Sraffian model and the Neumann one is that if one assumes the second model to be single production¹⁹ then the model developed by Neumann exactly replicates all the results derived from our previous study of the Sraffian model in the preceding section as far as one assumes irreducibility.

In the case of an irreducible technology one knows that by the Frobenius theorem, the associated spectral radius is maximum and is an eigenvalue.

Moreover, the left and right eigenvectors associated to this Frobenius root are the only positive vectors. From previous economic analysis of the Sraffian model it was shown that the set of prices will be all positive and all commodities basics. As claimed before, the Frobenius theorems uphold the economic applications and under these assumptions unify the results and the study of these economies.

Moreover, from this von Neuman type model, it has been shown that unless the economy is irreducible, the rate of technical and economic expansion would not coincide. If irreducibility is assumed then both rates coincide and additionally one can show that the price and output-intensities are up to a scalar unique and positive as the Frobenius theorem quarantees.

However, what about a reducible economy? For the Sraffian economy prices, we discussed that prices are not unique and even if positive, it can lead to some interpretation problems. We saw that a zero price will cancelled out over a subsystem of nonbasic goods with zero price. This has nothing to do with over-supplied commodities at all. The notion of zero price or zero output intensity has a total different explanation from the von Neumann economy. This is particularly important because it leads to the basis of both models and shows how the assumptions describe

15 In the original model, Neumann writes the model in a rather descriptive form instead of a planning form as Gale. Using λ and μ , he formulates the model as follows,

$$\lambda Ay \leq By, \text{ if } \lambda Ay < By \rightarrow p_j = 0$$

$$\mu A \geq Bp, \text{ if } \lambda Ap > Bp \rightarrow y_j = 0$$

and shows that with assumption $\alpha_{ij} + b_{ij} > 0$, $\lambda = \mu$ is unique and exists $p, y \geq 0$ not necessarily unique (see Neumann [12]).

16 Jachs [4] has found necessary and sufficient conditions to ensure exactly one solution.

17 This assumption is introduced simultaneously for obtaining $pBy > 0$ as mentioned above.

18 It might be of interest to compare this assumption with Neumann original ($a_{ij} + b_{ij} > 0$) assumption. In fact, irreducibility means that to some extent all sectors are eventually connected (at some time) and Neumann assumption means that each good as input or output is always employed. Gale's assumption actually builds up in a line close to that of Neumann's though in a less restrictive sense.

19 In this case, one has n activities and B transform into the Identity matrix.

different environments. In the Neumann case, zero prices or outputs in reducible economies are due to the excess of supply criteria or a larger cost than price. Then, it is obtained that $\mu^* \leq \lambda^*$.

The introduction of irreducibility at this level implies by Frobenius theorem that prices will be all positive as is the case for outputs and then the notion of zero price is given away. All “commodities are economic goods” as Menger would arm. Thus the strict inequality disappears and the system becomes an equality model. The Neumann model becomes the Sraffian model by definition though in a similar linear programming form. Thus, as noticed above both systems generate the same result.

It is not surprising that the Sraffian and Neumann models coincide in their solution and in their specification of the *reproductive perception of the system*. What the Neumann system actually confers is a solution that in the one hand allows for a price system equal to cost and in the other hand solves the actual production proportions between the different sectors such that production is sustainable. This is exactly the object and solution obtained by Sraffa and his use of y^* to construct the standard commodity assuming an irreducible matrix, i.e., basic commodities only. It is also important to recognize that there is no consideration at all for any kind of initial stocks being assumed in any model especially in the von Neumann one²⁰.

Writing the Neumann model in exactly the same way as before but with single production one can show the following result which exactly replicates the model of Sraffa.

Theorem 7. *If the matrix A is irreducible then $\mu^* = \lambda^*$ and the optimal intensity vector is positive and unique up to a multiplication by a positive number.*

4. CONCLUSION REMARKS

Departing from a rigorous derivation of the Sraffa model recurring to some theorems discovered by Frobenius and Perron at the beginning of the century, the paper introduces the concept of productive economies or surplus economies and shows the existence of the price system solution. Employing these results, the paper provides a clear derivation of the concept of the Standard Commodity and studies its implications for the theory of distribution.

The paper then sketches the von Neumann's balanced growth model and show how Sraffa's model is related this model. Showing the role of the assumptions, one establishes the relationship between both models, particularly the role of irreducibility to obtain uniqueness and unify the study of both models.

There are some extensions that might be of interest. The general case²¹ of these economies is particularly important in the sense of joint production economies. It has much of the same line of thought followed in the paper but it is not straightforward such relation. The most difficult issue at stake for this relation would be related to the existence of zero prices for the Neumann case when one is assuming irreducibility for (A, B) . This is related to the relation between the number of commodities and activities and to the input-output distinction.

The reducible case would be difficult to compare but it carries the distinction above considered between zero prices for each model. There is no mention to excess of supply associated to Sraffa. The distinction derived between basics and nonbasics is very important for the interpretation of zero price goods as previously analyzed.

Finally, it might also be of interest for further research to study the stability of these systems and study also the behavior of solutions to small changes in the underlying fundamentals.

20 This was actually a neoclassical ideological frustration that was attacked with the turnpike theorems' results.

21 This comparison would actually require to develop Sraffa analysis of joint production which would extend this already long-essay.

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