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FUZZY GAMES

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SUMMARY

Multilinear extensions f, of characteristic functions v for cooperative games, here are seen as characteristics functions tllemselves. To do tllis, we lcok a! !he points inside tlle n-dimensional unitary cube, as fuzzy sets defined over tlle se! I of n players for game v. We show tila! f fulfills tlle definition of..!'haracteristic functions. We also show tila! mos! of standard tlleorems fer cooperative games also hold for tllis new type of games: The fuzzy games.

RESUMEN

Las extensiones multilineales f, de las funciones características v. de los juegos cooperativos, son tratadas aquí como funciones características. Para esto, consideramos a los puntos en el interior del cubo unitario enedimensional, como conjuntos borrosos, definidos sobre el conjunto I de n jugadores. Mostramos que f satisface las condiciones de la definición de las funciones características. También mostramos que muchos de los teoremas basícos de la teoría de los juegos cooperativos también son válidos para este nuevo tipo de juegos: Los juegos borrosos.

O. Introduction

Some time ago, I was doing some research on the application of multilinear extension of games to multiple criteria decision methods. I realized that multilinear extensions might have the properties of cooperative games, if we look at the interior points of the n-dimensional unit cube as fuzzy sets. This paper is the development of those ideas.

In section 1 we give a brief introduction to cooperative games, only what is needed to understand the following sections. In section 2 we state the main concepts of multilinear extension of games, then in section 3 we set the problem that we are trying to solve. Section 4 is dedicated to some concepts of fuzzy sets theory. In section 5 we use fuzzy sets to prove that the properties stated in section 3 are true.

After the fundamental of these new games, the fuzzy games are stablished, we show in section 6 that most of standard theorems of cooperative games also hold for fuzzy games; that is, a new branch of game theory is set. Section 7 shows some paths for further development, specially the adaptation of Shapley's value's formula to this contexto

1. Cooperative games.

Definition 1.1 Let I be the set consisting of n players (n ~ 3). We say that a game is defined in characteristic function form if we can define a function v over all subsets of I (these subsets are called coalitions), such that v has the following properties:

v: P(I) ~ R , the real numbers set, such that, for all S and T subsets of 1:

(1.1) v(S vT) > v(S) + v(T) if S (\ T = 0

(1.2) v(|1|) = O.

How the function v is defined obviously depends on the object that we are trying to model. Intuitively, v must be defined such that, for any coalition S, v(S) is the maximum amount of wealth that the coalition can get, *relying exclusively on their own means.*

The first condition (1.1) is called superadditivity. It establishes that forming coalitions is for the advantage of the members. The second condition, (1.2) is basically a mathematical convenience.

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A game can have several possible outcomes. We will treat each one as an n-tuple x = (x1,x2""'x_n), where xi is interpreted as the payoff received by player i. For a vector x to be reasonably interpreted as the result or solution of the game, it will have to meet at least the following two conditions:

(1.3) xi ~ v({i}), for all i =1,...,n; and
(individual rationality)

(1.4) $\underset{1=1}{\overset{n}{\sim}}$X'I = v(I). (group rationality)

Definition 1.2 A tuple that satisfies (1.3) and (1.4) is called an **imputation.** Let's denote by **M(v)** the set of all imputations for game v.

NOTE: When v(I) =1, and because of (1.4), the set M(v) is the (n-1)-dimensional simplex; for example, when n=3 this simplex is the triangle with vertices (1,0,0), (0,1,0) and (0,0,1).

Definition 1.3: Let S be a coalition; let x and y be two imputations. We say that x is preferred over y by the coalition S if:

(1.5) xi> Yi forall i e S, and

(1.6) k xi ~ v(S).
ieS

Given two imputations x and y, we say that **x is preferred over** y , and we write xGy, if there exists a coalition S such that S prefers x over y. We also say that x **dominates y.**

Definition 1.4: The **eore** of the game, denoted by C(v), consists of all nondominated imputations.

Theorem 1.1: C(v) consists of all imputations x such that:

(1.7) kXi ~ v(S) V S", O, Ss;;;I
ieS

Note: In this section we will not prove any of the propositions, since they are well known results in Game Theory. Proofs can be found in several books, e.g., Roberts [1976].

There are several concepts found in the game theory literature that have been proposed as solutions of games. The core is an adequate concept as a solution of cooperative games, but unfortunately the set of solutions for (1.7) very often is empty. Hence, sometimes this is not a suitable concept for practical problems.

Another approach to the solution concept comes from the idea of stability.

Definition 1.5: Let B be a subset of M(v). We say that B is internally stable if V x,y in B neither xGy nor yGx. We say that B is externally stable if for every y in M(v)-B, there exists some x in B such that xGy. We say that B is **stable** if it is both internally and externally stable.

Theorem 1.2: The core is a subset of all stable sets.

Definition 1.6: A game is called **essential** if there are at least two disjoint coalitions S and T such that:

(1.8) v(S v T) > v(S) + v(T).

Theorem 1.3: vis essential if and only if v(I) >k v(i)
iel

Definition 1.7: A game is called **eonstant-sum if**

(1.9) v(S) + v(I-S) = v(I) V S s;;; I.

Theorem 1.4: Let v be an essential constant-sum game, then C(v) is empty.

Definition 1.8: Let v and w be two characteristic functions defined over the same set of players I. Let f be a bijective function from M(v) onto M(w); f is called an isomorphism from v to w if:

(1.10) xGy iff f(x)Gf(y) V x, Y e I.

Definition 1.9: A game is called:

(1.11) (zero)O-normalized if v({i})=O Viel.

(1.12) (one) **1-normalized** if v(I) = 1.

Theorem 1.5: If v is the characteristic function of a O-normalized game then:

(1.13) a. v(S) ~ O V S s;;; I.
(1.14) b. T ~ S => v(T) ~ V(S).

Definition 1.10: Two games are called S-**equivalent** if there exists r > and q1,...,qn real numbers such that

(1.14) w(S) = r'v(S) + k qi V S s;;; I
ieS

Theorem 1.6: If vis essential then it is equivalent to exactly one (0,1)-normalized game w, where:

n
(1.15) w(S) = (v(S) - k v({i}» / (v(I) - k v({i}»
ieS i=1

Theorem 1.7: Assume that w and vare the characteristic functions of two S-equivalent games. Then the two games are isomorphic.

Ideally, a solution concept should point to a single imputation, because this way we would know how much of the wealth in dispute (v(I» must be allocated to each player. The core and the stable set do not always point to a single imputation; that is, sometimes they contain more than one imputation; even in some cases the resulting sets are very large. Hence some people have looked for solution concepts that point to a single imputation. One of the best known is **Shapley's value.**

Let H be a function defined over the set of characteristic functions which in turn are defined over the set I. We want H such that H(v) is a n-tuple of real numbers. Let p be a permutation of the elements of I. For any characteristic function v, let w be the characteristic function defined as follows: w(S) = v(p(S». We also want H to satisfy the following axioms:

Axiom 1: Hi(w) = Hp(i)(v).

Axiom 2: n
~ HI'(v) = v(I).
1=1

Axiom 3: If v(S-{i}) = v(S) V S, then Hi(v) = O

Axiom 4: H(v+w) = H(v) + H(w) where v and w are characteristic functions.

Theorem 1.8: (Shapley) There exists a unique function satisfying axioms 1-4

(1.16) Hi(v)= kg(s)[v(S)-v(S-{i})] and
Ss;;;I

(1.17) g(s) =(s-1)I(n-s)! and s=ISI
n!

Note: Shapley's value is not always in the core of the game.

2. Multilinear extensions.

Let 1={1,.,n}. The characteristic function v is a function whose domain is 21, the set of all subsets of I. Each one of these subsets S can be identified with a tuple of O's and 1's, such that Xj is equal to 1 if j E S, and Xj is equal to O if j !i1S. We can think of these sets as the corners of the n-dimensional unitary cube.

For example, if n=3, the empty coalition is identified with (0,0,0); the coalition consisting of the first and third player is (1,0,1); the whole set of players is (1,1,1) and so on.

In summary, the domain of vis 2^1={O,1}1, the set of corners of the n-dimensional unitary cube.

The idea behind the multilinear extension is that we can extend the domain of v to the remaining points of the cube (interior and frontier). Each tuple (x1"'"x_n) may then be seen as some sort of fuzzy or random coalition.

Definition 2.1: [Owen, 1982] The **multilinear extension of v** is a function f defined as:

(2.1) f(xk"x_n) =
k {IT Xj. IT (1-xj)} v(S), for O ~Xj ~ 1, j=1,..,n.
Ss;;I jeS j!i1S

This function f is linear (geometrically speaking) in each component. For example, for x1 let us divide the subsets S into two classes: the ones which contain player 1 and the ones which do not. Then (2.1) can be expressed as:

(2.2) f(xk"xn) =
k rr Xj rr (1-xj)v(S) + k rr Xj rr (1-Xj) v(S)
Ss;;J jeS j!i1S Ss;;J jeS j!i1S
1eS 1!i1S

= x1 [k k rr Xjv(S)] + k m rr Xjv(S) = cX1 + d,
Ss;;J j".,1 Ss;;J
1eS 1!i1S

where k,m,c and d are expressions that don't inelude x1' this means that f can be expressed as a linear function of variable x1'

We also want to show that f is an extension of v; that is, they coincide at the corners of the cube. Let us define first the T-corner of the cube as the tuple:

(2.3) e(T)j = { 1 if jeT ; O if j!i1T

Hence, f(e(T» = k { rr e(T)j rr (1- e(T)j) } v(S).
Ss;;J jeS j!i1S

Now, if S".,T, then, for any j in S-T the first product is zero; for any j in T-S the second product is zero. That is, the. only non null term of the sum is for T=S. Also, since rr e(T)j rr (1-e(T)j) =1. We have:

Properties of v

- v(S v T) ~ v(S) + v(T)
  if S} T".,0
- v(0) = O

(2.4) f(e(T» = rr e(T)j . rr (1- e(T)j) v(T) = v(T)
jeS j!i1S

:. f is an extension of v. [Owen, 1982].

3. Formulation of the problem.

In section 2 we saw the function f as a multilinear extension of the function v, f was defined as: [Owen, 1982]

f(X1""Xn) =
k {rrXj"rr(1-Xj)}v(S).forO~xj~ 1,j=1, ..,n.
Ss;;J jeS j!i1S

We showed that this function coincides with v at the corners of the n-dimensional unitary cube (that is why it is an extension of v); but, what is the meaning of f(x) when x is an interior point of the cube?

It seems that Owen was interested in this function f for the purpose of proving Shapley's value formula by other means; he calls x a random coalition.

I think that a more appropriate name would be fuzzy coalition, since it is a "in between coalitions" object.

If these points x are the new type of coalition, and f is an extension of v, an obvious question is if falso extends the properties of v to the interior of the cube, that is:

Properties of f?

- f(x v y) ~ f(x) + f(y) (Property 1)
  ifx v y".,0
- f(0) = O (Property 2)

To prove or disprove these properties we need to state what we mean by fuzzy subset, union, intersection and empty set. In this sense we just follow the standard definitions of fuzzy set theory. [Kaufmann, 1975]

4. Fuzzy sets

Definition 4.1:
Let E be a set. Let μ be a function μ:E ~ [0,1], a fuzzy subset Z of E is the set of pairs Z = {e, μ(e)} Ve ε E.

Other notation is Z = {(e ı μ(e») V e e E. It is also common to use the notation μZ to emphasize the relation between Z and its function ..

Let A and S be fussy subsets of E.

Definition 4.2:
We say that A is ineluded in S if μA(e) ~ μS(e) V e ε E.
Notation A s;;; S

Definition 4.3:
Equality: μA(e) = μS(e) "lee E. Notation A = S

Definition 4.4:
Intersection:
μArlS(e) = Min {μA(e) , μS(e)} "leE E.

Notation A (\ S

Definition 4.5:
Union:
μAvS(e) = Max { μA(e) , μS(e)} "lee E.
Notation A vS

Definition 4.6:
Empty set:
μ 0(e) = O "lee E, Le. 0 = {(e,O)} "lee E.

Definition 4.7:
Difference
μA-S(e)=
Max{μA(e), μS(e)} - Min{μA(e), μS(e)} "lee E.
Notation A - S

NOTE.
Given any real numbers a and b, we have :
Max{a,b} - Min{a,b}= la-bl.

5. Cooperative games extended to fuzzy coalitions.

With these definitions at hand we can now go back to section 3 and look more elosely at our problem. For us, the set E is I, the set of n players.
Then, given x and y, points in the unitary cube we have:

Definition 5.1:
Let~: E ~[0,1],
such that ~U)= Xj . Similarly μyO)= Yj

Given this definition, now we can talk of basic properties of coalitions such as x s;;; y, x v y, X (\ y. What we need to prove now, is that the function f defined earlier fulfills the two basic properties described in section 3.

Proposition 5.1: The second property holds.

Proof
The property f(0) = O is true since f(O,...,0) =

k { rr o. rr(1-0)} v(S) = O

Se I jES j!i1S

In order to prove that the first property is also true, we need first to prove two auxiliary results.

Now, when x (\ y = 0 we have that Min{xj,Yj} = O V j e I. That is, either Xj = O or Yj = O.

Under these conditions the union has a very particular form, since Max {Xj'Yj} = Xt yj-

Also notice that O ~ Xi +Yi ~ 1 for the same reasons.

Then we have that x v y =

(Max{ x_1, y_1}, ..., Max{ x_n, y_n}) =

(X1 + YI' ..... , xn + Yn)' Hence

Proposition 5.2. f(x v y) = f(x + y)

Proof. f(x v y) =

k {rr Max{ xi' Yj}• rr (1-Max{ Xj' Yj}) } v(S).
Se I jeS j!i1S

= k { rr (y Yj) rr (1-(y Yj-»} v(S).
Se I jeS j!i1S

On the other hand we have that

f(x+y)= f(x1 + Y1,....·,X_n +Yn) =

k {rr(xt Yj) rr (1- *(xt* Yj »} v(S).
Se I jeS j!i1S

That is, the property f(x v y) = f(x + y) holds.

Also note that 1 - Xj - Yj = 1 - Xj - Yj + XjYj= (1- Xj)(1- Yj) since XjYj = 0, then tl1e unlon can be expressed as:

f(x v y) = k { rr (y Yj) rr (1- xj)(1- Yj)} v(S).
Se I jeS j!i1S

Proposition 5.3. f(x + y) ~ f(x) + f(y)

Proof. In order to prove this proposition we need the following property: [OWEN, 1982] [ALVARADO, 1987]

f(x) can be expressed as f(x) =

k CT rr Xj where CT are const~nts
Ts;;J jeT

and k CT = v(S)
Ts;;;S

f(x + y) = k CT rr (Xj + Yj) =
Ts;;J jeT

k CT (Tl Xj + rrYj + R)
Ts;;J jeT jeT

Where R is the sum of products of some Xj and Yk ' since all of them are positive so is R., hence

f(x + y) =

k CT rr Xj + k CT rrYj + Rk CT
Ts;;J jeT TS;;t jeT TS;;t

Sut letting S = I in k CT = v(S) we have that
TS;;;S

k CT = v(I) , hence
Ts;;J

f(x + y) =

k CT rr Xj + k CT rrYj + Rv(I).
Ts;;J jeT Ts;;J jeT

'The only remaining problem would be if v(I) is negative, but for a vast majority of cooperative games v(I) is positive; anyway, if it is not, then is always posible to find an isomorphic game for which the value of the whole set of players is positive. Under these asumptions we have that

f(x + y) = k CT rr Xj + k CT rrYj + Rv(I) =
Ts;;J jeT Ts;;J jeT

f(x) + f(y) + Rv(I)

:. f(x + y) ~ f(x) + f(y).

Proposition 5.4 The first property is true.

Proof: From Propositions 5.2 and 5.3 follows property 1.

Definition 5.2. Given a game in characteristic function form v, and given its multilinear extension f, we define the associated fuzzy game as the function f. Since it has the properties of characteristics functions, and the coalitions are the fuzzy sets, defined over the set of players I of v.

Sefore we start looking for other properties of function f, it is convenient to think about the usefulness of this kind of games. The analysis of cooperative games is based on the idea that each player elearly belongs or not to each one of the feasible coalitions.

Sut it may happens that in some situations that question cannot be answered elearly. That is, *maybe* this particular player wold be in that coalition. Obviuously in this case, the ideas of fuzzy sets becomes a useful tool.

The strategy will be as follows: First model the problem as an standard coooperative game, that is: define the set of players and the characteristic function v. Second, define the multilinear extension f of v. In section 6 we will see that most of the properties of v holds nicely also for f.

6. Other properties of fuzzy games.

Proposition 6.1: z is an imputation for game v <=> z is an imputation for game f.

Proof: If z is an imputation for v we have that $z_i \geq v(i)$ $\forall i$ and $\sum z_i = v(I)$.

But $f(0,0,\ldots,1,0,\ldots,0) = v(i)$ since equation (2.4),

also $f(1,1,\ldots,1) = v(I)$ again for (2.4)

Hence $z_i \geq v(i)$ $\forall i$ <=> $z_i \geq f(0,0,\ldots,1,0,\ldots,0)$ $\forall i$

$\sum z_i = v(I)$ if <=> $\sum z_i = f(1,1,\ldots,1)$.

Despite the last proposition is a very nice one, we cannot go along with all definitions and theorems mentioned in section 2, since we need first to state clearly what should we mean by phrases like " for all i in S".

This is so, because now we are talking about fuzzy sets and coalitions, hence we cannot talk yet of concepts like preference of one imputation over another. It is not straight forward to extend the definition (1.5) (1.6) of an imputation preferred ayer another by a coalition S.

A modification of the definition that show some promise will be:

An imputation z is preferred over imputation w by coalition x if:

(1.5)" $z_i x_i > w_i(1-x_i)$ $\forall i = 1,\ldots,n$. The left factor x_i shows the degree of belonging of i to coalition x, hence the right factor $(1-x_i)$ shows the degree of not belonging.

Another posible definition is $z_i > w_i$ $\forall$ i such that $x_i \geq \alpha$, with $0 \leq \alpha \leq 1$.

(1.6) $\sum_{i=1}^{n} z_i x_i \leq f(x)$

Proposition 6.2:

v is an essential game <=> f is an essential game.

Proof. $v(I) > \sum_{i=1}^{n} v(i)$ $f(1,\ldots,1) > \sum_{i=1}^{n} f(0,0,\ldots,1,0,\ldots,0)$

Proposition 6.3: If v is constant sum then f is constant sumo

Proof. Owen proved that if v is constant sum then $f(1-x_1,\ldots,1-x_n) = v(I) - f(x_1,\ldots,x_n)$ for any x in the cube. From the definition we have that f is constant sumo

Proposition 6.4: If v is a (0,1)-normalized game then f is (0,1)-normalized

Proof: Since v(i) =

$f(1,0,\ldots,0,1,0,\ldots,0)$, if $v(i) = 0$,

also $f(1,0,\ldots,0,1,0,\ldots,0) = 0$.

On the other hand $v(I) = f(1,1,\ldots,1) = 1$.

Proposition 6.5: If v is a 0-normalized game then

a. $f(x) \geq 0$ $\forall$ x in the unitary cube

b. $x \geq y$ => $f(x) \geq f(y)$.

Proof: See Theorem 1.4 and [Alvarado, 1988].

There are many definitions and theorems related to cooperative games in characteristic function form v. SO far we have proved that most of the basic theorems also holds for function f. Other theorems must wait *tor* a suitable definition of imputation dominance for fuzzy coalitions. Sut it is also clear that not all properties of vare inherited by f, for example v may be simmetric, but it doesn't follows that f is simmetric. Some counterexamples can be built

7. An open problem.

So far we have seen that multilinear extensions fulfill the definition of cooperative game's characteristic function formo Also we saw that this function f inherits most of the properties of function v. Hence, up to now the situation is as follows:

<u>characteristic function v</u>	<u>multilinear extension f</u>
• fulfills definition properties (1.1) and (1.2)	• idem, see Props 5.1 and 5.2.
• fulfills definition properties (1.3) and (1.4)	• idem, see Proposition 6.1
• fulfills definition properties (1.5) and (1.6)	• pending, a suitable definition needed
• satisfies theorem 1.1	• p'ending, needs (1.5) and (1.6)
• satisfies theorem 1.2	• pending, needs (1.5) and (1.6)
• satisfies theorem 1.3	• idem, see Proposition 6.2
• satisfies theorem 1.4	• idem, see Propositions 6.2 and 6.3
• satisfies theorem 1.5	• idem, see Propositions 6.4 and 6.5
• satisfies theorem 1.6	• idem, see Propositions 6.2, 6.4 and 6.5
• satisfies theorem 1.7	• idem, see Propositions 5.1 and 5.2
• satisfies theorem 1.8 (Shapley)	• OPEN QUESTION

Before any aatlempt to prove Shapley's value formula for function f, we must be aware of the underlying differences.

• Set of players I is finite, $|I|=n$ • Set of coalitions 2^I is finite, $|2^I|=2^n$	• Set of players I is finite $|I|=n$ • Set of coalitions is the n-dimensional unit cube, it is infinite and not numerable
• (1.16) $H_i(v) = \sum_{S \subseteq I} g(s)\,[v(S) - v(S-\{i\})]$	• The summatories must be transformed into integrals over suitable regions of the unit cube. Also a suitable substitution for g(s) must be found.

8. Conclusions.

We have just opened the door of a field in game theory that seems to be very interesting and promising, and also more close to real situations in social sciences, where objects are not elearly cut. We encourage the reader to try to fill the gaps still open, such as the concept of dominance and Shapley's value for fuzzy games.

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