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# ANALYSIS OF OPTIMAL CONTROL PROBLEMS FOR THE PROCESS OF WASTEWATER BIOLOGICAL TREATMENT

# ANÁLISIS DE PROBLEMAS DE CONTROL ÓPTIMO PARA EL PROCESO DE TRATAMIENTO BIOLÓGICO DE AGUAS RESIDUALES

Elina V. Grigorieva* Natalia V. Bondarenko†
Evgenii N. Khailov‡ Andrei Korobeinikov§

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*Department of Mathematics and Computer Sciences, Texas Woman's University, Denton, TX 76204, U.S.A. E-Mail: `EGrigorieva@mail.twu.edu`

†Department of Computational Mathematics and Cybernetics, Moscow State Lomonosov University, Moscow, 119992, Russia. E-Mail: `nataliabonda@mail.ru`

‡Same address as/*Misma dirección que:* N.V. Bondarenko. E-Mail: `khailov@cs.msu.su`

§Centre de Recerca Matemática, Campus de Bellaterra, Edifici C - 08193 Bellaterra, Barcelona, Spain. E-Mail: `AKorobeinikov@crm.cat`

**Abstract**

We consider a three-dimensional deterministic control model of the process of aerobic wastewater biotreatment. For this model, we formulate and solve two optimal control problems, each of which has a corresponding minimizing functional. For the first problem, the functional is a weighted sum of the pollutant concentration at the end of a fixed time interval and the cumulative biomass concentration over the interval. For the second problem, the functional is a weighted sum of the pollutant concentration at the end of the time interval and the cumulative oxygen and biomass concentrations over the interval. In order to solve these problems, we apply the Pontryagin Maximum Principle. The switching functions are analytically investigated and uniquely determine the type of the optimal controls for the considered problems. Their properties allow the simplification of the optimal control problems to that of finite-dimensional constrained minimization. Numerical solutions of the optimal control problems are also provided.

**Keywords:** wastewater treatment, nonlinear model, optimal control.

**Resumen**

Consideramos un modelo de control determinístico tridimensional del proceso de biotratamiento aeróbico de aguas residuales. Para este modelo, formulamos y resolvemos dos problemas de control óptimo, cada uno de los cuales tiene un funcional a minimizar. Para el primer problema, el funcional es una suma ponderada de la concentración del contaminante al final de un intervalo de tiempo fijo y la concentración acumulada de la biomasa sobre el intervalo. Para el segundo problema, el funcional es una suma ponderada de la concentración del contaminante al final del intervalo de tiempo y las concentraciones acumuladas de oxígeno y biomasa sobre el intervalo. Para resolver estos problemas, aplicamos el Principio del Máximo de Pontryagin. Las funciones de conmutación son investigadas analíticamente y determinan unívocamente el tipo de controles óptimos para los problemas considerados. Sus propiedades permiten la simplificación de los problemas de control óptimo para una minimización finitodimensional con restricciones. Se brindan las soluciones numéricas de los problemas de control óptimo.

**Palabras clave:** tratamiento de aguas residuales, modelo no lineal, control óptimo.

**Mathematics Subject Classification:** 49J15, 49N90, 93C10, 93C95.

# 1 Introduction

The objectives of the biological treatment of wastewater are ($i$) the elimination of pathogenic microorganisms and ($ii$) the reduction of the organic matter to an acceptable level. Autothermal thermophilic aerobic digestion (ATAD) is one method used to treat wastewater with non-pathogenic thermophilic bacteria [9, 2]. The bacteria consume the organic matter and kill pathogenic microorganisms with heat produced as a by-product of the metabolic synthesis. This process is effective but costly, as it requires continuous aeration. An intention of reduction of the high cost motivates intensive mathematical studies of this process. A possibility to control the process leads to a variety of optimal control problems, which are associated with this process.

Some of these problems, such as the minimization of a pollutant concentration at the terminal time and construction of the attainable set were considered in our earlier publications [3, 4]. The minimization of the total energy consumption was presented at the 2012 Joint AMS–MAA mathematics meeting in Boston, USA [5]. In addition, [3] contains a fairly complete review of the process of biological wastewater treatment and, in particular, modeling and control of the ATAD. In this investigation, we examine the minimization of a weighted sum of organic pollutants and pathogenic microorganisms at the end of the fixed treating time interval and the integral concentrations of oxygen and biomass over a finite time interval.

# 2 Mathematical model

A mathematical model of the process can be formulated as the following system of three ordinary differential equations with initial conditions:

$$\begin{cases} \dot{x}_1(t) = -x_1(t)x_2(t)x_3(t) + u(t)(m - x_1(t)), \ t \in [0, T], \\ \dot{x}_2(t) = -x_1(t)x_2(t)x_3(t), \\ \dot{x}_3(t) = x_1(t)x_2(t)x_3(t) - dx_3(t), \\ x_1(0) = x_1^0, \ x_2(0) = x_2^0, \ x_3(0) = x_3^0, \\ x_1^0 \in (0, m), \ x_2^0 > 0, \ x_3^0 > 0. \end{cases} \tag{1}$$

Here, $x_1(t)$ is the concentration of oxygen; $x_2(t)$ is the concentration of organic matter (pollutants); $x_3(t)$ is the concentration of the thermophilic aerobic bacteria. This model assumes that the reaction is described by the mass action law [6]. System (1) is considered on a fixed time interval

$[0, T]$, which corresponds with the duration of the treatment of a single batch. The first equation of the system describes the evolution of the oxygen concentration in the treated sludge: here the first term defines its consumption in the reaction and the second term characterizes the rate of aeration. In the latter, the function $u(t)$ is the air pumping rate, which also serves as the control function for this model. The second equation expresses the reduction of the pollutant concentration due to the reaction. Finally, the third equation of system (1) describes the growth of biomass of the thermophilic aerobic bacteria in the reaction; the bacteria are also dying due to natural reasons at a per capita rate $d$. The system is complemented with positive initial conditions and a constraint on the oxygen pumping rate.

System (1) is a controlled system, where function $u(t)$ is the control. The corresponding set of admissible controls $D(T)$ is composed of all Lebesgue measurable functions $u(t)$, which for almost all $t \in [0, T]$ satisfy the inequalities:

$$0 \leq u(t) \leq u_{\max},$$

where $u_{\max}$ is the maximum rate of aeration.

The following Lemmas describe the properties of the phase variables $x_1(t)$, $x_2(t)$ and $x_3(t)$ for system (1) and the upper boundary for $x_1(t)$.

**Lemma 1** *For an arbitrary control $u(\cdot) \in D(T)$ the corresponding solutions $x_1(t)$, $x_2(t)$, $x_3(t)$ of system* (1) *are defined for all $t \in [0, T]$ and satisfy inequalities*:

$$0 < x_1(t) < x_1^{\max},\ 0 < x_2(t) < x_2^{\max},\ 0 < x_3(t) < x_3^{\max},\ t \in (0, T],$$

*where*

$$x_1^{\max} = x_1^0 + m u_{\max} T,\ \ x_2^{\max} = x_2^0,\ \ x_3^{\max} = x_3^0 \mathrm{e}^{x_2^0 T (x_1^0 + m u_{\max} T)}.$$

**Lemma 2** *Inequality $x_1(t) < m$ holds for all $t \in [0, T]$ and all $u(\cdot) \in D(T)$.*

Proofs of Lemmas 1 and 2 are given in [3, 1]. These Lemmas allow us to assume that $x_1^{\max} = m$.

# 3 Optimal control problems

We are now ready to formulate optimal control problems for system (1).

The first problem, which we denote $(\widehat{P})$, is in a minimizing of a weighted sum of the pollutant concentration at the moment $T$ and the cumulative biomass over interval $[0, T]$:

$$\widehat{J}(u) = x_2(T) + \gamma \int_0^T x_3(t)dt \to \min_{u(\cdot)\in D(T)}.$$

The second problem, which is denoted $(\widetilde{P})$, is in a minimizing of a weighted sum of the pollutant concentration at the moment $T$ and the cumulative concentrations of the biomass and oxygen over interval $[0, T]$:

$$\widetilde{J}(u) = x_2(T) + \int_0^T \Big(\alpha x_1(t) + \beta x_3(t)\Big)dt \to \min_{u(\cdot)\in D(T)}.$$

Here $\alpha$, $\beta$ and $\gamma$ are given positive constants.

The existence of the optimal solutions for the optimal control problems $(\widehat{P})$ and $(\widetilde{P})$ follows from [7]. Let function $\widehat{u}_*(t)$ be the optimal control for problem $(\widehat{P})$, and $\widehat{x}_*(t) = (\widehat{x}^{\,*}_1(t), \widehat{x}^{\,*}_2(t), \widehat{x}^{\,*}_3(t))^\top$ be the corresponding optimal solution of system (1); respectively, let function $\widetilde{u}_*(t)$ be the optimal control for problem $(\widetilde{P})$, and $\widetilde{x}_*(t) = (\widetilde{x}^{\,*}_1(t), \widetilde{x}^{\,*}_2(t), \widetilde{x}^{\,*}_3(t))^\top$ be the corresponding optimal solution of the system (1). Here $^\top$ is transpose.

## 4 Pontryagin maximum principle

In order to analyze the optimal control problems $(\widehat{P})$ and $(\widetilde{P})$, we apply the Pontryagin Maximum Principle [8].

For problem $(\widehat{P})$, for the optimal control $\widehat{u}_*(t)$ and the corresponding optimal trajectory $\widehat{x}_*(t) = (\widehat{x}^{\,*}_1(t), \widehat{x}^{\,*}_2(t), \widehat{x}^{\,*}_3(t))^\top$ there is a non-trivial solution $\widehat{\psi}_*(t) = (\widehat{\psi}^{\,*}_1(t), \widehat{\psi}^{\,*}_2(t), \widehat{\psi}^{\,*}_3(t))^\top$ of the adjoint system

$$\begin{cases} \dot{\widehat{\psi}}{}^{\,*}_1(t) = \widehat{u}_*(t)\widehat{\psi}^{\,*}_1(t) + \widehat{x}^{\,*}_2(t)\widehat{x}^{\,*}_3(t)\Big(\widehat{\psi}^{\,*}_1(t) + \widehat{\psi}^{\,*}_2(t) - \widehat{\psi}^{\,*}_3(t)\Big), \\ \dot{\widehat{\psi}}{}^{\,*}_2(t) = \widehat{x}^{\,*}_1(t)\widehat{x}^{\,*}_3(t)\Big(\widehat{\psi}^{\,*}_1(t) + \widehat{\psi}^{\,*}_2(t) - \widehat{\psi}^{\,*}_3(t)\Big), \\ \dot{\widehat{\psi}}{}^{\,*}_3(t) = \widehat{x}^{\,*}_1(t)\widehat{x}^{\,*}_2(t)\Big(\widehat{\psi}^{\,*}_1(t) + \widehat{\psi}^{\,*}_2(t) - \widehat{\psi}^{\,*}_3(t)\Big) + d\widehat{\psi}^{\,*}_3(t) + \gamma, \\ \widehat{\psi}^{\,*}_1(T) = 0,\ \widehat{\psi}^{\,*}_2(T) = -1,\ \widehat{\psi}^{\,*}_3(T) = 0, \end{cases} \tag{2}$$

which satisfies the relationship

$$\widehat{u}_*(t) = \begin{cases} u_{\max} & ,\ \text{when}\quad \widehat{L}(t) > 0, \\ u \in [0, u_{\max}] & ,\ \text{when}\quad \widehat{L}(t) = 0, \\ 0 & ,\ \text{when}\quad \widehat{L}(t) < 0. \end{cases} \tag{3}$$

Here, $\widehat{L}(t) = \widehat{\psi}^{\,*}_1(t)$ is the switching function for problem $(\widehat{P})$; its definition takes into account inequality $m - \widehat{x}^{\,*}_1(t) > 0$ for $t \in [0, T]$, which follows from Lemma 2.

Likewise, for problem $(\widetilde{P})$, for the optimal control $\widetilde{u}_*(t)$ and the corresponding optimal trajectory $\widetilde{x}_*(t) = (\widetilde{x}^{\,*}_1(t), \widetilde{x}^{\,*}_2(t), \widetilde{x}^{\,*}_3(t))^\top$ there is a non-trivial solution $\widetilde{\psi}_*(t) = (\widetilde{\psi}^{\,*}_1(t), \widetilde{\psi}^{\,*}_2(t), \widetilde{\psi}^{\,*}_3(t))^\top$ of the adjoint system

$$\begin{cases} \dot{\widetilde{\psi}}^{\,*}_1(t) = \widetilde{u}_*(t)\widetilde{\psi}^{\,*}_1(t) + \\ \qquad + \widetilde{x}^{\,*}_2(t)\widetilde{x}^{\,*}_3(t)\Big(\widetilde{\psi}^{\,*}_1(t) + \widetilde{\psi}^{\,*}_2(t) - \widetilde{\psi}^{\,*}_3(t)\Big) + \alpha, \\ \dot{\widetilde{\psi}}^{\,*}_2(t) = \widetilde{x}^{\,*}_1(t)\widetilde{x}^{\,*}_3(t)\Big(\widetilde{\psi}^{\,*}_1(t) + \widetilde{\psi}^{\,*}_2(t) - \widetilde{\psi}^{\,*}_3(t)\Big), \\ \dot{\widetilde{\psi}}^{\,*}_3(t) = \widetilde{x}^{\,*}_1(t)\widetilde{x}^{\,*}_2(t)\Big(\widetilde{\psi}^{\,*}_1(t) + \widetilde{\psi}^{\,*}_2(t) - \widetilde{\psi}^{\,*}_3(t)\Big) + d\widetilde{\psi}^{\,*}_3(t) + \beta, \\ \widetilde{\psi}^{\,*}_1(T) = 0,\ \widetilde{\psi}^{\,*}_2(T) = -1,\ \widetilde{\psi}^{\,*}_3(T) = 0, \end{cases} \tag{4}$$

such that the relationship

$$\widetilde{u}_*(t) = \begin{cases} u_{\max} & \text{, when } \quad \widetilde{L}(t) > 0, \\ u \in [0, u_{\max}] & \text{, when } \quad \widetilde{L}(t) = 0, \\ 0 & \text{, when } \quad \widetilde{L}(t) < 0, \end{cases} \tag{5}$$

hold. Here $\widetilde{L}(t) = \widetilde{\psi}^{\,*}_1(t)$ is the switching function of problem $(\widetilde{P})$; this takes into account inequality $m - \widetilde{x}^{\,*}_1(t) > 0$ for $t \in [0, T]$, which follows from Lemma 2.

We have to define auxiliary functions:

$$\begin{aligned} &\widehat{G}(t) = \widehat{\psi}^{\,*}_1(t) + \widehat{\psi}^{\,*}_2(t) - \widehat{\psi}^{\,*}_3(t),\ \widehat{P}(t) = -\widehat{\psi}^{\,*}_3(t) - \gamma d^{-1}, \\ &\widehat{f}(t) = \widehat{x}^{\,*}_2(t)\widehat{x}^{\,*}_3(t) + \widehat{x}^{\,*}_3(t)\widehat{x}^{\,*}_1(t) - \widehat{x}^{\,*}_1(t)\widehat{x}^{\,*}_2(t),\ t \in [0, T] \end{aligned}$$

for problem $(\widehat{P})$, and similarly, the functions:

$$\begin{aligned} &\widetilde{G}(t) = \widetilde{\psi}^{\,*}_1(t) + \widetilde{\psi}^{\,*}_2(t) - \widetilde{\psi}^{\,*}_3(t) + \alpha\Big(\widetilde{x}^{\,*}_2(t)\widetilde{x}^{\,*}_3(t)\Big)^{-1}, \\ &\widetilde{P}(t) = -\widetilde{\psi}^{\,*}_3(t) + \alpha\Big(\widetilde{x}^{\,*}_2(t)\widetilde{x}^{\,*}_3(t)\Big)^{-1} - \beta d^{-1}, \\ &\widetilde{f}(t) = \widetilde{x}^{\,*}_2(t)\widetilde{x}^{\,*}_3(t) + \widetilde{x}^{\,*}_3(t)\widetilde{x}^{\,*}_1(t) - \widetilde{x}^{\,*}_1(t)\widetilde{x}^{\,*}_2(t),\ t \in [0, T] \end{aligned}$$

for problem $(\widetilde{P})$.

Now the adjoint system (2) for problem $(\widehat{P})$ can be reformulated as the following system of equations for the functions $\widehat{L}(t)$, $\widehat{G}(t)$, $\widehat{P}(t)$:

$$\begin{cases} \dot{\widehat{L}}(t) = \widehat{u}_*(t)\widehat{L}(t) + \widehat{x}_2^{\,*}(t)\widehat{x}_3^{\,*}(t)\widehat{G}(t), \ t \in [0,T], \\ \dot{\widehat{G}}(t) = \widehat{u}_*(t)\widehat{L}(t) + \widehat{f}(t)\widehat{G}(t) + d\widehat{P}(t), \\ \dot{\widehat{P}}(t) = -\widehat{x}_1^{\,*}(t)\widehat{x}_2^{\,*}(t)\widehat{G}(t) + d\widehat{P}(t), \\ \widehat{L}(T) = 0, \ \widehat{G}(T) = -1, \ \widehat{P}(T) = -\gamma d^{-1}. \end{cases} \tag{6}$$

Likewise, the adjoint system (4) for problem $(\widetilde{P})$ can be now formulated as the system of equations for the functions $\widetilde{L}(t)$, $\widetilde{G}(t)$, $\widetilde{P}(t)$:

$$\begin{cases} \dot{\widetilde{L}}(t) = \widetilde{u}_*(t)\widetilde{L}(t) + \widetilde{x}_2^{\,*}(t)\widetilde{x}_3^{\,*}(t)\widetilde{G}(t), \ t \in [0,T], \\ \dot{\widetilde{G}}(t) = \widetilde{u}_*(t)\widetilde{L}(t) + \widetilde{f}(t)\widetilde{G}(t) + d\widetilde{P}(t), \\ \dot{\widetilde{P}}(t) = -\widetilde{x}_1^{\,*}(t)\widetilde{x}_2^{\,*}(t)\widetilde{G}(t) + d\widetilde{P}(t) + \alpha\widetilde{x}_1^{\,*}(t)(\widetilde{x}_2^{\,*}(t))^{-1}, \\ \widetilde{L}(T) = 0, \ \widetilde{G}(T) = -1 + \alpha\Big(\widetilde{x}_2^{\,*}(T)\widetilde{x}_3^{\,*}(T)\Big)^{-1}, \\ \widetilde{P}(T) = \alpha\Big(\widetilde{x}_2^{\,*}(T)\widetilde{x}_3^{\,*}(T)\Big)^{-1} - \beta d^{-1}. \end{cases} \tag{7}$$

We use systems (6) and (7) for the analysis of the switching functions $\widehat{L}(t)$ and $\widetilde{L}(t)$, respectively.

# 5 Properties of the switching function for problem $(\widehat{P})$

Analysis of the Cauchy problem (6) leads to the following Lemmas, which describe properties of the switching function $\widehat{L}(t)$.

**Lemma 3** *The switching function $\widehat{L}(t)$ is not zero on any finite subinterval in $[0,T]$.*

**Remark 1.** By Lemma 3 and relationship (3), the optimal control $\widehat{u}_*(t)$ is a piecewise constant function, which takes values $\{0, u_{\max}\}$.

**Lemma 4** *The switching function $\widehat{L}(t)$ has at most two zeros on the interval $[0,T]$.*

**Remark 2.** The proof of this assertion is conducted by arguments, which are similar to the arguments presented in [3, 1].

**Lemma 5** *There exists the value* $\widehat{\theta} \in [0, T)$, *such that the inequality* $\widehat{L}(t) > 0$ *holds on the interval* $(\widehat{\theta}, T)$.

**Remark 3.** It follows from Lemma 5 and relationship (3), that $\widehat{u}_*(t) = u_{\max}$ on the interval $(\widehat{\theta}, T]$. This implies that for problem $(\widehat{P})$ the optimal control process must end with the maximum aeration rate.

## 6 Types of optimal control for problem $(\widehat{P})$

Lemma 4, Remarks 1 and 3, relationship (3) and the initial conditions of system (6) enable us to make conclusions about possible types of optimal control $\widehat{u}_*(t)$.

**Theorem 1** *For problem* $(\widehat{P})$, *the optimal control* $\widehat{u}_*(t)$ *can be*:
*either a constant function*

$$\widehat{u}_*(t) = u_{\max},\ t \in [0, T]; \tag{8}$$

*or a piecewise constant function with one switching of the type*

$$\widehat{u}_*(t) = \begin{cases} 0 & ,\ \textit{for} \quad 0 \leq t \leq \widehat{\theta}_*, \\ u_{\max} & ,\ \textit{for} \quad \widehat{\theta}_* < t \leq T, \end{cases} \tag{9}$$

*where* $\widehat{\theta}_* \in (0, T)$ *is the moment of switching.*

## 7 Solving problem $(\widehat{P})$

Problem $(\widehat{P})$ can be solved using the following procedure. For an arbitrary value $\theta \in [0, T]$ we define control $u_\theta(t)$ as

$$u_\theta(t) = \begin{cases} 0 & ,\ \text{for} \quad 0 \leq t \leq \theta, \\ u_{\max} & ,\ \text{for} \quad \theta < t \leq T. \end{cases} \tag{10}$$

It is easy to see that control $u_\theta(t)$ defined in this way includes all possible types (8), (9) of the optimal control $\widehat{u}_*(t)$.

System (1) can be integrated on interval $[0, T]$ with control $u_\theta(t)$. The functions $x_2^\theta(t)$ and $x_3^\theta(t)$, which correspond to this control, then should be substituted into functional $\widehat{J}(u)$. This yields a function

$$\widehat{F}(\theta) = \widehat{J}(u_\theta),\ \theta \in [0, T],$$

which is a function of a single variable $\theta \in [0, T]$, and hence problem $(\widehat{P})$ is now reduced to a constrained minimization problem

$$\widehat{F}(\theta) \to \min_{\theta \in [0,T]} . \tag{11}$$

Methods for numerical solution of such problem are well-known [10]. We provide and discuss some numerical results in Section 11.

# 8 Properties of the switching function for problem $(\widetilde{P})$

The analysis of the Cauchy problem (7) leads to the following conclusions about properties of the switching function $\widetilde{L}(t)$.

**Lemma 6** *The switching function $\widetilde{L}(t)$ is not zero on any finite subinterval in $[0, T]$.*

**Remark 4.** It is clear from Lemma 6 and relationship (5) that the optimal control $\widetilde{u}_*(t)$ is a piecewise constant function with values $\{0, u_{\max}\}$.

**Lemma 7** *The switching function $\widetilde{L}(t)$ has at most three zeros on the interval $[0, T]$.*

**Lemma 8** *For $\alpha > \widetilde{x}_2^*(T)\widetilde{x}_3^*(T)$ there exists the value $\widetilde{\theta} \in [0, T)$, such that the inequality $\widetilde{L}(t) > 0$ holds on the interval $(\widetilde{\theta}, T)$. For $\alpha < \widetilde{x}_2^*(T)\widetilde{x}_3^*(T)$ there exists the value $\widetilde{\theta} \in [0, T)$, such that the inequality $\widetilde{L}(t) < 0$ is valid on the interval $(\widetilde{\theta}, T)$.*

**Lemma 9** *If $\alpha = \widetilde{x}_2^*(T)\widetilde{x}_3^*(T)$, then the behavior of the function $\widetilde{L}(t)$ depends on the value of $(1 - \beta d^{-1})$. If $1 - \beta d^{-1} \leq 0$, then there exists the value $\widetilde{\theta} \in [0, T)$, such that the inequality $\widetilde{L}(t) < 0$ holds on the interval $(\widetilde{\theta}, T)$. If $1 - \beta d^{-1} > 0$, then there exists the value $\widetilde{\theta} \in [0, T)$, such that the inequality $\widetilde{L}(t) > 0$ is valid on the interval $(\widetilde{\theta}, T)$.*

**Remark 5.** By Lemmas 8 and 9 and relationship (5), there is an interval $(\widetilde{\theta}, T]$, such that, depending on values of $\alpha$ and $\beta$, the optimal control $\widetilde{u}_*(t)$ takes the value either 0, or $u_{\max}$. This implies, that, in contrast to problem $(\widehat{P})$ (see Remark 3), for problem $(\widetilde{P})$ the optimal process can end as at the maximum as at minimum aeration rate.

**Remark 6.** The proofs of Lemmas 6–9 are similar to the proofs of Lemmas 3–5.

## 9 Types of optimal control for problem $(\widetilde{P})$

Lemma 7, Remarks 4 and 5, and relationship (5), together with the initial conditions of system (7), allow us to make conclusion about types of the optimal control $\widetilde{u}_*(t)$, which are possible for problem $(\widetilde{P})$.

**Theorem 2** *For problem $(\widetilde{P})$, the optimal control $\widetilde{u}_*(t)$ can be: either a constant function of the type either*

$$\widetilde{u}_*(t) = 0,\ t \in [0,T], \tag{12}$$

*or*

$$\widetilde{u}_*(t) = u_{\max},\ t \in [0,T]; \tag{13}$$

*or a piecewise constant function with one switching of the type either*

$$\widetilde{u}_*(t) = \begin{cases} u_{\max} & ,\ \textit{for}\quad 0 \le t \le \widetilde{\theta}_*, \\ 0 & ,\ \textit{for}\quad \widetilde{\theta}_* < t \le T, \end{cases} \tag{14}$$

*or*

$$\widetilde{u}_*(t) = \begin{cases} 0 & ,\ \textit{for}\quad 0 \le t \le \widetilde{\theta}_*, \\ u_{\max} & ,\ \textit{for}\quad \widetilde{\theta}_* < t \le T, \end{cases} \tag{15}$$

*where $\widetilde{\theta}_* \in (0,T)$ is the moment of switching;*
*or a piecewise constant function with two switchings of the type either*

$$\widetilde{u}_*(t) = \begin{cases} 0 & ,\ \textit{for}\quad 0 \le t \le \widetilde{\theta}^{\,*}_1, \\ u_{\max} & ,\ \textit{for}\quad \widetilde{\theta}^{\,*}_1 < t \le \widetilde{\theta}^{\,*}_2, \\ 0 & ,\ \textit{for}\quad \widetilde{\theta}^{\,*}_2 < t \le T, \end{cases} \tag{16}$$

*or*

$$\widetilde{u}_*(t) = \begin{cases} u_{\max} & ,\ \textit{for}\quad 0 \le t \le \widetilde{\theta}^{\,*}_1, \\ 0 & ,\ \textit{for}\quad \widetilde{\theta}^{\,*}_1 < t \le \widetilde{\theta}^{\,*}_2, \\ u_{\max} & ,\ \textit{for}\quad \widetilde{\theta}^{\,*}_2 < t \le T, \end{cases} \tag{17}$$

*where $\widetilde{\theta}^{\,*}_1, \widetilde{\theta}^{\,*}_2 \in (0,T)$ are the moments of switching.*

## 10 Solving problem $(\widetilde{P})$

We now can describe a method of solving problem $(\widetilde{P})$. Let us define a set

$$\Lambda = \Big\{ \theta = (\theta_1, \theta_2, \theta_3) \in \mathrm{R}^3 : 0 \le \theta_1 \le \theta_2 \le \theta_3 \le T \Big\}.$$

For an arbitrary point $\theta \in \Lambda$ we can define a control $u_\theta(t)$ as

$$u_\theta(t) = \begin{cases} 0 & , \text{ for } \quad 0 \le t \le \theta_1, \\ u_{\max} & , \text{ for } \quad \theta_1 < t \le \theta_2, \\ 0 & , \text{ for } \quad \theta_2 < t \le \theta_3, \\ u_{\max} & , \text{ for } \quad \theta_3 < t \le T. \end{cases} \tag{18}$$

It is easy to see that, assuming that the corresponding values $\theta_1$, $\theta_2$ and $\theta_3$ are given, control $u_\theta(t)$ includes all possible types (12)–(17) of the optimal control $\widetilde{u}_*(t)$.

We now substitute control $u_\theta(t)$ into system (1) and integrate this system over interval $[0, T]$. This yields functions $x_1^\theta(t)$, $x_2^\theta(t)$ and $x_3^\theta(t)$, which correspond to this control, and which we then substitute into functional $\widetilde{J}(u)$. The result is a function of three variables

$$\widetilde{F}(\theta_1, \theta_2, \theta_3) = \widetilde{J}(u_\theta), \ (\theta_1, \theta_2, \theta_3) \in \Lambda.$$

Problem $(\widetilde{P})$ is now reduced to a problem of constrained minimizing

$$\widetilde{F}(\theta_1, \theta_2, \theta_3) \to \min_{(\theta_1, \theta_2, \theta_3) \in \Lambda}, \tag{19}$$

which can be solved by the same methods as the problem (11).

## 11 Numerical results

The following examples demonstrate the methods of solving problems $(\widehat{P})$, $(\widetilde{P})$, which were described above.

**Example 1. (E-1)** The initial conditions and parameters of system (1) are:

$$x_1^0 = 1.0000, \quad x_2^0 = 1.0000, \quad x_3^0 = 1.0000,$$
$$m = 2.0000, \quad d = 1.0000, \quad u_{\max} = 4.0000, \quad T = 1.0000.$$

**Example 2. (E-2)** The initial conditions and parameters of system (1) are:

$$x_1^0 = 0.0002, \quad x_2^0 = 30.0000, \quad x_3^0 = 0.0300,$$
$$m = 0.0050, \quad d = 0.2400, \quad u_{\max} = 4.0000, \quad T = 6.0000.$$

**Example 3. (E-3)** The initial conditions and parameters of system (1) are:

$$x_1^0 = 0.0019, \quad x_2^0 = 2.4980, \quad x_3^0 = 0.0874,$$
$$m = 0.0480, \quad d = 0.2400, \quad u_{\max} = 4.0000, \quad T = 12.0000.$$

Numerical solutions of problem (11) for Examples 1–3 are shown in Table 1. Here $\theta_*$ is the moment of switching of the piecewise constant optimal control $u^*_\theta(t) = u_*(t)$, defined by formula (10); $\widehat{J}_*$ is the minimum value of the functional $\widehat{J}(u)$ and simultaneously the minimum of the function $\widehat{F}(\theta)$ of the problem (11).

| | $\gamma = 0.005$ | $\gamma = 0.5$ | $\gamma = 2.5$ | $\gamma = 5.0$ | $\gamma = 10.0$ |
|---|---|---|---|---|---|
| **E-1** | $\theta_* = 0.000$ | $\theta_* = 0.000$ | $\theta_* = 0.413$ | $\theta_* = 0.663$ | $\theta_* = 0.818$ |
| | $\widehat{J}_* = 0.2242$ | $\widehat{J}_* = 0.7004$ | $\widehat{J}_* = 2.5572$ | $\widehat{J}_* = 4.6998$ | $\widehat{J}_* = 8.9063$ |
| **E-2** | $\theta_* = 0.000$ | $\theta_* = 2.600$ | $\theta_* = 5.382$ | $\theta_* = 5.658$ | $\theta_* = 5.814$ |
| | $\widehat{J}_* = 29.9849$ | $\widehat{J}_* = 30.0459$ | $\widehat{J}_* = 30.2395$ | $\widehat{J}_* = 30.4794$ | $\widehat{J}_* = 30.9591$ |
| **E-3** | $\theta_* = 0.000$ | $\theta_* = 8.592$ | $\theta_* = 11.352$ | $\theta_* = 11.640$ | $\theta_* = 11.808$ |
| | $\widehat{J}_* = 2.4424$ | $\widehat{J}_* = 2.6692$ | $\widehat{J}_* = 3.3639$ | $\widehat{J}_* = 4.2311$ | $\widehat{J}_* = 5.9654$ |

**Table 1:** Results of solving of problem $(\widehat{P})$ for Examples 1–3.

Numerical solutions of problem (19) for Examples 1–3 are shown in Table 2. Here $\theta^*_1$, $\theta^*_2$, $\theta^*_3$ are the moments of switching of the piecewise constant optimal control $u^*_\theta(t) = u_*(t)$, defined by formula (18); $\widetilde{J}_*$ is the minimum value of the functional $\widetilde{J}(u)$ and simultaneously the minimum of the function $\widetilde{F}(\theta_1, \theta_2, \theta_3)$ of the problem (19).

| | $\alpha = 0.001$ | $\alpha = 0.001$ | $\alpha = 0.5$ | $\alpha = 2.5$ | $\alpha = 5.0$ |
|---|---|---|---|---|---|
| | $\beta = 5.0$ | $\beta = 50.0$ | $\beta = 5.0$ | $\beta = 50.0$ | $\beta = 100.0$ |
| **E-1** | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ |
| | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ |
| | $\theta^*_3 = 0.660$ | $\theta^*_3 = 0.950$ | $\theta^*_3 = 1.000$ | $\theta^*_3 = 1.000$ | $\theta^*_3 = 1.000$ |
| | $\widetilde{J}_* = 4.6582$ | $\widetilde{J}_* = 42.2092$ | $\widetilde{J}_* = 43.9751$ | $\widetilde{J}_* = 87.4041$ | $\widetilde{J}_* = 87.4041$ |
| **E-2** | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ |
| | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ |
| | $\theta^*_3 = 5.640$ | $\theta^*_3 = 5.940$ | $\theta^*_3 = 6.000$ | $\theta^*_3 = 6.000$ | $\theta^*_3 = 6.000$ |
| | $\widetilde{J}_* = 30.4817$ | $\widetilde{J}_* = 34.8297$ | $\widetilde{J}_* = 30.4829$ | $\widetilde{J}_* = 34.8304$ | $\widetilde{J}_* = 39.6610$ |
| **E-3** | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ | $\theta^*_1 = 0.000$ |
| | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ | $\theta^*_2 = 0.000$ |
| | $\theta^*_3 = 11.520$ | $\theta^*_3 = 11.760$ | $\theta^*_3 = 11.760$ | $\theta^*_3 = 11.760$ | $\theta^*_3 = 11.760$ |
| | $\widetilde{J}_* = 4.1973$ | $\widetilde{J}_* = 19.5011$ | $\widetilde{J}_* = 4.2039$ | $\widetilde{J}_* = 19.5338$ | $\widetilde{J}_* = 36.5708$ |

**Table 2:** Results of solving of problem $(\widetilde{P})$ for Examples 1–3.

Numerical experiments for problem $(\widehat{P})$ showed that for Examples 1–3 the corresponding optimal control $\widehat{u}_*(t)$ is either constant function, or piecewise constant function with one switching of the types (8), (9). Analogical experiments for problem $(\widetilde{P})$ showed that for Examples 1–3 the corresponding optimal control $\widetilde{u}_*(t)$ is either constant function, or piecewise constant function with one switching of the types (12), (15).

In Figures 1-4 for Example 2 the graphs of the optimal controls $\widehat{u}_*(t)$, $\widetilde{u}_*(t)$ are presented for different values of constants $\gamma$ and $\alpha$, $\beta$, respectively.

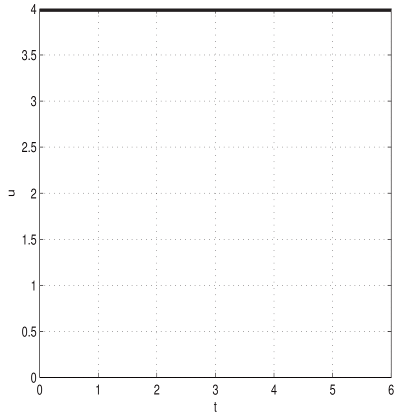


**Figure 1:** Optimal control $\widehat{u}_*(t)$ for $\gamma = 0.005$.

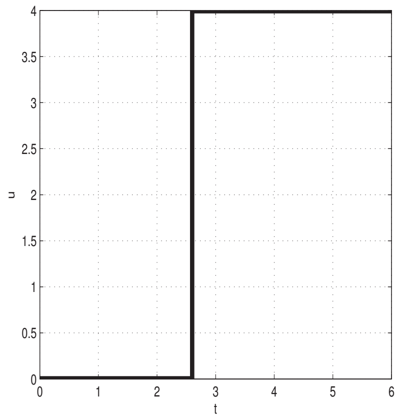


**Figure 2:** Optimal control $\widehat{u}_*(t)$ for $\gamma = 0.5$.

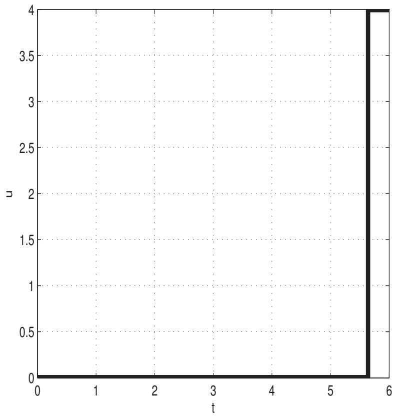


**Figure 3:** Optimal control $\widetilde{u}_*(t)$ for $\alpha = 0.001$, $\beta = 5.0$.

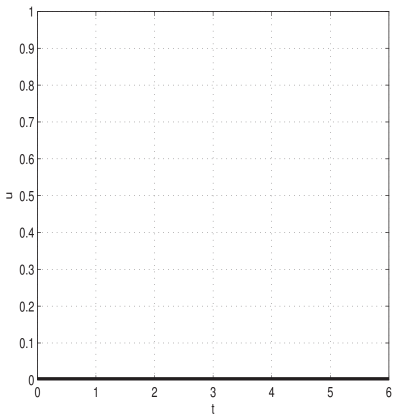


**Figure 4:** Optimal control $\widetilde{u}_*(t)$ for $\alpha = 0.5$, $\beta = 5.0$.

# 12 Conclusion

In this paper we consider two optimal control problems for a nonlinear three-dimensional system of ordinary differential equation that model the process of wastewater biotreatment. The objective functionals define either the weighted sum of the concentration of pollutant at the end of the time interval and the cumulative biomass throughout the interval, or the weighted sum of the pollutant at the end of the time interval and the cumulative biomass and oxygen throughout the interval. The Pontryagin Maximum Principle enables us to establish the structure of the optimal controls for these problems and to simplify the problems to a finite-dimensional constrained minimization. Results of numerical experiments demonstrate the optimal aeration rates at different parameters of the model.

## References

[1] Bondarenko, N.V.; Grigorieva, E.V.; Khailov, E.N. (2010) "Attainable set of three-dimensional nonlinear system describing the wastewater treatment process", in: Yu.S. Osipov & A.V. Kryazhimskii (Eds.) *Problems of Dynamical Control*, **5**, MAX Press, Moscow: 28–41.

[2] Gomez, J.; de Gracia, M.; Ayesa, E.; Garcia-Heras, J.L. (2007) "Mathematical modelling of autothermal thermophilic aerobic digesters", *Water Research* **41**(5): 959–968.

[3] Grigorieva, E.; Bondarenko, N.; Khailov, E.; Korobeinikov, A. (2012) "Finite-dimensional methods for optimal control of autothermal thermophilic aerobic digestion", in: K.Y. Show & X. Guo (Eds.) *Industrial Waste*, InTech, Croatia: 91–120.

[4] Grigorieva, E.V.; Bondarenko, N.V.; Khailov, E.N.; Korobeinikov, A. (2012) "Three-dimensional nonlinear control model of wastewater biotreatment", *Neural, Parallel, and Scientific Computations* **20**: 23–36.

[5] Grigorieva, E.V.; Khailov, E.N.; Korobeinikov, A. (2012) "Reduction of the operation cost via optimal control of an industrial wastewater biotreatment process", in: `http://jointmathematicsmeetings.org/amsmtgs/2138_abstracts/1077-g5-1378.pdf`.

[6] Krasnov, K.S.; Vorob'ev, N.K.; Godnev, I.N.; et al. (1995) *Physical Chemestry 2*. Vysshaya Shkola, Moscow.

[7] Lee, E.B.; Marcus, L. (1967) *Foundations of Optimal Control Theory*. John Wiley & Sons, New York.

[8] Pontryagin, L.S.; Boltyanskii, V.G.; Gamkrelidze, R.V.; Mishchenko, E.F. (1962) *Mathematical Theory of Optimal Processes.* John Wiley & Sons, New York.

[9] Rojas J.; Burke, M.; Chapwanya, M.; Doherty, K.; Hewitt, I.; Korobeinikov, A.; Meere, M.; McCarthy, S.; O'Brien, M.; Tuoi, V.T.N.; Winstenley, H.; Zhelev, T. (2010) "Modeling of autothermal thermophilic aerobic digestion", *Mathematics-in-Industry Case Studies* **2**: 34–63.

[10] Vasil'ev, F.P. (2002) *Optimization Methods.* Factorial Press, Moscow.