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LENSING PROPERTIES OF THE EINASTO PROFILE IN TERMS OF THE MEIJER G FUNCTION

PROPIEDADES DEL EFECTO LENTE PARA EL PERFIL DE EINASTO EN TÉRMINOS DE LA FUNCIÓN G DE MEIJER

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Abstract

In N-body simulations of cold dark matter, it has been found that three-parameter models, particularly the Einasto profile, yield better fits to a wide range of dark matter haloes than two parameter models like the Navarro-Frenk-White profile. Recently, the analytical properties of the Einasto profile has been studied, allowing closed expressions for its surface mass density and lensing properties in terms of the Fox H and Meijer G functions, using a Mellin transform formalism. These expressions are valid for all values of the Einasto index in terms of the Fox H function, and valid for integer and half-integer values of Einasto index in terms of the Meijer G function. In this paper, we derive expressions for lensing properties of the Einasto profile for all rational values of the Einasto index in terms of the Meijer G function. Equivalency between these expressions and other recent results is also discussed

Keywords: cosmology; dark matter; Meijer G function.

Resumen

En simulaciones de N-cuerpos de materia oscura fría, se ha encontrado que modelos de tres parámetros, particularmente el perfil de Einasto, ofrece mejores ajustes para un amplio rango de halos de materia oscura que los modelos de dos parámetros como el perfil Navarro-Frenk-White. Recientemente, las propiedades analíticas del perfil de Einasto han sido estudiadas, lográdose expresiones cerradas para su densidad de masa superficial y propiedades de lente gravitational en términos de la función H de Fox, usando el formalismo de la transformada de Mellin. Estas expresiones son válidas para todos los valores del índice de Einasto en términos de la función H de Fox, y válidos para valores enteros y semi-enteros del índice de Einasto en términos de la función G de Meijer. En este artículo, se determinan expresiones para las propiedades de lente gravitational del perfil de Einasto para todos los valores racionales del índice de Einasto en términos de la función G de Meijer. La equivalencia entre estas expresiones y otros resultados recientes también es discutida.

Palabras clave: cosmología; materia oscura; función G de Meijer.

Mathematics Subject Classification: 33C60; 33E20.

1 Introduction

The standard theory of cosmological structure formation is the Λ cold dark matter (ΛCMD) model, which had sucessfully provided crucial information about the structure of the Universe at large scales [23]. However, the observations of low surface brightness (LSB) and dwarf galaxies indicate that the density profile at small radii is characterised by a central core. On the other side, numerical simulations predict density profiles with a central cusp (region with too high dark matter density) at small radii. This disagreement between observations and simulations is the so-called cusp-core problem (see [8] for a full-review).

Gravitational lensing, a phenomena predicted by Einstein's general relativity theory that consists in the bending of light by a massive object, is presently playing an important role in astrophysics, in probing the matter distribution of objects like galaxies and galaxy clusters [22, 17, 20]. Since lensing is sensitive at small radii, it could provide a piece in finding a solution to the cusp-core problem.

Results from numerical simulations [4] favoured non-singular three-parameters models over singular two-parameter models, such as the Navarro-Frenk-White profile [18, 19]. One of these favoured models is the Einasto profile [11]:

$$\rho(r) = \rho_{\mathrm{s}} \exp\left\{-d_n\left[\left(\frac{r}{\mathrm{r_s}}\right)^{1/n} - 1\right]\right\}, \tag{1}$$

where r is the spatial radius, n is the shape parameter n which we call the Einasto index, and according to N-body simulations is in the range $5 \lesssim n \lesssim 8$, $\mathrm{r_s}$ represents the radius of the sphere that contains half of the total mass, ρ_{s} is the mass density at $r = \mathrm{r_s}$, and d_n is a numerical constant that ensures that $\mathrm{r_s}$ is indeed the half-mass radius. Defining the central density $\rho_0 = \rho_s\, e^{d_n}$ and length scale $h = \mathrm{r_s}/d_n^n$, we express the density profile compactly as

$$\rho(r) = \rho_0 \exp\left[-\left(\frac{r}{h}\right)^{1/n}\right]. \tag{2}$$

The use of the Einasto profile has rapidly increased during the last years (see e.g., [7, 9, 16, 10]). Therefore, a detailed investigation of the properties of Einasto model is relevant. A complete study of the properties of the Einasto model has been done by Cardone et al. [6]. They provide analytical expressions for quantities such as the mass profile and gravitational potential, and discuss the dynamical structure for both isotropic and anisotropic cases.

A complete general set of analytical formulae for lensing properties of the Einasto profile valid for general values of the Einasto index has been provided by Retana-Montenegro et al. [21] (hereafter RM12). They extended the analytical study of the Einasto model using some of the techniques employed for studying the analytical properties of the Sérsic model [2, 3]. To find these properties, the Mellin transform-method was used, and expressions in terms of the Fox H function were obtained. These analytical expressions will help to increase the application of the Einasto profile in cosmological studies and maybe, it could help to find a solution to the cusp-core problem.

In this paper, we present a set of analytical expressions valid for all rational values of the Einasto index in terms of the Meijer G function, that are complementary to the results of RM12. In section 2 we briefly describe the Mellin transform formalism. In section 3 we derive expressions for the surface mass density, cumulative mass, deflection angle, and deflection potential. Finally, discussion and conclusions of our results can be seen in section 4.

2 Mellin transform technique

The Mellin transform technique [14, 1, 12] consists in that one-dimensional definite integrals

$$f(z) = \int_0^\infty g(t, z)\,\mathrm{d}t, \tag{3}$$

can expressed as the Mellin convolution

$$f(z) \equiv \int_0^\infty f_1(t)\, f_2\left(\frac{z}{t}\right) \frac{\mathrm{d}t}{t}, \tag{4}$$

of the functions f_1 and f_2. The Mellin convolution theorem, which states that the Mellin transform of a Mellin convolution of two functions is the pointwise product of their Mellin transforms, can be applied to eq. (4) and inverting the Mellin transform of the Mellin convolution, $f\,(z)$ can be expressed as the inverse Mellin transform of the pointwise product of the f_1 and f_2 Mellin transforms. The Mellin transform is defined by

$$\mathfrak{M}_f(u) = \phi(u) = \int_0^\infty f(z)\, z^{u-1}\,\mathrm{d}z, \tag{5}$$

and the inverse Mellin transform by

$$\mathfrak{M}_\phi^{-1}(z) = f(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \phi(u)\, z^{-u}\,\mathrm{d}u, \tag{6}$$

where the integration path is a vertical line in the complex plane.

Then, the integral (3) may be written as

$$f(z) = \frac{1}{2\pi i}\int_{\mathcal{L}} \mathfrak{M}_{f_1}(u)\, \mathfrak{M}_{f_2}(u)\, z^{-u}\,\mathrm{d}u. \tag{7}$$

With the requirement that f_1 and f_2 are of hypergeometric type, their Mellin transforms can be written as products of $\Gamma\,(a + Au)$ or $[\Gamma\,(a + Au)]^{-1}$, with

$\Gamma(v)$ the gamma fuction, and A being real. The resulting integral (7) is of the Mellin-Barnes, type and it can be then compared and written as a Fox H function when $A \neq 1$ or as a Meijer G function when $A = 1$.

The Meijer G function [15] is defined by

$$G^{m,n}_{p,q}\left[\begin{array}{c}\mathbf{a}\\ \mathbf{b}\end{array}\middle|\, z\right] \equiv$$
$$\frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{j=1}^{m}\Gamma(b_j+s)\prod_{j=1}^{n}\Gamma(1-a_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j-s)\prod_{j=n+1}^{p}\Gamma(a_j+s)}\, z^{-s}\,\mathrm{d}s. \tag{8}$$

The definition for the Fox H function can be found in [13].

3 Lensing properties

The first step to study the lensing properties of a model is to calculate the deprojection in the lens plane, which corresponds to the surface mass density $\Sigma(x)$. All the other lensing properties for a specific model can be found by simple integration of this quantity.

For a spherically symmetric lens, the surface mass density can be written as an Abel transform [5]

$$\Sigma(\xi) \equiv 2\int_{\xi}^{\infty} \frac{\rho(r)\; r\,\mathrm{d}r}{\sqrt{r^2-\xi^2}}, \tag{9}$$

where r is the spatial radius, and ξ is the distance measured from the lens centre.

Applying the Mellin transform technique to eq. (9), we find [21]

$$\Sigma(x) = 2n\sqrt{\pi}\,\rho_0\, h\, x\,\frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\Gamma(2ny)\,\Gamma\left(-\frac{1}{2}+y\right)}{\Gamma(y)}\left[x^2\right]^{-y}\mathrm{d}y, \tag{10}$$

with $x = \xi/h$.

Using the fact that $n = p/q$ is a irrational number when p and q are integer numbers, we obtain

$$\Sigma(x) = \sqrt{\pi}\,\rho_0\, h\, x\,\frac{1}{2\pi i}\int_{\mathcal{L}} \frac{q\,\Gamma(q+2p\,y)\,\Gamma\left(-\frac{1}{2}+q\,y\right)}{\Gamma(1+q\,y)}\left[x^{2q}\right]^{-y}\mathrm{d}y. \tag{11}$$

Substituting the three gamma functions in eq. (11) and using the Gauss multiplication formula

$$\Gamma(2ny) = (2n)^{-\frac{1}{2}+2ny}\,(2\pi)^{\frac{1}{2}-n}\,\Gamma(y)\prod_{j=1}^{2n-1}\Gamma\left(\frac{j}{2n}+y\right), \tag{12}$$

and then comparing that final result with the definition of the Meijer G function we find

$$\Sigma\,(x) = \sqrt{\frac{p}{q}}\frac{\rho_0\, h}{(2\pi)^{p-1}}\, x\, G^{2p+q-1,0}_{q-1,2p+q-1}\left[\begin{array}{c|c}\mathbf{a} & \dfrac{x^{2q}}{(2p)^{2p}}\\ \mathbf{b} & \end{array}\right], \tag{13}$$

where $\mathbf{a}$ is the vector of $q-1$ components given by

$$\mathbf{a} = \left\{\frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q}\right\}, \tag{14}$$

and $\mathbf{b}$ is the vector of size $2p+q-1$ given by

$$\mathbf{b} = \left\{\frac{1}{2p}, \frac{2}{2p}, \ldots, \frac{2p-1}{2p}, -\frac{1}{2q}, \frac{1}{2q}, \frac{3}{2q}, \ldots, \frac{2q-3}{2q}\right\}. \tag{15}$$

The cumulative surface mass can be simply found by integrating the surface mass density

$$M\,(\xi) \equiv 2\pi \int_0^{\xi} \Sigma(\xi')\, \xi'\, \mathrm{d}\xi'. \tag{16}$$

Inserting eq. (13) into eq. (16) and performing the integration (eq. 07.34.21.0003.01 at the Wolfram Functions Site[1] (WFS)), we obtain

$$M\,(x) = \sqrt{\frac{p}{q}}\frac{\rho_0\, h^3}{2q\,(2\pi)^{p-2}}\, x^3\, G^{2p+q-1,1}_{q,2p+q}\left[\begin{array}{c|c}1-\frac{3}{2q}, \mathbf{a} & \dfrac{x^{2q}}{(2p)^{2p}}\\ \mathbf{b}, -\frac{3}{2q} & \end{array}\right]. \tag{17}$$

Light rays coming from a distant source object can be deflected when passing close to a lens or deflector. Assuming a thin and circulary symmetric lens, the equation that describes the trayectory of the light rays passing a gravitational lens system is called the reduced lens equation [22]

$$y = x - \alpha\,(x)\,, \tag{18}$$

where $\alpha\,(x)$ is the deflection angle.

This deflection angle, is also an integral of the surface mass density

$$\alpha(x) \equiv \frac{2}{x}\int_0^x x'\, \frac{\Sigma(x')}{\Sigma_{\rm crit}}\, \mathrm{d}x', \tag{19}$$

with $\Sigma_{\rm crit}$ the critical surface mass density (see [22]).

[1] http://functions.wolfram.com/HypergeometricFunctions/MeijerG/

Combining eqs. (13) and (19), we get

$$\begin{aligned} \alpha\left(x\right) &= \frac{\kappa_{\mathrm{c}}}{2q\ \left(2\pi\right)^{p-1}\sqrt{\frac{p}{q}}\Gamma\left(\frac{p}{q}\right)}\, x^2 \times \\ &\times G_{q,2p+q}^{2p+q-1,1}\left[\begin{array}{c} 1-\frac{3}{2q}, \mathbf{a} \\ \mathbf{b}, -\frac{3}{2q} \end{array} \middle| \frac{x^{2q}}{\left(2p\right)^{2p}}\right], \end{aligned} \tag{20}$$

where we defined the central convergence as

$$\kappa_{\mathrm{c}} \equiv \frac{\Sigma\left(0\right)}{\Sigma_{\mathrm{crit}}} = \frac{2\,\rho_0\, h\, n\, \Gamma\left(n\right)}{\Sigma_{\mathrm{crit}}}. \tag{21}$$

The deflection potential for a circulary symmetric lens is [22]

$$\psi(x) \equiv 2\int_0^x x'\, \frac{\Sigma(x')}{\Sigma_{\mathrm{crit}}}\, \ln\left(\frac{x}{x'}\right)\, \mathrm{d}x'. \tag{22}$$

We can easily calculate this deflection potential using eqs. (13) and (22), to get

$$\begin{aligned} \psi\left(x\right) &= \frac{\kappa_{\mathrm{c}}}{4q^2\ \left(2\pi\right)^{p-1}\sqrt{\frac{p}{q}}\Gamma\left(\frac{p}{q}\right)}\, x^3 \times \\ &\times G_{q,2p+q}^{2p+q-1,1}\left[\begin{array}{c} 1-\frac{3}{2q}, 1-\frac{3}{2q}, \mathbf{a} \\ \mathbf{b}, -\frac{3}{2q}, -\frac{3}{2q} \end{array} \middle| \frac{x^{2q}}{\left(2p\right)^{2p}}\right]. \end{aligned} \tag{23}$$

The equivalency between eqs. (13), (17), (20), and (23) and the results of RM12 for integer values of n can be checked using eqs. 07.34.17.0007.01 and 07.34.17.0013.01 at WFS. Also, the case for half-integer values of n can be also proved.

4 Conclusions

We derived analytical expressions for surface mass density $\Sigma\left(x\right)$, cumulative surface mass $M\left(x\right)$, deflection angle $\alpha\left(x\right)$, deflection potential $\psi\left(x\right)$ in terms of the Meijer G function, for all values of the Einasto index n. These expressions are complementary to the results of RM12, and permit to compute with arbitrary precision the lensing properties of the Einasto profile. Recently, Novak et al. [24] used a numerical adaptation for the deprojected Sérsic model properties written in terms of the Meijer G function [2] to study homology of galaxies for a sample of simulated galaxy merger remnants. This work proves the usefulness of analytical expressions expressed as Meijer G functions in calculating the properties of density profiles.

It is to be noted, that the results presented here require a fast numerical implementation of the Meijer G function, because the parameter vectors tend to be large in the relevant range for n, increasing the time required to compute the values of the function. Current software with numerical implementations of the Meijer G function includes `Maple`, `Mathematica`, `Sage` and `mpmath`.

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