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RELATIONS OF K-TH DERIVATIVE OF DIRAC DELTA IN HYPERCONE WITH ULTRAHYPERBOLIC OPERATOR

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Abstract

In this paper we prove that the generalized functions $\delta^{(k)}(P_+)-\delta^{(k)}(P)$, $\delta^{(k)}(P_-)-\delta^{(k)}(-P)$ and $\delta_1^{(k)}(P)-\delta_2^{(k)}(P)$ are concentrated in the vertex of the cone $P=0$ and we find their relationship with the ultrahyperbolic operator iterated $(k+1-\frac{n}{2})$ times under condition $k \geq \frac{n}{2}-1$.

Keywords: distributions, generalized functions, distributions spaces, properties of distributions.

Resumen

En este trabajo se prueba que las funciones generalizadas $\delta^{(k)}(P_+)-\delta^{(k)}(P)$, $\delta^{(k)}(P_-)-\delta^{(k)}(-P)$ y$\delta_1^{(k)}(P)-\delta_2^{(k)}(P)$ están concentradas en el vértice del cono $P=0$ y encontramos sus relaciones con el operador ultrahiperbólico iterado $(k+1-\frac{n}{2})$ veces bajo la condición $k \geq \frac{n}{2}-1$.

Palabras clave: distribuciones, functiones generalizadas, espacios de distribuciones, propiedades de distribuciones.

AMS Subject Classification: 46F.

1 Introduction

Let $x=(x_1,x_2,\cdots,x_n)$ be a point of the n-dimensional Euclidean space R^n.

Consider a quadratic form in n variables defined by

$$P=P(x)=x_1^2+\ldots+x_p^2-x_{p+1}^2-\ldots-x_{p+q}^2 \tag{1}$$

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where $p+q=n$ is the dimension of the space.

We call $\varphi(x)$ the C$^{\infty}$ functions with compact support defined from $\mathbb{R}^n$ to $\mathbb{R}$ ([2],page 4).

From [1], page 253, formula (2), the distribution P_+^{λ} is defined by

$$\left(P_+^{\lambda},\varphi\right)=\int_{P>0}(P(x))^{\lambda}\varphi(x)dx \tag{2}$$

where λ is a complex number and $dx=dx_1dx_2\dots dx_n$. For Real$(\lambda)\geq 0$, this integral converges and is analytic function of λ. Analytic continuation to Real$(\lambda)<0$ can be used to extend the definition of $\left(P_+^{\lambda},\varphi\right)$. Further from [1], page 254, we have,

$$\left(P_+^{\lambda},\varphi\right)=\int_0^{\infty}u_q^{\lambda+\frac{p+q}{2}-1}\Phi_{\lambda}(u)du \tag{3}$$

where

$$\begin{aligned}
{}_q\Phi_{\lambda}(u) &= \frac{1}{4}\int_0^{\infty}t^{\frac{q-2}{2}}(1-t)^{\lambda}\phi_1(u,tu)dt && (4)\\
\phi(r,s) &= \phi_1(u,v) && (5)\\
\phi(r,s) &= \int\varphi d\Omega_p d\Omega_q, && (6)\\
r &= \sqrt[2]{x_1^2+\cdots+x_p^2}\ , && (7)\\
s &= \sqrt[2]{x_{p+1}^2+\dots x_{p+q}^2}\ , && (8)
\end{aligned}$$

$d\Omega_p$ and $d\Omega_q$ are elements of surface are on the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively.

Similarly we can also defined the generalized P$_-^{\lambda}$ by

$$\left(P_-^{\lambda},\varphi\right)=\int_{-P>0}(-P(x))^{\lambda}\varphi(x)dx. \tag{9}$$

Further we obtain

$$\left(P_-^{\lambda},\varphi\right)=\int_0^{\infty}v_p^{\lambda+\frac{p+q}{2}-1}\Phi_{\lambda}(v)dv \tag{10}$$

where

$${}_p\Phi_{\lambda}(u)=\frac{1}{4}\int_0^{\infty}t^{\frac{p-2}{2}}(1-t)^{\lambda}\phi_1(vt,v)dt. \tag{11}$$

From (1) the $P=0$ hypersurface is a hypercone with a singular point (the vertex) at the origin.

On the other hand, from [1], page 249, we have,

$$\left(\delta^{(k)}(P),\varphi\right)=\int_0^{\infty}\left[\left(\frac{\partial}{2s\partial s}\right)^k\left\{s^{q-2}\frac{\phi(r,s)}{2}\right\}\right]_{s=r}r^{p-1}dr \tag{12}$$

and

$$\left(\delta^{(k)}(P),\varphi\right) = (-1)^k \int_0^\infty \left[\left(\frac{\partial}{2r\partial r}\right)^k \left\{r^{p-2}\frac{\phi(r,s)}{2}\right\}\right]_{r=s} s^{q-1}ds \tag{13}$$

where $\phi(r,s)$ is defined by the equation (6).

Also from [1], page 250, the generalized functions $\delta_1^{(k)}(P)$ and $\delta_2^{(k)}(P)$ are defined by

$$\left(\delta_1^{(k)}(P),\varphi\right) = \int_0^\infty \left[\left(\frac{\partial}{2s\partial s}\right)^k \left\{s^{q-2}\frac{\phi(r,s)}{2}\right\}\right]_{s=r} r^{p-1}dr \tag{14}$$

and

$$\left(\delta_2^{(k)}(P),\varphi\right) = (-1)^k \int_0^\infty \left[\left(\frac{\partial}{2r\partial r}\right)^k \left\{r^{p-2}\frac{\phi(r,s)}{2}\right\}\right]_{r=s} s^{q-1}ds \tag{15}$$

where $\phi(r,s)$ is $r^{1-p}s^{1-q}$ multiplied by the integral of φ over the surface $x_1^2+x_2^2+\cdots x_p^2 = r^2$ and $x_{p+1}^2 + x_{p+2}^2 + \cdots x_{p+q}^2 = s^2$.

The integrals converges and coincide for

$$k < \frac{p+q-2}{2}. \tag{16}$$

If, on the other hand,

$$k \geq \frac{p+q-2}{2} \tag{17}$$

these integrals must be understood in the sen se of their regularization (see [1], page 250).

Now in general $\delta_1^{(k)}(P)$ and $\delta_2^{(k)}(P)$ may not be the same generalized function.

Note that the definition of these generalized functions implies that in any case

$$\delta_2^{(k)}(P) = (-1)^k \delta_1^{(k)}(-P). \tag{18}$$

From [1], page 278, the following formulae are valid,

$$\delta^{(k)}(P_+) = (-1)^k k! \mathcal{R}\rceil s_{\lambda=-k-1} P_+^\lambda \tag{19}$$

and

$$\delta^{(k)}(P_-) = (-1)^k k! \mathcal{R}\rceil s_{\lambda=-k-1} P_-^\lambda\ . \tag{20}$$

On the other hand, from [1], page 278, for odd n, as well as for even n and $k < \frac{n}{2} - 1$ we have,

$$\delta^{(k)}(P_+) = \delta_1^{(k)}(P) = \delta^{(k)}(P) \tag{21}$$

and

$$\delta^{(k)}(P_-) = \delta_1^{(k)}(-P). \tag{22}$$

While in the case of even dimension and $k \geq \frac{n}{2} - 1$

$$\delta^{(k)}(P_+) - \delta_1^{(k)}(P) \tag{23}$$

and

$$\delta^{(k)}(P_-)-\delta_1^{(k)}(-P) \tag{24}$$

are generalized functions concentrated at the vertex of the $P=0$ cone ([1],page 279).

From [1], page 279 we have:

If p and q are both even and if $k \geq \frac{n}{2}-1$, then

$$(-1)^k\delta^{(k)}(P_+)-\delta^{(k)}(P_-)=a_{q,n,k}L^{k+1-\frac{n}{2}}\left\{\delta(x)\right\} \tag{25}$$

while in all other cases

$$\delta^{(k)}(P_-)=(-1)^k\delta^{(k)}(P_+)\ . \tag{26}$$

In (25)

$$a_{q,n,k}=\frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!} \tag{27}$$

and L^j is a linear homogeneous diferential operation iterated j times defined by the following formula

$$L^j=\left\{\frac{\partial^2}{\partial x_1^2}+\dots\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\dots-\frac{\partial^2}{\partial x_{p+q}^2}\right\}^j\ . \tag{28}$$

The operator $L=\left\{\frac{\partial^2}{\partial x_1^2}+\dots\frac{\partial^2}{\partial x_p^2}-\frac{\partial^2}{\partial x_{p+1}^2}-\dots-\frac{\partial^2}{\partial x_{p+q}^2}\right\}$ is often called ultahyperbolic.

From [1], page 255, $\left(P_+^\lambda,\varphi\right)$ has two sets of singularities namely

$$\lambda=-1,-2,-3,\dots \tag{29}$$

and

$$\lambda=-\frac{n}{2},-\frac{n}{2}-1,\dots \tag{30}$$

and from [1], pages 256-269 and page 352 we have ([4], page 139, formula (2.27)):

$$\mathcal{R}\rceil s_{\lambda=-k-1}P_+^\lambda = \frac{(-1)^k}{k!}\delta_1^{(k)}(P) \text{ if } p \text{ is even and } q \text{ odd,} \tag{31}$$

$$\mathcal{R}\rceil s_{\lambda=-k-1}P_+^\lambda = \frac{(-1)^k}{k!}\delta_1^{(k)}(P) \text{ if } p \text{ is odd and } q \text{ even,} \tag{32}$$

$$\mathcal{R}\rceil s_{\lambda=-\frac{n}{2}-k}P_+^\lambda = 0 \text{ if } p \text{ is even and } q \text{ odd} \tag{33}$$

and

$$\mathcal{R}\rceil s_{\lambda=-\frac{n}{2}-k}P_+^\lambda=\frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^k k!\Gamma(\frac{n}{2}+k)}L^k\left\{\delta(x\right\} \text{ if } p \text{ is odd and } q \text{ even.} \tag{34}$$

where L^k is defined by the formula (28).

Similarly $\left(P_-^\lambda,\varphi\right)$ has singularities in the same points that $\left(P_+^\lambda,\varphi\right)$ and taking into account all that we have above about P_+^λ remains true also for P_-^λ except that p and q must interchanged, and in all the formulae $\delta_1^{(k)}(P)$ must be peplaced by

$$\delta_1^{(k)}(-P)=(-1)^k\delta_2^{(k)}(P) \tag{35}$$

and (L) by $(-L)$ (see ([1]), pages 279 and 352) we have,

$$\mathcal{R}]s_{\lambda=-k-1}P_{-}^{\lambda} = \frac{(-1)^k}{k!}\delta_1^{(k)}(-P) \text{ if } p \text{ is odd and } q \text{ even,} \quad (36)$$

$$\mathcal{R}]s_{\lambda=-k-1}P_{-}^{\lambda} = \frac{(-1)^k}{k!}\delta_1^{(k)}(-P) \text{ if } p \text{ is even and } q \text{ odd,} \quad (37)$$

$$\mathcal{R}]s_{\lambda=-\frac{n}{2}-k}P_{-}^{\lambda} = 0 \text{ if } p \text{ is odd and } q \text{ even} \quad (38)$$

and

$$\mathcal{R}]s_{\lambda=-\frac{n}{2}-k}P_{-}^{\lambda} = \frac{(-1)^{\frac{p}{2}}\pi^{\frac{n}{2}}}{4^k k!\Gamma(\frac{n}{2}+k)}(-L)^k\left\{\delta(x\right\} \text{ if } p \text{ is even and } q \text{ odd.} \quad (39)$$

If the dimension n of the space is even and p and q are even, P_{+}^{λ} has simple poles at $\lambda = -\frac{n}{2} - k$,where k is non-negative integer, and the residues are given by ([1], p.268 and [4], p.141)

$$\mathcal{R}]s_{\lambda=-\frac{n}{2}-k,k=0,1,2,..}P_{+}^{\lambda} = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}\delta_1^{(\frac{n}{2}+k-1)}(P) + \quad (40)$$

$$+\frac{(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^k k!\Gamma(\frac{n}{2}+k)}L^k\left\{\delta(x\right\}, \quad (41)$$

where L^k is defined by (28).

If, on the other hand, p and q are odd, P_{+}^{λ} has pole of order 2 at $\lambda = -\frac{n}{2} - k$ and from [1], p.269 and [4], p.143, we have

$$\mathcal{R}]s_{\lambda=-\frac{n}{2}-k}P_{+}^{\lambda} = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}\delta_1^{(\frac{n}{2}+k-1)}(P) + +\frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}\left[\psi(\frac{p}{2}) - \psi(\frac{n}{2})\right] \cdot L^k\left\{\delta(x)\right\}, \quad (42)$$

where

$$\psi(x) = \frac{\Gamma^{'}(x)}{\Gamma(x)}. \quad (43)$$

and $\Gamma(x)$ is the function gamma defined by

$$\Gamma(x) = \int_0^{\infty} e^{-z}z^{x-1}dz. \quad (44)$$

([3], Vol.I, p.344).

For integral and half-integral values of the argument, $\psi(x)$ is given by

$$\psi(k) = -\gamma + 1 + \frac{1}{2} + \cdots + \frac{1}{k-1}, \quad (45)$$

$$\psi(k+\frac{1}{2}) = -\gamma - 2\ln(2) + 2\left(1 + \frac{1}{3} + \cdots + \frac{1}{2k-1}\right), \quad (46)$$

where γ is Euler's constant.

Similarly

$$\mathcal{R}\rceil s_{\lambda=-\frac{n}{2}-k}P_-^\lambda = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}\delta_1^{(\frac{n}{2}+k-1)}(-P) + +\frac{(-1)^{\frac{p}{2}}\pi^{\frac{n}{2}}}{4^k k!\Gamma(\frac{n}{2}+k)}(-L)^k\{\delta(x\} \tag{47}$$

if p and q are even, and

$$\mathcal{R}\rceil s_{\lambda=-\frac{n}{2}-k}P_-^\lambda = \frac{(-1)^{\frac{n}{2}+k-1}}{\Gamma(\frac{n}{2}+k)}\delta_1^{(\frac{n}{2}+k-1)}(-P) + +\frac{(-)1^{\frac{p+1}{2}}\pi^{\frac{n}{2}-1}}{2^{2k}k!\Gamma(\frac{n}{2}+k)}\cdot\left[\psi(\frac{q}{2})-\psi(\frac{n}{2})\right](-L)^k\{\delta(x)\} \tag{48}$$

if p and q are odd

2 Relations of k-th derivative of Dirac delta in hypercone with ultrahyperbolic operator

In this paragraph we prove that generalized functions $\delta^{(k)}(P_+) - \delta_1^{(k)}(P)$ and $\delta^{(k)}(P_-) - \delta_1^{(k)}(-P)$ are concentrated in the vertex of the cone $P = 0$.

Theorem 1 *Let k be non-negative integer and n even dimension of the space then the following formulae are valid,*

$$\delta^{(k)}(P_+) - \delta_1^{(k)}(P) = B_{k,p,q}L^{k-\frac{n}{2}+1} \text{ if } k \geq \frac{n}{2} - 1 \tag{49}$$

where

$$B_{k,p,q} = \frac{(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!} \text{ for } p \text{ and } q \text{ are both even,} \tag{50}$$

and

$$B_{k,p,q} = \frac{(-1)^k(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}. \left[\psi(\tfrac{p}{2})-\psi(\tfrac{n}{2})\right].L^{k-\frac{n}{2}+1}\{\delta(x)\} \text{ for } p \text{ and } q \text{ are both odd.} \tag{51}$$

PROOF: From (41),(47) and considering the formulae (19) and (20) under conditions $k \geq \frac{n}{2}-1$,and when p and q are even, we have

$$\delta^{(k)}(P_+) - \delta_1^{(k)}(P) = (-1)^k a_{q,n,k}L^{k-\frac{n}{2}+1}\{\delta(x)\}. \tag{52}$$

where $a_{q,n,k}$ is defined by (27).
Similarly from (42), (48) and considering the formulae (19) and (20) under conditions $k \geq \frac{n}{2}-1$,and when p and q are odd, we have

$$\delta^{(k)}(P_+) - \delta_1^{(k)}(P) = \frac{(-1)^k(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}. \left[\psi(\tfrac{p}{2})-\psi(\tfrac{n}{2})\right].L^{k-\frac{n}{2}+1}\{\delta(x)\} \text{ for p and q are both odd.} \tag{53}$$

From (52) and (53) we obtain the formula (49),(50) and (51) which proves the theorem. ■

The formula (49) represent a relation between $\delta^{(k)}(P_+) - \delta_1^{(k)}(P)$ and the ultrahyperbolic operator iterated $k - \frac{n}{2} + 1$ times under condition $k \geq \frac{n}{2} - 1$.

Theorem 2 *Let k be non-negative integer and n even dimension of the space, then the following formulae are valid:*

$$\delta^{(k)}(P_-) - \delta_1^{(k)}(-P) = D_{k,p,q} L^{k-\frac{n}{2}+1} \{\delta(x)\} \tag{54}$$

where

$$D_{k,p,q} = \frac{(-1)(-1)^{\frac{q}{2}} \pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k - \frac{n}{2} + 1)!} \text{ for } p \text{ and } q \text{ are both even,} \tag{55}$$

and

$$D_{k,p,q} = \frac{(-1)^{\frac{q+1}{2}} \pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!} \left[\psi(\tfrac{q}{2}) - \psi(\tfrac{n}{2})\right] .L^{k-\frac{n}{2}+1} \{\delta(x)\} \text{ for } p \text{ and } q \text{ are both odd} \tag{56}$$

PROOF: From (41),(47) and considering the formulae (19) and (20) under conditions $k \geq \frac{n}{2} - 1$,and when p and q are even, we have:

$$\delta^{(k)}(P_-) - \delta_1^{(k)}(-P) = (-1) a_{q,n,k} L^{k-\frac{n}{2}+1} \{\delta(x)\} \tag{57}$$

where $a_{q,n,k}$ is defined by (27)
Similarly from (42), (48) and considering the formulae (19) and (20) under conditions $k \geq \frac{n}{2} - 1$, and when p and q are odd, we have:

$$\delta^{(k)}(P_-) - \delta_1^{(k)}(-P) = \frac{(-1)^{\frac{q+1}{2}} \pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!} \left[\psi(\tfrac{q}{2}) - \psi(\tfrac{n}{2})\right] .L^{k-\frac{n}{2}+1} \{\delta(x)\} \text{ for } p \text{ and } q \text{ are both odd} \tag{58}$$

From the formulae (57) and (58) we obtain the formulae (54),(55) and (56) which proves the theorem. ■

The formula (54) represents a relation between $\delta^{(k)}(P_-) - \delta_1^{(k)}(-P)$ with the ultrahyperbolic operator iterated $k - \frac{n}{2} + 1$ times under condition $k \geq \frac{n}{2} - 1$.

Theorem 3 *Let k be non-negative integer and n even dimension of the space then the following formulae are valid,*

$$\delta_1^{(k)}(P) - \delta_2^{(k)}(P) = A_{k,p,q} L^{k-\frac{n}{2}+1} \{\delta(x)\} \tag{59}$$

where

$$A_{k,p,q} = \frac{(-1)(-1)^k(-1)^{\frac{q}{2}} \pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k - \frac{n}{2} + 1)!} \text{ for } p \text{ and } q \text{ are both even,} \tag{60}$$

and

$$D_{k,p,q} = \frac{(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}. \tag{61}$$

$$\left[\psi(\tfrac{q}{2}) - \psi(\tfrac{p}{2})\right].L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\} \text{ for } p \text{ and } q \text{ are both odd}$$

PROOF: From (49) and (54) using (25), (50) and (60) under conditions $k \geq \frac{n}{2} - 1$, and when p and q are even, we have,

$$\frac{(-1)(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\} = \delta^{(k)}(P_+) - (-1)^k\delta^{(k)}(P_-) =$$

$$\delta_1^{(k)}(P) - \delta_2^{(k)}(P) + \frac{(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\} + \tag{62}$$

$$\frac{(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\}$$

therefore

$$\delta_1^{(k)}(P) - \delta_2^{(k)}(P) = \frac{(-1)(-1)^k(-1)^{\frac{q}{2}}\pi^{\frac{n}{2}}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\}. \tag{63}$$

Similarly from (49) and (54) using (26), (51) and (56) under conditions $k \geq \frac{n}{2} - 1$,and when p and q are odd, we have

$$\delta_1^{(k)}(P) - \delta_2^{(k)}(P) = \delta^{(k)}(P_+) - (-1)^k\delta^{(k)}(P_-) +$$

$$+\frac{(-1)^k(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}.\left[\psi(\tfrac{n}{2}) - \psi(\tfrac{p}{2}) + \psi(\tfrac{q}{2}) - \psi(\tfrac{n}{2})\right].L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\} = \tag{64}$$

$$= \frac{(-1)^k(-1)^{\frac{q+1}{2}}\pi^{\frac{n}{2}-1}}{4^{k-\frac{n}{2}+1}(k-\frac{n}{2}+1)!}.\left[\psi(\tfrac{q}{2}) - \psi(\tfrac{p}{2})\right].L^{k-\frac{n}{2}+1}\left\{\delta(x)\right\}$$

From the formulae (63) and (64) we obtain the formulae (59), (60) and (61) which proves the theorem. ■

The formula (59) represent a relation between $\delta_1^{(k)}(P) - \delta_2^{(k)}(P)$ with the ultrahyperbolic operator iterated $k - \frac{n}{2} + 1$ times under condition $k \geq \frac{n}{2} - 1$.

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