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SOME EXACT SOLUTIONS FOR A UNIDIMENSIONAL FOKKER-PLANCK EQUATION BY USING LIE SYMMETRIES

ALGUNAS SOLUCIONES EXACTAS PARA LA ECUACIÓN UNIDIMENSIONAL DE FOKKER-PLANCK USANDO SIMETRÍAS DE LIE

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Abstract

The Fokker Planck equation appears in the study of diffusion phenomena, stochastics processes and quantum and classical mechanics. A particular case from this equation, $u_t - u_{xx} - xu_x - u = 0$, is examined by the Lie group method approach. From the invariant condition it was possible to obtain the infinitesimal generators or vectors associated to this equation, identifying the corresponding symmetry groups. Exact solution were found for each one of this generators and new solution were constructed by using symmetry properties.

Keywords: Lie groups; partial differential equations; invariant solutions; Fokker Planck equation.

Resumen

La ecuación de Fokker Planck aparece en el estudio de fenómenos de difusión, procesos estocásticos y mecánica clásica y cuantica. Un caso particular de esta ecuación, $u_t - u_{xx} - xu_x - u = 0$, es analizada empleando el método de los grupos de Lie. De la condición de invariación fue posible obtener los generadores infinitesimales ó vectores de la ecuación identificando los correspondientes grupos de simetría. Se obtuvieron soluciones exactas para cada uno de estos generadores y se construyeron nuevas soluciones aplicando propiedades de simetría.

Palabras clave: grupos de Lie; ecuaciones diferenciales parciales; soluciones invariantes; ecuación de Fokker Planck.

Mathematics Subject Classification: 35A30.

1 Introduction

The Fokker-Planck equation also known as the Kolmogorov forward equation (diffusion equation), describes the time evolution of the probability density function of the velocity of a particle, and can be generalized to other observables as well [9]. Nicolay Bogoliubov and Nikolay Krylov were the first to derivate this equation in the study of classical and quantum mechanics. In the general case the time-dependent probability distribution is given by the equation:

$$\frac{\partial u}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}[A_i(x_1, \cdots, x_N)u] + \sum_{i=1}^{N}\sum_{j=1}^{N} \frac{\partial^2}{\partial x_i \partial x_j}[B_{ij}(x_i, \cdots, x_N)u] \tag{1}$$

where A is the drift vector and B the diffusion tensor.

The foregoing equation is frequently studied to model certain phenomena in which parameters defined by probability distribution functions (e.g., Brownian motion) appear.

To find exact solutions to this equation can be useful to study interesting physics phenomena but it is only possible for some particular cases [2], [13].

In this article several exacts solutions are found by the Lie group method, for the one spatial dimensional x, with $A = -x$ and $B = 1$:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(ux), \tag{2}$$

that is

$$u_t - u_{xx} - xu_x - u = 0. \tag{3}$$

In late 1800s, mathematicians Sophus Lie and Felix Klein, inspired by Evariste Galois' pioneering work in polynomial equations, decided to apply his results to differential equations. This new line of inquiry was based on the Erlangen program, addressing such equations in geometric terms. From this perspective, differential equations are studied as objects isomorphic to $\mathbb{R}^n$ space, seeking properties that remain invariant under a specified group of transformations in the same space. Lie identified groups of continuous transformations which, when acting on differential equations, left such equations invariant (i.e., Lie groups) [3], [6], [4].

Thus, in the simplest case of first order equations:

$$f(x, y, y') = 0, \tag{4}$$

the study focuses on groups of transformations that have the following form:

$$\begin{aligned} x_1 &= \phi(x, y, \alpha) \\ y_1 &= \varphi(x, y, \alpha) \\ y_1' &= \frac{dy_1}{dx_1} = \theta(x, y, y', \alpha), \end{aligned} \tag{5}$$

which, when acting on the variables in the differential equation (4), transform it in such manner that:

$$f(x, y, y') = f(x_1, y_1, y_1'),$$

i.e., the original equation and the equation written in the new coordinates provided by such group are indistinguishable, which is tantamount to transforming solutions to the original equation into solutions for the transformed equation. Such group of transformations is often referred to as symmetry.

For example the group of transformations of the plane corresponding to translation with respect to the y axis:

$$x_1 = x$$
$$y_1 = y + \alpha,$$

may be extended in order to act on derivatives of a first order ordinary differential equation by:

$$y_1' = \frac{dy_1}{dx_1} = \frac{dy}{dx},$$

this extended group leaves the following separable equation invariant:

$$\frac{dy}{dx} - f(x) = 0,$$

because:

$$\frac{dy}{dx} - f(x) = \frac{dy_1}{dx_1} - f(x_1).$$

The Lie theory proves that if a group of symmetries in an ordinary differential equation is known, then a one to one change in coordinates exists which transforms such group into a group of translations with respect to one of the axes and the equation into one of separable variables. Likewise, such group may be used to identify an integrating factor of the differential equation that converts it into an exact equation.

In the case of ordinary differential equations of the n order, a group of symmetries of the form (5), extended to act on derivatives up to the n order can be used to reduce the order of differential equations by one unit [5].

Invariability of partial differential equations (PDE) under a one parameter group of symmetries, permits reduction in the number of independent variables by one unit, especially useful in the case of second order equations with one dependent and two independent variables that can be transformed into ordinary equations conserving some of the solutions of the PDE [7], [10], [11]. This article deals with the application of the Lie Group Method to the Fokker-Planck equation (3). First a discussion of the method is presented. Then, the Lie symmetries are found for the unidimensional Fokker-Planck equation and finally with these symmetries the associated exact solutions were calculated and new solutions were obtained.

2 Description of the method

The second order PDE could be written as:

$$F(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}) = 0, \tag{6}$$

where the equation can be seen as a submanifold in space isomorphic to $\mathbb{R}^8$. One element of this space is the octuple $(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx})$.

2.1 Prolongation of a function

Definition 2.1 *Given a smooth function:*

$$u = f(t, x), \qquad f : \mathbb{R}^2 \to \mathbb{R}, \tag{7}$$

there is an induced function $f^{(2)} = Pr^{(2)} f(t, x)$ *referred to as the second extension of f, which is defined as:*

$$Pr^{(2)} f = (f, f_t, f_x, f_{tt}, f_{tx}, f_{xx}). \tag{8}$$

In this context a solution to (6) is a smooth function $u = f(t, x)$, such that

$$F(t, x, Pr^2 f(t, x)) = 0. \tag{9}$$

2.2 Symbol of a group

Definition 2.2 *Given the Lie group G*

$$t_1 = \phi_1(t, x, u, \alpha) \tag{10}$$

$$x_1 = \phi_2(t, x, u, \alpha) \tag{11}$$

$$u_1 = \phi_3(t, x, u, \alpha), \tag{12}$$

that acts on solutions to the differential equation (6), the infinitesimal symbol or generator of the group is defined by means of the operator:

$$v = \frac{\partial}{\partial \alpha}\phi_1(t, x, u, \alpha)\bigg|_{\alpha=o} \frac{\partial}{\partial x} + \frac{\partial}{\partial \alpha}\phi_2(t, x, u, \alpha)\bigg|_{\alpha=o} \frac{\partial}{\partial y} + \frac{\partial}{\partial \alpha}\phi_3(t, x, u, \alpha)\bigg|_{\alpha=o} \frac{\partial}{\partial u}. \tag{13}$$

If what is known is the symbol v, it is possible to recover the equations of the corresponding group through the Taylor series expansion t_1, x_1 y u_1:

$$t_1 = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} v^n x = \phi_1(t, x, \alpha)$$

$$x_1 = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} v^n y = \phi_2(t, x, \alpha)$$

$$u_1 = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} v^n u = \phi_3(t, x, \alpha),$$

a process, which is referred to as exponentiation of the group G and which is denoted as $G = \exp(\alpha v)$. In practice, the equations of the group are obtained by solving the system of equations:

$$\int_t^{t_1} \frac{dt}{f(t,x)} = \int_x^{x_1} \frac{dx}{g(t,x)} = \int_u^{u_1} \frac{dt}{h(t,x)} = \int_0^{\alpha} d\alpha,$$

where

$$\begin{aligned} f(t,x,u) &= \frac{\partial}{\partial \alpha} \phi_1(t,x,u,\alpha)\bigg|_{\alpha=o} \\ g(t,x,u) &= \frac{\partial}{\partial \alpha} \phi_2(t,x,u,\alpha)\bigg|_{\alpha=o} \\ h(t,x,u) &= \frac{\partial}{\partial \alpha} \phi_3(t,x,u,\alpha)\bigg|_{\alpha=o}. \end{aligned}$$

2.3 Prolongation of the group action

Let G be a local group of transformations on an open subset $\mathbb{R}^2 \times \mathbb{R}$. There is a G induced local action over the prolonged space of variables,

$$(t, x, u^{(2)}) = (t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}),$$

referred to as the second extension of G, denoted, $Pr^{(2)}G$, defined in such a way that it transforms the derivatives of (7) into the corresponding derivatives of the function transformed by the group.

2.4 Symbol of the prolonged group

Definition 2.3 *Let M be an open subset of $\mathbb{R}^2 \times \mathbb{R}$ and suppose that v is an infinitesimal generator of a group $G = \exp(\alpha v)$, then the second extension of v, denoted as $Pr^{(2)}v$, is defined as the infinitesimal generator of the second prolongation of G, i.e.,*

$$Pr^{(2)}v\Big|_{(t,x,u^{(2)})} = \frac{d}{d\alpha}\Big|_{\alpha=0} Pr^{(2)}[\exp(\alpha v)](t,x,u^{(2)}). \qquad (14)$$

The symbol takes the form:

$$v = f(t,x,u)\frac{\partial}{\partial t} + g(t,x,u)\frac{\partial}{\partial y} + h(t,x,u)\frac{\partial}{\partial u}, \qquad (15)$$

for a group that operates in an open $M \subset \mathbb{R}^2 \times \mathbb{R}$, with first and second extension of v:

$$Pr^{(1)}v = v + h^t(t,x,u^{(1)})\frac{\partial}{\partial u_t} + h^x(t,x,u^{(1)})\frac{\partial}{\partial u_{x,}} \qquad (16)$$

$$Pr^{(2)}v = Pr^{(1)}v + h^{tt}(t,x,u^{(2)})\frac{\partial}{\partial u_{tt}} + h^{tx}(t,x,u^{(2)})\frac{\partial}{\partial u_{tx}} + h^{xx}(t,x,u^{(2)})\frac{\partial}{\partial u_{xx,}} \qquad (17)$$

where,

$$\begin{aligned} h^{(t)} &= \frac{\partial h}{\partial t} + \left(\frac{\partial h}{\partial u} - \frac{\partial f}{\partial t}\right)u_t - \frac{\partial g}{\partial t}u_x - \frac{\partial f}{\partial u}u_t^2 - \frac{\partial g}{\partial u}u_t u_x. \\ h^{(x)} &= \frac{\partial h}{\partial x} + \left(\frac{\partial h}{\partial u} - \frac{\partial g}{\partial x}\right)u_x - \frac{\partial f}{\partial x}u_t - \frac{\partial g}{\partial u}u_x^2 - \frac{\partial f}{\partial u}u_t u_x. \end{aligned} \qquad (18)$$

$$\begin{aligned} h^{(tt)} = {} & \frac{\partial^2 h}{\partial t^2} + \left(2\frac{\partial^2 h}{\partial t \partial u} - \frac{\partial^2 f}{\partial t^2}\right)u_t - \frac{\partial^2 g}{\partial t^2}u_x + \left(\frac{\partial h}{\partial u} - 2\frac{\partial f}{\partial t}\right)u_{tt} - \\ & - 2\frac{\partial g}{\partial t}u_{tx} + \left(\frac{\partial^2 h}{\partial u^2} - 2\frac{\partial^2 f}{\partial t \partial u}\right)u_t^2 - 2\frac{\partial^2 g}{\partial t \partial u}u_t u_x - \frac{\partial^2 f}{\partial u^2}u_t^3 - \\ & - \frac{\partial^2 g}{\partial u^2}u_t^2 u_x - 3\frac{\partial f}{\partial u}u_t u_{tt} - \frac{\partial g}{\partial u}u_x u_{tt} - 2\frac{\partial g}{\partial u}u_t u_{tx}. \end{aligned} \qquad (19)$$

$$\begin{aligned} h^{(tx)} = &\frac{\partial^2 h}{\partial t \partial x} + \left(\frac{\partial^2 h}{\partial t \partial u} - \frac{\partial^2 g}{\partial t \partial x}\right) u_x + \left(\frac{\partial^2 h}{\partial x \partial u} - \frac{\partial^2 f}{\partial t \partial x}\right) u_t - \\ &- \frac{\partial g}{\partial t} u_{xx} + \left(\frac{\partial h}{\partial u} - \frac{\partial f}{\partial t} - \frac{\partial g}{\partial x}\right) u_{tx} - \frac{\partial f}{\partial x} u_{tt} - \frac{\partial^2 g}{\partial t \partial u} u_x^2 + \\ &+ \left(\frac{\partial^2 h}{\partial u^2} - \frac{\partial^2 f}{\partial t \partial u} - \frac{\partial^2 g}{\partial x \partial u}\right) u_t u_x - \frac{\partial^2 f}{\partial x \partial u} u_t^2 - \frac{\partial^2 g}{\partial u^2} u_t u_x^2 - \frac{\partial^2 f}{\partial u^2} u_t^2 u_x - \\ &- 2\frac{\partial g}{\partial u} u_x u_{tx} - 2\frac{\partial f}{\partial u} u_t u_{tx} - \frac{\partial f}{\partial u} u_x u_{tt} - \frac{\partial g}{\partial u} u_t u_{xx}. \end{aligned}$$

$$\begin{aligned} h^{(xx)} = &\frac{\partial^2 h}{\partial x^2} + \left(2\frac{\partial^2 h}{\partial x \partial u} - \frac{\partial^2 g}{\partial x^2}\right) u_x - \frac{\partial^2 f}{\partial x^2} u_t + \left(\frac{\partial h}{\partial u} - 2\frac{\partial g}{\partial x}\right) u_{xx} - \\ &- 2\frac{\partial f}{\partial x} u_{tx} + \left(\frac{\partial^2 h}{\partial u^2} - 2\frac{\partial^2 g}{\partial x \partial u}\right) u_x^2 - 2\frac{\partial^2 f}{\partial x \partial u} u_t u_x - \frac{\partial^2 g}{\partial u^2} u_x^3 - \\ &- \frac{\partial^2 f}{\partial u^2} u_t u_x^2 - 3\frac{\partial g}{\partial u} u_x u_{xx} - \frac{\partial f}{\partial u} u_t u_{xx} - 2\frac{\partial f}{\partial u} u_x u_{tx}. \end{aligned} \tag{20}$$

See [2],[10].

The condition to be satisfied so that a differential equation will be invariant under a group of transformations, or, which is necessary so that the solutions to a differential equation transformed by a group, be, in turn, solutions to the differential equation transformed by such group, is stated in the following theorem.

Teorema 2.1 *Let*

$$F(t, x, Pr^{(2)}u(t, x)) = 0, \tag{21}$$

be a maximum rank second order equation defined in an open subset $M \subset \mathbb{R}^2 \times \mathbb{R}$*. If* G *is a local group of transformations acting on* M *and*

$$Pr^{(2)}v[F(t, x, Pr^{(2)}u(t, x)] = 0, \qquad \textit{whenever} \qquad F(t, x, Pr^{(2)}u(t, x)) = 0,$$

for each infinitesimal generator v *of* G*, then* G *is a group of symmetries of the differential equation [10].*

2.5 Search of symmetry groups

The previous theorem together with the prolongation formula (14), defines an effective way to calculate the symmetries of a second order differential equation in one dependent and two independent variables, a formulation that can be generalized for higher orders.

In applying the theorem to such differential equation, an equation equal to zero results in which t, x and u and their first and second order partial derivatives may appear, as well as the f, g and h functions of a hypothetical group of symmetries and their first and second order partial derivatives with respect to t, x and u. Eliminating any dependencies among the derivatives of u resulting from the differential equation and equating the coefficients of the partial derivatives from u to zero, we arrive at a system of numerous PDEs denominated determinant equations (some of which repeat the same information) which can almost always be easily solved.

This system's general solution determines the most general form for the differential equation's symmetry group. As with ordinary differential equations, the set of generators found forms a Lie algebra, and the general group of symmetries can be found by exponentiation of these generators.

3 The Fokker-Planck equation

The Jacobian matrix

$$(0, -u_x, -1, 1, -x, 0, 0, -1), \tag{22}$$

derived from the Fokker-Planck equation, (3)

$$u_t - u_{xx} - xu_x - u = 0,$$

is never annulled; consequently, the submanifold defined by the differential equation in $\mathbb{R}^8$ is of maximum rank, i.e., the submanifold does not have singularities.

Applying theorem (2.1) and the prolongation formula (17) to the differential equation (3) it follows

$$\left[h\frac{\partial}{\partial u} + h^{tt}\frac{\partial}{\partial u_{tt}} + h^{xx}\frac{\partial}{\partial u_{xx}}\right](u_t - u_{xx} - xu_x - u) = 0, \tag{23}$$

.

Then h^{tt} and h^{xx} are substituted from expressions (19) and (20). Next $u_{xx} = u_t - xu_x - u$ is replaced whenever it appears in (23) and terms of each of the derivatives of u are grouped. The coefficients of each of the resulting monomials are equated to zero leading to a numerous system of partial differential equations. These system was solved by using the specialized software Lie 5.1 to find the functions f, g and h that were used to define the symbols or vectors (table (1)) of the Fokker-Planck equation symmetry groups.

Table 1: Symbols of symmetry groups.

Symbol	Symbol
$v_1 = w(t,x)\frac{\partial}{\partial u}$	$v_5 = -e^{-2t}u\frac{\partial}{\partial u} - xe^{-2t}x\frac{\partial}{\partial x} + e^{-2t}\frac{\partial}{\partial t}$
$v_2 = -\frac{\partial}{\partial t}$	$v_6 = -e^{t}ux\frac{\partial}{\partial u} + e^{t}\frac{\partial}{\partial x}$
$v_3 = -e^{2t}ux^2\frac{\partial}{\partial u} + e^{2t}x\frac{\partial}{\partial x} + e^{2t}\frac{\partial}{\partial t}$	$v_7 = -u\frac{\partial}{\partial u}$
$v_4 = -e^{-t}\frac{\partial}{\partial x}$	

In the foregoing table, $w(t;x)$ is a function that satisfies the Fokker Plank equation and $v_2, v_3, \cdots, v_7$, constitute a basis for the Lie algebra group of symmetries. As expected from algebras theory, the Lie bracket is a closed operation between these generators.

The corresponding one-parameter groups found by exponentiation of such symbols are respectively:

$$
\begin{aligned}
(t_1,x_1,u_1) &= (t, x, u + \alpha_1 F(t,x))\\
(t_1,x_1,u_1) &= (t + \alpha_2, x, u)\\
(t_1,x_1,u_1) &= (\ln\left(e^{-2t} - 2\alpha_3\right)^{-\frac{1}{2}}, x\left(e^{-2t} - 2\alpha_3\right)^{-\frac{1}{2}} e^{-t}, ue^{-\frac{x^2}{2}\left[\left(e^{-2t} - 2\alpha_3\right)^{-1} e^{-2t} - 1\right]})\\
(t_1,x_1,u_1) &= (t, x - \alpha_4, u)\\
(t_1,x_1,u_1) &= (\ln\left(e^{2t} + 2\alpha_5\right)^{1/2}, x\left(e^{2t} + 2\alpha_5\right)^{-1/2} e^{t}, u\left(e^{2t} + 2\alpha_5\right)^{1/2} e^{-t})\\
(t_1,x_1,u_1) &= (t, x + \alpha_6, ue^{-\alpha_6(x + \frac{\alpha_6}{2})})\\
(t_1,x_1,u_1) &= (t, x, ue^{-\alpha_7}).
\end{aligned}
\tag{24}
$$

For example a funtion invariant under the group with vector v_6:

$$v_6 = -e^{t}ux\frac{\partial}{\partial u} + e^{t}\frac{\partial}{\partial x},$$

must satisfy the condition:

$$v_6[f] = -\frac{\partial}{\partial t}f = 0. \tag{25}$$

The well known characteristics method [6], [12], assures that the (25) equation is equivalent to the following differential equations system:

$$\frac{du}{-e^{t}ux} = \frac{dx}{e^{t}} = \frac{dt}{0} = d\alpha,$$

from

$$\frac{dt}{0} = d\alpha,$$

it follows that t is an invariant

$$t_1 = t.$$

From

$$\frac{dx}{e^t} = d\alpha,$$

the integrals

$$\int_x^{x_1} dx = e^t \int_0^\alpha d\alpha$$

it is found that

$$x_1 = x + \alpha_6.$$

From

$$\frac{du}{-e^t ux} = d\alpha,$$

separation of variables and the integration from u to u_1 and from 0 to α leads to:

$$u_1 = ue^{-\alpha_6(x+\alpha_6/2)}.$$

The theory of Lie assures that a group symmetry transforms solution into solutions, this implies that if $u = f(x,t)$ is a solution of the differential equation then u_1 will also be a solution.

For instance, the group of symmetries obtained from the vector v_6

$$(t_1, x_1, u_1) = (t, x + \alpha_6, ue^{-\alpha_6(x+\frac{\alpha_6}{2})})$$

transforms a solution $f(x,t)$ into a new solution,

$$u_1 = f(x,t)e^{-\alpha_6(x+\alpha_6/2)},$$

replacing x, t, u for their transformations x_1, t_1, u_1, it follows

$$u_1 = f(x_1 - \alpha_6, t_1)e^{-\alpha_6(x_1-\alpha_6+\alpha_6/2)},$$

$$u_1 = f(x_1 - \alpha_6, t_1)e^{-\alpha_6 x_1}K,$$

where K is a constant.

According to the symmetry group corresponding to the vector v_7, the product of a scalar by a solution is also a solution. In particular if x_1, t_1 are equal to x, t then $u_1 = u$ and

$$u = f(x - \alpha_6, t)e^{-\alpha_6 x},$$

This is a new solution of differential equation (3) obtained from a known one.

New solutions obtained from the other generatos are listed below:

1. $u = f(t, x) + \alpha F(t, x),$
2. $u = f(t - \alpha, x),$
3. $u = f\left(\ln(e^{-2t} + 2\alpha)^{-1/2}, x(e^{-2t} + 4\alpha)^{-1/2}) (e^{-2t} + 2\alpha)^{1/2}\right) e^{-\frac{1}{2}\left(x^2(e^{-2t}+4\alpha)^{-1}(e^{-2t}+2\alpha)^2 e^{2t}-1\right)},$
4. $u = f(t, x + \alpha),$
5. $u = f\left(\ln(e^{2t} - 2\alpha)^{1/2}, x(e^{2t} - 2\alpha)^{-1/2}e^t\right) e^t(e^{2t} - 2\alpha)^{-1/2},$
6. $u = f(t, x - \alpha)e^{-\alpha x},$
7. $u = f(t, x)e^{-\alpha},$

where $F(t, x)$ and $f(t, x)$ are solutions of the differential equation (3).

3.1 Invariant solutions

Since a PDE is invariant when transformed under a group of symmetries, if the solutions of the differential equation are transformed into solutions under such group action, it is reasonably possible that certain differential equation solutions will also be invariant under such group. The present study explores the possibility of using such assumption to reduce the PDE to an ordinary differential equation, which may be solved in such a manner as to provide exact solutions, invariant under the different groups. The reduction method will now be applied with each of the symmetry groups found for the finite dimension algebra $(v_2, \ldots, v_7)$. Again, the condition

$$v_2[f] = -\frac{\partial}{\partial t} f = 0, \tag{26}$$

and the characteristics method assures that the (26) equation is equivalent to the following differential equations system:

$$\frac{du}{0} = \frac{dt}{1} = \frac{dx}{0} = d\alpha,$$

from which the invariants $y = x$ and $\eta = u$ are derived, if it is assumed that the invariant solution sought is in the form

$$\eta = w(y),$$

consequently, replacing this condition in the differential equation under study (3),

$$u_t = 0, \qquad u_x = w_y, \qquad u_{xx} = w_{yy},$$

is obtained, thus

$$w_{yy} + yw_y + w = 0, \tag{27}$$

which is an ordinary differential equation (ODE) in y and w. In such equation, the substitution

$$v = w(y)e^{ky^2},$$

leads to

$$v_{yy} + (1-4k)yv_y + (4k^2 - 2k)y^2 v + (1-2k)v = 0,$$

which for $k = 1/4$ y $\nu = 0$, takes the form of the Weber Equation,

$$v''(y) + (\nu + \frac{1}{2} - \frac{1}{4}y^2)v(y) = 0,$$

whose solution is known and can be expressed in terms of parabolic cylinder functions

$$v(y) = c_1 D_\nu(y) + c_2 D_{-\nu-1}(y),$$

where

$$D_\nu(y) = 2^{\nu/2+1/4} y^{-1/2} W_{\nu/2+1/4,-1/4}(\frac{1}{2}y^2), \quad \text{y}$$

and $W_{l,m}$; is the Whittaker function (see Weber Equation [1], [14]), [15]).

An invariant solution to the (3) equation under the v_2 symmetry group is then:

$$u(x) = (c_1 D_0(x) + c_2 D_{-1}(ix))\, e^{-\frac{x^2}{4}}. \tag{28}$$

In a similar manner, the characteristic system

$$v_3 = -e^{2t}ux^2\frac{\partial}{\partial u} + e^{2t}x\frac{\partial}{\partial x} + e^{2t}\frac{\partial}{\partial t},$$

for the generator

$$\frac{du}{-e^{2t}ux^2} = \frac{dt}{e^{2t}} = \frac{dx}{e^{2t}x} = d\alpha,$$

leads to the invariants

$$y = xe^{-t}, \qquad \text{y} \quad \nu = ue^{\frac{x^2}{2}},$$

taking $\nu = w(y)$, it follows that,

$$\begin{aligned}
u &= we^{-\frac{x^2}{2}},\\
u_t &= -xe^{-t}e^{-\frac{x^2}{2}}w_y,\\
u_x &= (e^{-t}w_y - xw)e^{-\frac{x^2}{2}},\\
u_{xx} &= (e^{-2t}w_{yy} - 2xe^{-t}w_y + x^2w - w)e^{-\frac{x^2}{2}}.
\end{aligned}$$

Replacing the foregoing in 3), the reduced equation

$$w_{yy} = 0,$$

is arrived at, leading to general solution

$$w(y) = c_1y + c_2,$$

i.e., an invariant solution under the v_3 group generator is:

$$u(t,x) = \left[axe^{-t} + b\right]e^{-\frac{x^2}{2}}. \tag{29}$$

The invariants $y = t$ y $\eta = u$, are obtained for

$$v_4 = -e^{-t}\frac{\partial}{\partial x}.$$

consequently, an invariant solution under such group will take the form:

$$u = w(y),$$

resulting in

$$\begin{aligned}
u_t &= w_y,\\
u_x &= 0,\\
u_{xx} &= 0,
\end{aligned}$$

replacing in (3), it follows:

$$w_y = w,$$

resolved as

$$w = ce^y,$$

one invariant solution under the v_4 symmetry group is then:

$$u = ce^t. \tag{30}$$

Similarly, the generator

$$v_5 = -e^{-2t}u\frac{\partial}{\partial u} - xe^{-2t}x\frac{\partial}{\partial x} + e^{-2t}\frac{\partial}{\partial t},$$

leads to the invariant

$$y = xe^t, \qquad \text{y} \quad \nu = ue^{-t},$$

assuming that $\nu = w(y)$, then

$$\begin{aligned} u &= we^t, \\ u_t &= (yw_y + w)e^t, \\ u_x &= e^{2t}w_y, \\ u_{xx} &= e^{3t}w_{yy}, \end{aligned}$$

is arrived at. Replacing such equation in (3), the reduced equation

$$w_{yy} = 0,$$

is obtained, leading to general solution

$$w(y) = c_1 y + c_2,$$

i.e.,

$$u(t,x) = \left[c_1 xe^t + c_2\right] e^t, \tag{31}$$

is an invariant solution under the v_5 generator group.

The generator

$$v_6 = -e^t ux\frac{\partial}{\partial u} + e^t\frac{\partial}{\partial x},$$

leads to the invariants

$$y = t, \qquad \text{y} \quad \nu = ue^{\frac{x^2}{2}},$$

assuming $\nu = w(y)$,

$$u = we^{-\frac{x^2}{2}},$$
$$u_t = w_y e^{-\frac{x^2}{2}},$$
$$u_x = -xwe^{-\frac{x^2}{2}},$$
$$u_{xx} = w(x^2-1)e^{-\frac{x^2}{2}},$$

is arrived at and, replacing such equation in (3), reduced equation

$$w_y = 0,$$

is obtained, with general solution

$$w(y) = c,$$

As a result,

$$u(t,x) = ce^{-\frac{x^2}{2}}, \tag{32}$$

is an invariant solution to the equation (3) under the v_6 generator group.

Finally, considering the group generators:

$$v_1 = -w(t,x)\frac{\partial}{\partial u},$$

and

$$v_7 = -u\frac{\partial}{\partial u},$$

the condition to be met by an invariant surface $F(t,x,u) = u - s(t,x) = 0$ is:

$$vF(t,x,u) = -f\frac{\partial s(t,x)}{\partial t} - g\frac{\partial s(t,x)}{\partial x} + h\frac{\partial s(t,x)}{\partial u} = 0,$$

which is equivalent to,

$$f\frac{\partial s(t,x)}{\partial t} + g\frac{\partial s(t,x)}{\partial x} = h,$$

since f and g are equal to cero and $h \neq 0$, we conclude that no surfaces are invariant under these groups. It is important to note that v_1 implies the linearity of the solutions.

Linear combinations of generators will now be addressed. The characteristic system for the linear combination $v_8 = -av_2 - bv_7$, i.e.,

$$v_8 = a\frac{\partial}{\partial t} + bu\frac{\partial}{\partial u}$$

has the invariant

$$y = x$$

and

$$w = ue^{-\frac{b}{a}t},$$

assuming

$$u = w(y)e^{\frac{b}{a}t},$$

the resulting EDO is

$$w_{yy} + yw_y + (1 - \frac{b}{a})w = 0,$$

due to the change of variable

$$v = we^{ky^2},$$

for $k = 4$, the EDO is transformed into the Webber Equation

$$v_{yy} + (-\frac{b}{a} + \frac{1}{2} - \frac{1}{4}y^2)v = 0,$$

with the following solution:

$$v(y) = k_1 D_{-\frac{b}{a}}(y) + k_2 D_{-\frac{b}{a}-1}(iy),$$

then

$$u = e^{-\frac{1}{4}y^2} e^{\frac{b}{a}t}(k_1 D_{-\frac{b}{a}}(y) + k_2 D_{-\frac{b}{a}-1}(iy)), \tag{33}$$

proceeding in a very similarly fashion, invariable solutions are obtained under invariant generators

$$v_9 = -av_2 + bv_6, \tag{34}$$

and

$$v_{10} = -av_2 - bv_4 - cv_7. \tag{35}$$

A summary of the results obtained from equations (28) to (35) appears in table (2).

Table 2: Invariant solutions for the Fokker-Planck equation.

Generator	Invariant Solution
v_2	$u = [c_1 D_0(x) + c_2 D_{-1}(ix)]\, e^{-\frac{x^2}{4}}$
v_3	$u = \left[axe^{-t} + b\right] e^{-\frac{x^2}{2}}$
v_4	$u = ce^t$
v_5	$u = \left[c_1 xe^t + c_2\right] e^t$
v_6	$u = ce^{-\frac{x^2}{2}}$
v_7	No integral surfaces were found
$v_8 = -av_2 - bv_7$	$u = e^{-\frac{1}{4}x^2} e^{\frac{b}{a}t} \left[c_1 D_{-\frac{b}{a}}(x) + c_2 D_{-\frac{b}{a}-1}(ix)\right]$
$v_9 = -av_2 + bv_6$	$u = e^{\frac{-(x-\frac{b}{a}e^t)^2}{4}} \left[c_1 D_0(x - \frac{b}{a}e^t) + c_2 D_{-1}(i(x - \frac{b}{a}e^t))\right] e^{\frac{-x^2}{2}}$
$v_{10} = -av_2 - bv_4 - cv_7.$	$u = e^{-\frac{(x+\frac{b}{a}e^{-t})^2}{4}} \left[c_1 D_{-\frac{c}{a}}(x + \frac{b}{a}e^{-t}) + c_2 D_{-1}(i(x + \frac{b}{a}e^{-t}))\right] e^{\frac{c}{a}t}$

4 Conclusions

1. The Fokker-Planck equation is invariant under a Lie group generated by six vector fields. It was possible to find invariant solutions and build new solutions based on known solutions in five of these group generators or symbols.

2. The total number of possible linear combinations between the finite dimension sub algebras v_2 to v_7 is 63. Exploring each of these options is tedious. Because many of the calculations may be redundant since the same solution may be included within the solutions found for other one-parameter groups, the need for an algorithm applicable to the group algebra that minimizes required calculations is obvious. The study of these algorithms (e.g., Ovsiannikov's optimal algebras) may prove a good complement to the foregoing.

3. The Lie group method applied to differential equations provides an algorithmic way to construct invariant solutions to a large number of partial differential equations. For non-linear equations, these solutions may be the only ones available, making them very important for modeling physical situations and for comparison with numerical solutions. Specialized software currently exists to deal with the voluminous calculation of symmetries required and should be taken into account for more practical Lie theory management, especially with respect to higher-order equations or differential equation systems. It is suggested that interested readers consult Hereman's review of this specialized software [8].

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