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SOME PROPERTIES AND INEQUALITIES RELATED TO THE k^{th} INVERSE MOMENT OF A POSITIVE BINOMIAL VARIATE

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Abstract

We present some known properties of $b_k(n,p)$ the k^{th} inverse moment of a positive binomial variate. Additional properties of this function and new bounds are derived.

Keywords: Probability, statistics, life contingencies, Bernstein polynomials.

Resumen

Se presentan algunas propiedades conocidas del k-simo momento inverso $b_k(n,p)$ de una variable aleatoria binomial positiva. Se derivan otras desigualdades nuevas y algunas propiedades de esta funcin.

Palabras clave: Probabilidad, estadstica, contingencias de vida, polinomios de Bernstein.

Mathematics Subject Classification: 62P05, 12E10, 26D05, 47N30.

1 Preliminary technical results

The following known or simple results are needed in Sections 2 and 3.

Lemma 1.1 *(Jensen's inequality).* Let X be a random variable in (a,b) with finite expectation $\mu = E[X]$. For any function f such that $f''(x) > 0$, for all $x \in (a,b)$, we have $E[f(X)] \geq f(\mu)$.

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Corollary 1.1 For any $\alpha > 0$ and a positive random variable X such that $E[X] < \infty$, the inequality $E\left[X^{-\alpha}\right] \geq E[X]^{-\alpha}$ *holds.*

Definition 1.1 *Bernstein polynomials.* For a function f defined on the closed interval $[0,1]$ the expression

$$B_n^f(x) \quad = \quad \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}, \tag{1}$$

is called the Bernstein polynomial of order n of the function f.

Lemma 1.2 (Bernstein [2]). For a function f bounded on $[0,1]$, the relation

$$\lim_{n\to\infty} B_n^f(x) \quad = \quad f(x),$$

holds at each point of continuity x of f, futhermore, the relation holds uniformly on $[0,1]$ if f is continuous on this interval.

Remarkable phenomena can occur for unbounded functions. As an example, take the function

$$F(x) = \begin{cases} |c-x|^{-\gamma} & \text{if } 0 \leq x \leq 1,\ x \neq c, \\ 0 & \text{if } x = c, \end{cases} \tag{2}$$

where $0 \leq c \leq 1$ and $\gamma > 0$. In this case the behaviour of $B_n^F(x)$ depends on c as follows.

Lemma 1.3 (Lorentz [7]). Let F be defined by (2). For almost all $c \in [0,1]$, $B_n^F(x) \to F(x)$ for all $x \neq c$; and there is a set of c of values for which the sequence $B_n^F(x)$ is unbounded for any x different from 0, c or 1.

The proof of the first part of Lemma 1.3 is based on the fact that for $k > 2$, c must satisfy the inequality $|v/n - c| \geq n^{-k}$ for all large n and all integer number $v > 0$. If $c = 0$ this inequality is clearly true and the next corollary follows.

Corollary 1.2 . Let F be defined by (2) with $c = 0$. Then $B_n^F(x) \to F(x)$ for all $0 < x \leq 1$.

Lemma 1.4 Let $n \geq 2$ be an integer.

(i) If $0 < p \leq 1$ then $(1+np)(1-p)^n < 1$.

(ii) If $n \geq 3$ and $2/n \leq p \leq 1$ then $[2(n-1)p+1](1-p)^{n-1} < 1$.

(iii) If $r > 0$ and $0 \leq p \leq 1$ then $p^r(1-p)^n \leq \left(\frac{r}{n+r}\right)^r\left(1-\frac{r}{n+r}\right)^n < \left(\frac{r}{n}\right)^r e^{-r}$.

(iv) If $0 < p \leq 1$ then $\left[\frac{n(n-1)}{2}p^2+(n-1)p+1\right](1-p)^{n-1} < 1$.

(v) If $0 < z < 1$ then $\dfrac{z}{\sqrt{1-z}} + \ln(1-z) > 0$.Also, for $n \geq 1$ and $0 < x < n$

$$\frac{x}{n-x} + \ln\left(1 - \frac{x}{n}\right) > 0. \tag{3}$$

(vi) For $0 < q < 1$, $p = 1 - q$ and $k = 1, 2, \dots$

$$q^{-k} \geq u(k,p) := 1 + k\left(\frac{p}{q}\right) + \frac{1}{2}k(k-1)\left(\frac{p}{q}\right)^2 \geq v(k,p) := 1 + k\left(\frac{p}{q}\right). \tag{4}$$

Proof. In (i)-(iv) we may assume if necessary that $p < 1$.
(i) Since the function $f(p) = (1+np)(1-p)^n$ is decreasing in $]0,1[$, then $f(p) < f(0) = 1$.
(ii) Let $f(p) = [2(n-1)p + 1](1-p)^{n-1}$. Then $f'(p) < 0$ for $\frac{1}{2n} < p \leq 1$. Since $1/(2n) < 2/n$ then $f(p) \leq f(2/n) = \left(\frac{5n-4}{n-2}\right)\left(1 - \frac{2}{n}\right)^n$, for $2/n \leq p \leq 1$. From this we see that $f(p) < 1$ for $n = 3, 4$. If $n \geq 5$ then $f(p) < \left(\frac{5(5)-4}{5-2}\right)e^{-2} < 1$.
(iii) The first inequality is due to the fact that the function $u(p) = p^r(1-p)^n$, for $0 \leq p \leq 1$, has a global maximum at $p = r/(n+r)$. The second inequality follows since the function $h(x) = (1 - 1/x)^x$ is increasing for $x \geq 1$ with $\lim_{x\to\infty} h(x) = e^{-1}$.
(iv) Consider the function $h(p) = \left[\frac{1}{2}n(n-1)p^2 + (n-1)p + 1\right](1-p)^{n-1}$ for $0 < p < 1$. Then $h'(p) = -\frac{1}{2}n(n^2-1)p^2(1-p)^{n-2} < 0$. This yields $h(p) < h(0) = 1$ for $0 < p \leq 1$.
(v) The function $f(z) = \frac{z}{\sqrt{1-z}} + \ln(1-z)$ for $0 \leq z < 1$ is such that $f'(z) > 0$ for $0 < z < 1$. Therefore, $f(z) > f(0) = 0$. This proves the first inequality. Using this with $z = x/n$, and since $1 - z < \sqrt{1-z}$ for $0 < z < 1$, we conclude that (3) is also true.
(vi) It follows from the identity$q^{-k} = (1 + p/q)^k = \sum_{j=0}^{k} \binom{k}{j} \left(\frac{p}{q}\right)^j$. ∎

Corollary 1.3 (i) $K_n(x) = \left(1 - \dfrac{x}{n}\right)^{1/x}$ is decreasing in $]0, n]$ for $n \geq 1$.

(ii) $F_n(x) = \left[\dfrac{(n-x)(n-x-1)}{n(n-1)}\right]^{1/x}$ is decreasing in $]0, n-1]$ for $n \geq 2$.

Proof. (i) If $R_n(x) = \ln[K_n(x)]$ then from (3) $R_n'(x) = -\frac{\frac{x}{n-x} + \ln\left(1 - \frac{x}{n}\right)}{x^2} < 0$.
(ii) It is true since $K_n(x) > 0$ and $F_n(x) = K_n(x)K_{n-1}(x)$. ∎

We now apply these results to the inverse moments calculation of a positive binomial random variable.

2 The k^{th} inverse moment of a positive binomial variate

The inverse moments of positive binomial random variables are used, among other fields, in actuarial sciences and life testing. We present here a brief introduction to the subject.

Definition 2.1 *The positive binomial distribution.* Let X be a binomial random variable with parameters (n, p), i.e. its probability function is given by

$$f_X(m) = \binom{n}{m} p^m q^{n-m}, \quad m = 0, 1, \dots, n,$$

where $q = 1 - p$. Then a random variable Y is called a positive binomial distribution with parameters (n, p) if $f_Y(m) = f_{X|X>0}(m)$, i.e.

$$f_Y(m) = \frac{\binom{n}{m} p^m q^{n-m}}{1 - q^n}, \quad m = 1, 2, \ldots, n.$$

In this case its k^{th} inverse moment (c.f. [6]) is denoted by $b_k(n, p)$, that is, for $k = 0, 1, 2, \ldots$

$$b_k(n, p) = E[Y^{-k}] = \frac{1}{1 - q^n} \sum_{m=1}^{n} \frac{1}{m^k} \binom{n}{m} p^m q^{n-m}.$$

We use a representation of $b_k(n, p)$, which up to a factor can be denoted by $B_k(n, p)$, given by

$$B_k(n, p) = n^k (1 - q^n) b_k(n, p) = \sum_{m=1}^{n} \frac{n^k}{m^k} \binom{n}{m} p^m q^{n-m}. \tag{5}$$

We also denote $B_k(0, p) = b_k(0, p) = 0$ and observe that $B_0(n, p) = 1 - q^n$.

The inverse moments $b_1(n, p)$ and $b_2(n, p)$ are important in actuarial mathematics (c.f. [1] and [9]). Using (5) one can compute $B_k(n, p)$ directly for any finite n. Unfortunately, for large n, the binomial probabilities are curbersome to find. In view of this difficulty, recursive formulas are used to accelerate the computations. Alternatively, some approximations are also used.

Grab and Savage [5] provided the recursive equation:

$$b_1(n + 1, p) = \frac{q(1 - q^n)}{(1 - q^{n+1})} b_1(n, p) + \frac{1}{n + 1}. \tag{6}$$

Govindarajulu [4] provided the following recursive relationship:

$$b_2(n + 1, p) = \frac{q(1 - q^n)}{(1 - q^{n+1})} \left[\frac{b_1(n, p)}{n + 1} + b_2(n, p) \right] + \frac{1}{(n + 1)^2}. \tag{7}$$

To approximate $b_k(n, p)$ the formula from Mendenhall and Lehman [8] can be used:

$$b_k(n, p) \approx \frac{1}{n^k} \prod_{r=1}^{k} \frac{a + b - r}{a - r}, \tag{8}$$

where $a = (n - 1)p$ and $b = (n - 1)q$. Other approximations to $b_k(n, p)$ are found in [9], [10] and [11].

Replacing (8) in (5) for $k = 1$ and $k = 2$ yields, respectively

$$B_1(n, p) \approx \frac{(1 - q^n)(n - 2)}{a - 1},$$
$$B_2(n, p) \approx \frac{(1 - q^n)(n - 2)(n - 3)}{(a - 1)(a - 2)}.$$

3 Some additional properties of the $B_k(n,p)$ polynomials

Section 2 shows the importance of Bernstein polynomials in the inverse moments calculation of positive binomial random variables. We give here some new results that can help to obtain numerical values of Bernstein polynomials.

3.1 Jensen's Inequality

Corollary 3.1 The polynomials $B_k(n,p)$ given by (5) satisfy the inequality

$$B_k(n,p) \geq \frac{(1-q^n)^{k+1}}{p^k}.$$

Proof. It follows from Corollary 1.1 that

$$E[Y^{-k}] = b_k(n,p) = \frac{B_k(n,p)}{n^k(1-q^n)} \geq E[Y]^{-k} = \left[\frac{np}{1-q^n}\right]^{-k}. \qquad \blacksquare$$

3.2 Recursive equations for $B_k(n,p)$

A recursive equation for $B_k(n,p)$ similar to those given by (6) and (7) are easily derived for any k as stated by the following proposition.

Proposition 3.1 The polynomials $B_k(n,p)$ given by (5), satisfy the recursive equation:

$$B_k(n+1,p) \quad = \quad q\left(\frac{n+1}{n}\right)^k B_k(n,p) + B_{k-1}(n+1,p),$$

with $B_0(n+1,p) = 1 - q^{n+1}$ and $n = 0, 1, \ldots$ and $B_k(1,p) = p$.

Proof. Using the relation $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$ we have

$$\begin{aligned}
B_k(n+1,p) &= \sum_{m=1}^{n+1} \frac{(n+1)^k}{m^k}\binom{n+1}{m} p^m q^{n+1-m}, \\
&= \sum_{m=1}^{n} \frac{(n+1)^k}{m^k}\left[\binom{n}{m} + \binom{n}{m-1}\right] p^m q^{n+1-m} + p^{n+1}, \\
&= \sum_{m=1}^{n} \frac{(n+1)^k}{m^k}\binom{n}{m} p^m q^{n+1-m} + \sum_{m=1}^{n} \frac{(n+1)^k}{m^k}\binom{n}{m-1} p^m q^{n+1-m} + p^{n+1}, \\
&= q\left(\frac{n+1}{n}\right)^k \sum_{m=1}^{n} \frac{n^k}{m^k}\binom{n}{m} p^m q^{n-m} + \sum_{m=1}^{n+1} \frac{(n+1)^{k-1}}{m^{k-1}}\binom{n+1}{m} p^m q^{n+1-m}, \\
&= q\left(\frac{n+1}{n}\right)^k B_k(n,p) + B_{k-1}(n+1,p). \qquad \blacksquare
\end{aligned}$$

Using Proposition 3.1 we get the following result that expresses B_k in terms of B_{k-1}.

Corollary 3.2 The polynomials $B_k(n,p)$ given by (5), satisfy the recursive equation:

$$B_k(n+1,p) \quad = \quad \sum_{m=0}^{n} B_{k-1}(m+1,p)\left(\frac{n+1}{m+1}\right)^k q^{n-m}.$$

Replacing $B_0(n,p) = 1 - q^n$ in Corollary 3.2 for $k = 1$ we obtain the following interesting special case, a counterpart to the result cited without proof by Stephan [10] for $b_1(n,p)$.

Corollary 3.3 The polynomials $B_1(n,p)$ can be written as powers of $q = 1 - p$:

$$B_1(n,p) = \sum_{m=0}^{n-1} \frac{n}{n-m} q^m - q^n \sum_{m=1}^{n} \frac{n}{m}.$$

3.3 Limit properties of the $B_k(n,p)$ polynomials

Corollary 3.4 The relation $\lim_{n\to\infty} B_k(n,p) = 1/p^k$ holds point-wise in $]0,1]$.

Proof. Observe that $B_k(n,p) = B_n^F(p)$, where $F(p)$ is given by (2) with $c = 0$ and $\gamma = k$, while $B_n^F(p)$ is defined by (1). Then the result follows from Corollary 1.2. ∎

3.4 Further properties of the $B_1(n,p)$ polynomials

By studying the first and second derivatives of $B_1(n,p)$ we find some inequalities as well as some inflexion points and convexity properties for these polynomials.

Proposition 3.2 For each $n \geq 2$ there exists a unique $0 < \alpha_n < 1$ such that

(i) $B_1(n,\alpha_n) = \dfrac{1-(1-\alpha_n)^n}{\alpha_n}$.

(ii) $B_1(n,p) < B_1(n,\alpha_n)$, $\forall\, p \in\ [0,1], p \neq \alpha_n$.

(iii) $B_1(n,p)$ increases in $[0,\alpha_n]$ and decreases in $]\alpha_n,1]$.

(iv) $B_1(n,p) < \dfrac{1-q^n}{p}$, $\forall\, p \in\,]0,\alpha_n[$ and $B_1(n,p) > \dfrac{1-q^n}{p}$, $\forall\, p \in\,]\alpha_n,1[$.

(v) The following inequalities hold for $n \geq 3$:

$$\frac{1}{n} \quad \leq \quad \alpha_n \leq u_n = \frac{\sqrt{3(n-1)(7n-11)} - 3(n-1)}{\sqrt{3(n-1)(7n-11)} + (n-1)(n-5)} \leq \frac{2}{n+1}. \qquad (9)$$

(vi) $\{\alpha_n\}$ is decreasing.

(vii) $\lim_{n\to\infty} \alpha_n = 0$.

Proof. For $0 < p < 1$

$$\begin{aligned} B_1'(n,p) &= n\frac{d}{dp}\left[\sum_{k=0}^{n-1}\frac{q^k}{n-k} - q^n\sum_{k=1}^{n}\frac{1}{k}\right], \text{ from Corollary 3.3,} \\ &= n\left[\sum_{k=0}^{n-1}\frac{-kq^{k-1}}{n-k} + nq^{n-1}\sum_{k=1}^{n}\frac{1}{k}\right], \text{ since } q = 1-p, \qquad (10) \\ &= -nq^{n-1}f(p), \end{aligned}$$

where

$$f(p) = \sum_{k=0}^{n-1}\frac{k}{n-k}q^{-(n-k)} - \sum_{k=1}^{n}\frac{n}{k} = \frac{B_1(n,p) - \frac{1-q^n}{p}}{q^n}. \qquad (11)$$

This function is continuous, strictly increasing in $]0,1[$, with $\lim_{p\to 0^+} f(p) = -n < 0$ and

$$\lim_{p\to 1^-} f(p) = +\infty. \qquad (12)$$

Thus, there exists a unique $\alpha_n \in]0,1[$ such that

$$f(p) < f(\alpha_n) = 0 < f(t), \qquad (13)$$

for all $p \in]0,\alpha_n[$ and all $t \in]\alpha_n, 1[$. The proof of (i)-(iv) follows from (13) and (11).
(v) Write (11) as

$$f(p) = \sum_{k=1}^{n}\left[\frac{(n-k)q^{-k} - n}{k}\right], \qquad (14)$$

then we get that $B_1'(n,p) > 0$ if $(n-k)q^{-k} - n < 0$, for all $k = 1,\ldots,n$. From Corollary 1.3 - (i), we get that $q > \max_{\{k=1,\ldots,n\}}\left(\frac{n-k}{n}\right)^{1/k} = 1 - 1/n$. This yields $p < 1/n$ and so $1/n \le \alpha_n$, which proves the first inequality of (9). Using of (4) in (14) yields

$$f(p) \ge \sum_{k=1}^{n}\left\{\frac{(n-k)u(k,p) - n}{k}\right\} \ge \sum_{k=1}^{n}\left\{\frac{(n-k)v(k,p) - n}{k}\right\},$$

or

$$\begin{aligned} f(p) &\ge \underbrace{n\left[\frac{1}{12}(n-1)(n-2)\left(\frac{p}{q}\right)^2 + \frac{1}{2}(n-1)\left(\frac{p}{q}\right) - 1\right]}_{A_2}, \qquad (15) \\ &\ge \underbrace{n\left[\frac{1}{2}(n-1)\left(\frac{p}{q}\right) - 1\right]}_{A_1}. \end{aligned}$$

Solving $A_2 > 0$ for $n \geq 3$ we get that $f(p) > 0$, i.e. $B_1'(n,p) < 0$, for $p > u_n$. This proves the second inequality of (9). Now solving $A_1 > 0$ for $n \geq 3$ we get $B_1'(n,p) < 0$ for $p > \frac{2}{n+1}$. This means that $\alpha_n \leq \frac{2}{n+1}$. The last inequality of (9) follows since $A_2 \geq A_1$.

(vi) By (iv) it is sufficient to prove that $B_1(n+1,\alpha_n) > \frac{1-(1-\alpha_n)^{n+1}}{\alpha_n}$. Now let $p = \alpha_n$ and $q = 1-\alpha_n$, then

$$\begin{aligned} B_1(n+1,p) &= q\left(\frac{n+1}{n}\right)B_1(n,p) + 1 - q^{n+1}, \text{ by Proposition 3.1,} \\ &= q\left(\frac{n+1}{n}\right)\left(\frac{1-q^n}{p}\right) + 1 - q^{n+1}, \text{ by (i),} \\ &= \frac{1-q^{n+1}}{p} + \frac{q}{np}\left[1-(np+1)q^n\right], \\ &> \frac{1-q^{n+1}}{p}, \text{ by Lemma 1.4 - (i).} \end{aligned}$$

(vii) It follows from (v). ■

Table 1: Values of α_n, lower and upper bounds and $1.5/n$

n	$1/n$	α_n	u_n	$2/(n+1)$	$1.5/n$
5	0.200000	0.288677	0.292893	0.333333	0.300000
10	0.100000	0.147419	0.152067	0.181818	0.150000
20	0.050000	0.074439	0.077545	0.095238	0.075000
50	0.020000	0.029946	0.031394	0.039216	0.030000
100	0.010000	0.015001	0.015761	0.019802	0.015000
200	0.005000	0.007507	0.007897	0.009950	0.007500
500	0.002000	0.003005	0.003163	0.003992	0.003000
1000	0.001000	0.001503	0.001582	0.001998	0.001500
2000	0.000500	0.000751	0.000791	0.001000	0.000750
5000	0.000200	0.000301	0.000316	0.000400	0.000300

From this empirical study we see that $\alpha_n \approx 1.5/n$; Table 1 reports its values together with α_n and its lower and upper bounds $1/n$, u_n and $2/(n+1)$.

Proposition 3.3 For each $n \geq 2$ the equation

$$B_1(n,p) = \frac{1}{p}, \tag{16}$$

has a unique solution β_n in $]0,1[$ such that

(i) $\alpha_n < \beta_n < 2.16/n$, where α_n is given by Proposition 3.2.

(ii) $B_1(n,p) < \frac{1}{p}$, $\forall\, p \in\,]0,\beta_n[$ and $B_1(n,p) > \frac{1}{p}$, $\forall\, p \in\,]\beta_n, 1[$.

(iii) $\{\beta_n\}$ is decreasing.

(iv) $\lim_{n\to\infty} \beta_n = 0$.

Proof. Since $\frac{1-q^n}{p} < \frac{1}{p}$ it follows from Proposition 3.2 that (16) has no solution in $]0, \alpha_n]$. Thus, (16) is equivalent to $g(p) = 0$, where

$$g(p) = f(p) - \frac{1}{p}, \tag{17}$$

for $p \in [\alpha_n, 1[$ and $f(p)$ defined by (11). The function g is continuous in $[\alpha_n, 1[$ and since $f(p)$ and $-1/p$ are strictly increasing then $g(p)$ is also strictly increasing. Besides from (13), $g(\alpha_n) = f(\alpha_n) - 1/\alpha_n = -1/\alpha_n < 0$ and by (12)

$$\lim_{p\to 1^-} g(p) \quad = \quad +\infty. \tag{18}$$

Therefore, there exists a unique $\beta_n \in]\alpha_n, 1[$ such that

$$g(p) \quad < \quad g(\beta_n) = 0 < g(t), \tag{19}$$

for all $p \in]\alpha_n, \beta_n[$ and all $t \in]\beta_n, 1[$. This completes the proof of (i)-(ii), except for the inequality $\beta_n \le 2.16/n$. To prove it, use (11) and (15) to get

$$B_1(n,p) \ge \underbrace{\frac{1-q^n}{p} + nq^n \left[\frac{1}{12}(n-1)(n-2)\left(\frac{p}{q}\right)^2 + \frac{1}{2}(n-1)\left(\frac{p}{q}\right) - 1\right]}_{T_n(p)}. \tag{20}$$

If $n = 2$ then $\beta_2 < 1 < 2.16/2$. Thus, from (20) it is sufficient to prove that for $n \ge 3$ and $p = 2.16/n$ we have $T_n(p) > 1/p$. Equivalently, this can be written as

$$\frac{1}{12}n(n-1)(n-2)\left(\frac{p}{q}\right)^2 + \frac{1}{2}n(n-1)\left(\frac{p}{q}\right) - n - \frac{1}{p} > 0,$$

or, after replacing $p = 2.16/n$, approximately as

$$\frac{0.006n(n+300.35)(n-2.12)}{(n-2.16)^2} > 0,$$

which is obviously true for $n \ge 3$.
(iii) By (ii) it is sufficient to prove that $B_1(n+1, \alpha_n) > \frac{1}{\alpha_n}$. Then let $p = \alpha_n$ and $q = 1-\alpha_n$. We have

$$\begin{aligned}
B_1(n+1,p) &= q\left(\frac{n+1}{n}\right)B_1(n,p) + 1 - q^{n+1}, \text{ by Proposition 3.1,} \\
&= (1-p)\left(1 + \frac{1}{n}\right)\frac{1}{p} + 1 - q^{n+1}, \text{ by (16),} \\
&= \frac{1}{p} + \frac{q}{np}\left(1 - npq^n\right), \\
&> \frac{1}{p}, \text{ by Lemma 1.4 - (iii) when } r = 1.
\end{aligned}$$

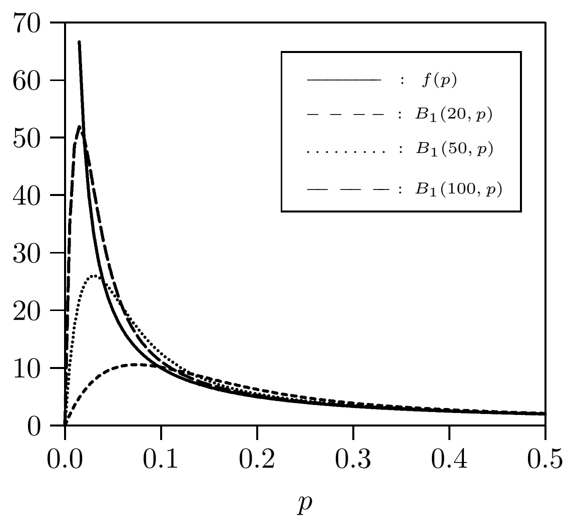
Table 2: Values of α_n, β_n and $2.16/n$

n	α_n	β_n	$2.16/n$
5	0.288677	0.364920	0.432000
10	0.147419	0.191705	0.216000
20	0.074439	0.098133	0.108000
50	0.029946	0.039797	0.043200
100	0.015001	0.019989	0.021600
200	0.007507	0.010017	0.010800
500	0.003005	0.004012	0.004320
1000	0.001503	0.002007	0.002160
2000	0.000751	0.001004	0.001080
5000	0.000301	0.000402	0.000432

(iv) It follows from (i). ∎

Table 2 illustrates Proposition 3.3 - (i) numerically. We see that for large values of n the upper bound $2.16/n$ can be taken as an approximation of β_n. Figure 1 illustrates Propositions 3.4, 3.2 and 3.3 for values of p between 0 and 0.5 and for $k = 1$.

Figure 1. Graphs of $B_1(n,p)$ and $f(p) = 1/p$



Proposition 3.4 For $n \geq 2$ and $r > 0$ the equation

$$B_1(n,p) = \frac{1}{p} + rq^n, \tag{21}$$

has a unique solution ${}_r\gamma_n$ in $]0,1[$ such that

(i) $\beta_n < {}_r\gamma_n$, where β_n is given by Proposition 3.3. If $0 < r \leq 3$, then ${}_r\gamma_n < 2.25/n$.

(ii) $B_1(n,p) < \dfrac{1}{p} + rq^n,\ \forall\, p \in\,]0, {}_r\gamma_n[$ and $B_1(n,p) > \dfrac{1}{p} + rq^n,\ \forall\, p \in\,]{}_r\gamma_n, 1[$.

(iii) $\{{}_r\gamma_n\}$ is decreasing as a function of n.

Proof. (i) - (ii) The inequality $\frac{1}{p} < \frac{1}{p} + rq^n$ and Proposition 3.3 show that (21) has no solution in $]0, \beta_n]$. Therefore, (21) can be written as $h(p) = 0$, where $h(p) = g(p) - r$, for $p \in [\beta_n, 1[$ and g is defined by (17). This function h is strictly increasing and continuous in $[\beta_n, 1[$. Besides, (19) implies that $h(\beta_n) = g(\beta_n) - r = -r < 0$, and from (18) it follows that $\lim_{p\to 1^-} h(p) = +\infty$. Then, there exists a unique ${}_r\gamma_n \in\,]\beta_n, 1[$ such that $h(p) < h({}_r\gamma_n) = 0 < h(t)$, for all $p \in\,]\beta_n, {}_r\gamma_n[$ and all $t \in\,]{}_r\gamma_n, 1[$. To prove the last inequality of (i) it is sufficient to prove that $B_1(n,p) > \frac{1}{p} + 3q^n$, for $p = 2.25/n$ and $n \geq 3$. From (20) this reduces to

$$\frac{1}{12}n(n-1)(n-2)\left(\frac{p}{q}\right)^2 + \frac{1}{2}n(n-1)\left(\frac{p}{q}\right) - n - \frac{1}{p} - 3 > 0.$$

Replacing $p = 2.25/n$ the latter becomes approximately

$$\frac{0.10(n-2.19)(n^2 - 11.69n + 67.77)}{(n-2.25)^2} > 0,$$

which is true for $n \geq 3$.
(iii) For $p = {}_r\gamma_n$ and $q = 1 - {}_r\gamma_n$ we have

$$\begin{aligned}
B_1(n+1,p) &= q\left(\frac{n+1}{n}\right)\left(\frac{1}{p} + rq^n\right) + 1 - q^{n+1}, \quad \text{by Propositon 3.1 and (21)},\\
&= \frac{1}{p} + rq^{n+1} + \frac{q}{np}\left(1 + rpq^n - npq^n\right),\\
&> \frac{1}{p} + rq^{n+1} + \frac{q}{np}\left(1 - npq^n\right),\\
&> \frac{1}{p} + rq^{n+1}, \quad \text{by Lemma 1.4 -(iii)}.
\end{aligned}$$

This and (ii) complete the proof. ■

As an illustration of Proposition 3.4 - (i), Table 3 reports values of ${}_r\gamma_n$ for the special cases where r is given (in actuarial symbols) by

$$r \;=\; {}_tr_x \;=\; \left(\frac{\ddot{s}_{\overline{t}|}}{\ddot{s}_{x:\overline{t}|} - \ddot{s}_{\overline{t}|}}\right)\frac{{}_tq_x}{{}_tp_x} - 1,$$

and ${}_tp_x$ is derived using the same mortality law used by Bowers et. al. [3], p. 78, i.e. $\mu_x = A + Bc^x$, where $A = 0.0007$, $B = 0.00005$ and $c = 10^{0.04}$, for $x = 20, 30, 40, 50, 60$, $t = 65 - x$ and $i = 0.06$. For comparison we also include β_n and $2.25/n$. We note that since values ${}_tr_x$ are small then values of ${}_r\gamma_n$ are close to β_n.

Proposition 3.5 For each $n \geq 3$ there exists a unique $0 < \delta_n < 1$ such that

Table 3: Values of β_n, $2.25/n$ and ${}_r\gamma_n$ for $r = {}_tr_x$

		$x=20$	$x=30$	$x=40$	$x=50$	$x=60$	
	${}_tp_x$	0.783335	0.792934	0.808959	0.841699	0.920114	
	${}_tr_x$	0.138972	0.201205	0.301175	0.447811	0.536277	
n	β_n	${}_r\gamma_n$					$2.25/n$
2	0.767592	0.774291	0.777179	0.781682	0.787994	0.791644	
5	0.364920	0.367624	0.368826	0.370744	0.373532	0.375198	0.450000
10	0.191705	0.192551	0.192928	0.193534	0.194419	0.194951	0.225000
20	0.098133	0.098367	0.098472	0.098640	0.098886	0.099034	0.112500
50	0.039797	0.039837	0.039854	0.039883	0.039925	0.039950	0.045000
100	0.019989	0.019999	0.020003	0.020011	0.020021	0.020028	0.022500
200	0.010017	0.010020	0.010021	0.010023	0.010025	0.010027	0.011250
300	0.006683	0.006684	0.006685	0.006685	0.006687	0.006687	0.007500
400	0.005014	0.005015	0.005015	0.005016	0.005016	0.005017	0.005625
500	0.004012	0.004013	0.004013	0.004013	0.004014	0.004014	0.004500

(i) $B_1(n,\delta_n) = \dfrac{1}{\delta_n} + \dfrac{(1-\delta_n) - [2(n-1)\delta_n + 1](1-\delta_n)^n}{(n-1)\delta_n^2}$.

(ii) $B_1(n,p) < \dfrac{1}{p} + \dfrac{q - [2(n-1)p+1]q^n}{(n-1)p^2}$, $\forall\, p \in\,]0,\delta_n[$,

$B_1(n,p) > \dfrac{1}{p} + \dfrac{q - [2(n-1)p+1]q^n}{(n-1)p^2}$, $\forall\, p \in\,]\delta_n,1[$.

(iii) $B_1(n,p)$ is concave in $]0,\delta_n]$ and convex in $]\delta_n,1]$.

(iv) The following inequalities hold for $n \geq 4$:

$$\frac{2}{n} \leq \delta_n \leq v_n = \frac{\sqrt{4(n-2)(13n-35)} - 4(n-2)}{\sqrt{4(n-2)(13n-35)} + (n-2)(n-7)} \leq \frac{9}{2n+5}, \tag{22}$$

$$\beta_n < \delta_n, \text{ where } \beta_n \text{ is given by Proposition 3.3.} \tag{23}$$

(v) $\{\delta_n\}$ is decreasing.

(vi) $\lim\limits_{n\to\infty} \delta_n = 0$.

Proof. We prove that $B_1''(n,p)$ changes sign exactly once in $]0,1[$. This is not true for $n < 3$ since B_1 is a polynomial of degree n. For $0 < p < 1$ we have from (10)

$$\frac{d^2B_1(n,p)}{dp^2} = n\left[\sum_{k=0}^{n-1} \frac{k(k-1)q^{k-2}}{n-k} - n(n-1)q^{n-2}\sum_{k=1}^{n}\frac{1}{k}\right] = nq^{n-2}h(p), \tag{24}$$

where

$$h(p) = \sum_{k=0}^{n-1} \frac{k(k-1)}{n-k} q^{-(n-k)} - n(n-1)\sum_{k=1}^{n}\frac{1}{k}. \tag{25}$$

Then h is continuous, strictly increasing in $]0,1[$, with $\lim_{p\to 1^-} h(p) = +\infty$ and

$$\lim_{p\to 0^+} h(p) \;=\; \sum_{k=0}^{n-1} \frac{k(k-1)}{n-k} - n(n-1)\sum_{k=1}^{n}\frac{1}{k} \;<\; \sum_{k=0}^{n-1}\frac{n(n-1)}{n-k} - n(n-1)\sum_{k=1}^{n}\frac{1}{k} \;=\; 0.$$

Therefore, there exists a unique $\delta_n \in\,]0,1[$ such that $h(p) < h(\delta_n) = 0 < h(t)$, for all $p \in\,]0,\delta_n[$ and all $t \in\,]\delta_n, 1[$. To prove (i)-(iii) we still need to verify that for $0 < p < 1$,

$$B_1''(n,p) = 0 \Leftrightarrow B_1(n,p) = \frac{1}{p} + \frac{q - [2(n-1)p+1]q^n}{(n-1)p^2}. \tag{26}$$

From (24) we see that $B_1''(n,p) = 0$ if and only if

$$\begin{aligned}
0 \;&=\; \sum_{k=1}^{n} \frac{(n-k)(n-k-1)}{k} q^{n-k} - n(n-1)q^n \sum_{k=1}^{n}\frac{1}{k},\\
&=\; \sum_{k=1}^{n} \frac{n(n-1) + (-2n+1)k + k^2}{k} q^{n-k} - n(n-1)q^n\sum_{k=1}^{n}\frac{1}{k},\\
&=\; (n-1)\left[\sum_{k=1}^{n}\frac{n}{k}q^{n-k} - \sum_{k=1}^{n}\frac{n}{k}q^n\right] + \sum_{k=1}^{n}\frac{(-2n+1)k}{k}q^{n-k} + \sum_{k=1}^{n} kq^{n-k},\\
&=\; (n-1)B_1(n,p) + (-2n+1)\frac{(1-q^n)}{p} + \sum_{k=0}^{n-1}(n-k)q^k,\\
&=\; (n-1)B_1(n,p) + (-2n+1)\frac{(1-q^n)}{p} + n\frac{(1-q^n)}{p} - q\frac{d}{dq}\left(\sum_{k=0}^{n-1} q^k\right),\\
&=\; (n-1)\left[B_1(n,p) - \frac{1}{p}\right] + (n-1)\frac{q^n}{p} - q\left(\frac{-npq^{n-1} + 1 - q^n}{p^2}\right),\\
&=\; (n-1)\left[B_1(n,p) - \frac{1}{p}\right] - \frac{q - [2(n-1)p+1]q^n}{p^2}.
\end{aligned}$$

This yields (26).

(iv) We may write (25) as

$$h(p) = \sum_{k=1}^{n} \frac{(n-k)(n-k-1)q^{-k} - n(n-1)}{k}. \tag{27}$$

We can get $h(p) < 0$ and thus, $B_n''(p) < 0$, if the condition $(n-k)(n-k-1)q^{-k} - n(n-1) < 0$, for all $k = 1,\ldots,n-2$ is added to (27). This condition is equivalent to $q > \left[\frac{(n-k)(n-k-1)}{n(n-1)}\right]^{1/k}$, for $k = 1,\ldots,n-2$. From Corollary 1.3 - (ii), we obtain $q > \max_{\{k=1,\ldots,n-2\}}\left[\frac{(n-k)(n-k-1)}{n(n-1)}\right]^{1/k} = \frac{n-2}{n}$ or $p < 2/n$. This proves the left hand side inequality of (22). To prove the other inequalities in (22), use (4) in (27) to get

$$\begin{aligned} h(p) &\geq \sum_{k=1}^{n}\left[\frac{(n-k)(n-k-1)u(k,p)-n(n-1)}{k}\right], \\ &\geq \sum_{k=1}^{n}\left[\frac{(n-k)(n-k-1)v(k,p)-n(n-1)}{k}\right], \end{aligned}$$

or

$$\begin{aligned} h(p) &\geq n(n-1)\underbrace{\left[\frac{1}{24}(n-2)(n-3)\left(\frac{p}{q}\right)^2+\frac{1}{3}(n-2)\left(\frac{p}{q}\right)-\frac{3}{2}\right]}_{D_2}, \\ &\geq n(n-1)\underbrace{\left[\frac{1}{3}(n-2)\left(\frac{p}{q}\right)-\frac{3}{2}\right]}_{D_1}. \end{aligned}$$

Solving $D_2 > 0$ for $n \geq 4$ we get $h(p) > 0$, i.e. $B''(p) > 0$, for $p > v_n$. This proves the second inequality of (22). Now solving $D_1 > 0$ for $n \geq 4$ we get $B_1''(n,p) > 0$ for $p > \frac{9}{2n+5}$. This leads to $\delta_n \leq \frac{9}{2n+5}$. The last inequality of (22) follows since $D_2 \geq D_1$. From (i), the first inequality of (22), Lemma 1.4 - (ii) and Proposition 3.3 - (ii) we obtain (23).
(v) From (ii) it is sufficiet to prove that for $p = \delta_n$ and $q = 1-\delta_n$

$$B_1(n+1,p) > \underbrace{\frac{1}{p}+\frac{q}{np^2}\left[1-(2np+1)q^n\right]}_{L_n(p)}.$$

To this purpose, use Proposition 3.1 and (i). After some algebraic manipulations we get:

$$\begin{aligned} B_1(n+1,p) &= \left(\frac{n-1}{n}+\frac{2}{n}\right)\left\{\frac{q}{p}+\frac{q}{(n-1)p^2}\left[q-(2np+1)q^n+2pq^n\right]\right\}+1-q^{n+1}, \\ &= L_n(p)+\frac{2q^2}{n(n-1)p^2}\left\{1-\left[\frac{n(n-1)}{2}p^2+(n-1)p+1\right]q^{n-1}\right\}, \\ &> L_n(p), \text{ from Lemma 1.4 -(iv).} \end{aligned}$$

(vi) It follows from (iv). ■

Proposition 3.5 - (iv) is illustrated numerically by Table 4. It appears that $2.87/n$ provides an adequate approximation for δ_n when n is large.

Proposition 3.6 Let $n \geq 2$. The function

$$\varphi_n(p) = \begin{cases} \dfrac{n(1-q^n)}{(n-1)p} - B_1(n,p) & \text{if } 0 < p \leq 1, \\ \dfrac{n^2}{n-1} & \text{if } p = 0, \end{cases}$$

has the following properties:

Table 4: Values of δ_n, lower and upper bounds and $2.87/n$

n	$2/n$	β_n	δ_n	υ_n	$9/(2n+5)$	$2.87/n$
5	0.400000	0.364920	0.530865	0.537525	0.600000	0.574000
10	0.200000	0.191705	0.277070	0.292359	0.360000	0.287000
20	0.100000	0.098133	0.141104	0.153010	0.200000	0.143500
50	0.040000	0.039797	0.057023	0.062978	0.085714	0.057400
100	0.020000	0.019989	0.028606	0.031797	0.043902	0.028700
200	0.010000	0.010017	0.014326	0.015977	0.022222	0.014350
500	0.004000	0.004012	0.005736	0.006410	0.008955	0.005740
1000	0.002000	0.002007	0.002869	0.003208	0.004489	0.002870
2000	0.001000	0.001004	0.001435	0.001605	0.002247	0.001435
5000	0.000400	0.000402	0.000574	0.000642	0.000900	0.000574

(i) For $n \geq 3$, φ_n is strictly decreasing in $[0, \delta_n[$ and strictly increasing in $]\delta_n, 1]$. Therefore, for all $p \in [0,1]$, $p \neq \delta_n$ we have

$$\varphi_n(p) \quad > \quad \varphi_n(\delta_n) \;=\; \frac{1}{(n-1)\delta_n^2}\left\{2\delta_n - 1 + [(n-2)\delta_n + 1](1-\delta_n)^n\right\}, \tag{28}$$

where δ_n is the one given by Proposition 3.5.

(ii) $\varphi_n(p) > 0$, for all $0 \leq p \leq 1$ and $2 \leq n \leq 5$.

(iii) $\varphi_n(p) > \left(\frac{nq}{n-1}\right)\varphi_{n-1}(p)$, for all $\frac{n}{2(n-1)} \leq p \leq 1$.

(iv) For $n \geq 6$, φ_n has two roots ϵ_n and ε_n in $]0,1[$ such that $\frac{1}{n-1} \leq \epsilon_n$, $1/2 < \varepsilon_n < 7/12$, and

$$\begin{array}{lllllllll} 0 & = & \varphi_n(\epsilon_n) & < & \varphi_n(p) & < & \varphi_n(0) & = & \frac{n^2}{n-1}, \quad \text{for all } 0 < p < \epsilon_n, \\ & & \varphi_n(\delta_n) & < & \varphi_n(p) & < & \varphi_n(\varepsilon_n) & = & 0, \quad \text{for all } \epsilon_n < p < \varepsilon_n, \\ 0 & = & \varphi_n(\varepsilon_n) & < & \varphi_n(p) & < & \varphi_n(1) & = & \frac{1}{n-1}, \quad \text{for all } \varepsilon_n < p < 1. \end{array}$$

If $n \geq 10$ we also have $\epsilon_n \leq \beta_n$, where β_n is given by Proposition 3.3.

Proof. (i) We may write φ_n as

$$\varphi_n(p) = -\left(\frac{n}{n-1}\right)\sum_{k=1}^{n}\frac{(n-k-1)q^{n-k} - (n-1)q^n}{k}. \tag{29}$$

This and (24) yield $\varphi_n'(p) = \left(\frac{q}{n-1}\right)B_n''(p)$. Proposition 3.5 completes the proof.
(ii) It is easily seen that the result holds for $n = 2$. For $3 \leq n \leq 5$ we have $\delta_n > 1/2$ or $2\delta_n - 1 > 0$. Then from (28) we get $\varphi_n(p) \geq \varphi_n(\delta_n) > 0$, for all $0 \leq p \leq 1$.

(iii) The result holds for $p = 1$. Let $\frac{n}{2(n-1)} \leq p < 1$. Since the function $\frac{n-(n-1)p}{n(1-p)}$ is increasing in $]0,1[$, we get $\frac{n-(n-1)p}{nq} \geq \frac{n-1}{n-2}$. From this and Proposition 3.1

$$\begin{aligned}
\varphi_n(p) &= \frac{n(1-q^n)}{(n-1)p} - \left\{ q\left[\frac{n}{n-1}\right] B_1(n-1,p) + 1 - q^n \right\}, \\
&= \frac{nq}{n-1}\left\{\left[\frac{n-(n-1)p}{nq}\right]\left[\frac{1-q^n}{p}\right] - B_1(n-1,p)\right\}, \qquad (30)\\
&> \frac{nq}{n-1}\left\{\left[\frac{n-(n-1)p}{nq}\right]\left[\frac{1-q^{n-1}}{p}\right] - B_1(n-1,p)\right\}, \\
&\geq \frac{nq}{n-1}\left[\frac{(n-1)(1-q^{n-1})}{(n-2)p} - B_1(n-1,p)\right] = \frac{nq}{n-1}\varphi_{n-1}(p).
\end{aligned}$$

(iv) Let $0 \leq p \leq \alpha_n$. From Proposition 3.2 - (iv) and since for these values of p, $\varphi_n(p) > \frac{1-q^n}{p} - B_1(n,p) \geq 0$, we get $\varphi_n(p) > 0$. For $n = 6$, by direct computation we can see that if $p > 0.52$ then $\varphi_6(p) > 0$. Since for $n \geq 7$ we have $\frac{n}{2(n-1)} \leq 7/12$, it follows from (iii) that $\varphi_n(p) > 0$, for $7/12 \leq p \leq 1$. Now take $\delta_n \leq p \leq 1/2$. Since in this interval φ_n is increasing we have $\varphi_n(p) \leq \varphi_n(1/2)$. From (30) and Proposition 3.5 - (ii), we get

$$\begin{aligned}
\varphi_n(p) &= \frac{nq}{n-1}\left\{\left[\frac{n-(n-1)p}{nq}\right]\left[\frac{1-q^n}{p}\right] - B_1(n-1,p)\right\}, \\
&< \frac{nq}{n-1}\left\{\left[\frac{n-(n-1)p}{nq}\right]\frac{1}{p} - \frac{1}{p} - \frac{q-[2(n-2)p+1]q^{n-1}}{(n-2)p^2}\right\}, \\
&= \frac{nq}{(n-1)p}\left\{\left[\frac{n-(n-1)p}{nq}\right] - 1 - \frac{q-[2(n-2)p+1]q^{n-1}}{(n-2)p}\right\}, \\
&= \frac{1}{(n-1)(n-2)p^2}\underbrace{\left\{-2p^2 + n(2p-1) + n[2(n-2)p+1]q^n\right\}}_{H}.
\end{aligned}$$

If $p = 1/2$ then $H = -\frac{1}{2} + \frac{n(n-1)}{2^n} < 0$ for $n \geq 6$. Hence $\varphi_n(1/2) < 0$ and so $\varphi_n(p) < 0$ for $\delta_n \leq p \leq 1/2$. Now, since $\varphi_n(0) > 0$, $\varphi_n(1/2) < 0$, $\varphi_n(1) > 0$ and φ_n is a continuous function with $\varphi_n'(p)$ vanishing only at δ_n, then φ_n has exactly two roots ϵ_n and ε_n in $]0,1[$ such that $\alpha_n < \epsilon_n < \delta_n$ and $1/2 < \varepsilon_n < 7/12$. We need to prove that $\frac{1}{n-1} \leq \epsilon_n \leq \beta_n$. The first inequality follows from (29). In fact it is then sufficient to prove that for $p < \frac{1}{n-1}$

$$\sum_{k=1}^{n} \frac{(n-k-1)q^{-k} - (n-1)}{k} < 0.$$

This is true since by Corollary 1.3 - (i), $(n-k-1)q^{-k} - (n-1) < 0$ for $q > \frac{n-2}{n-1}$ or $p < \frac{1}{n-1}$. Now we have to prove that $\epsilon_n \leq \beta_n$ for $n \geq 10$. Let $\beta_n < p < 1 - n^{-1/n}$. Then from Proposition 3.3 - (ii), we have

$$\begin{aligned}
\varphi_n(p) &= \frac{n(1-q^n)}{(n-1)p} - B_1(n,p) < \frac{n(1-q^n)}{(n-1)p} - \frac{1}{p}, \\
&= \frac{1}{(n-1)p}(1-nq^n) < 0, \quad \text{since } p < 1 - n^{-1/n}.
\end{aligned}$$

The inequalities are easily proved using ϵ_n, ε_n and (i). ■

We observe that $\lim_{n\to\infty} \epsilon_n = 0$. Also, from Proposition 3.6 - (iii) it appears that ε_n is close to $\frac{n}{2(n-1)}$. In fact our empirical studies show that this expression provides a good approximation for ε_n. This is shown in Table 5 which reports values of ϵ_n and ε_n. We see that ε_n tends to 1/2 as n increases.

In summary, the inequalities of Proposition 3.6 - (iv) may be written in terms of $B_1(n,p)$ and $b_1(n,p)$, as in the following corollary. For small values of np the inequalities of Propositions 3.2 and 3.3 and of Corollary 3.1 are superior; the latter have been included instead.

Table 5: Values of ϵ_n, ε_n, $1/(n-1)$ and $n/[2(n-1)]$

n	$1/(n-1)$	ϵ_n	ε_n	$n/[2(n-1)]$
6	0.200000	0.360000	0.513290	0.600000
10	0.111111	0.182082	0.558385	0.555556
20	0.052632	0.081842	0.528293	0.526316
50	0.020408	0.031047	0.510437	0.510204
100	0.010101	0.015270	0.505104	0.505051
200	0.005025	0.007574	0.502526	0.502513
500	0.002004	0.003015	0.501004	0.501002
1000	0.001001	0.001505	0.500501	0.500501
2000	0.000500	0.000752	0.500250	0.500250
5000	0.000200	0.000301	0.500100	0.500100

Corollary 3.5 Let $n \geq 2$ and $\alpha_n, \beta_n, \epsilon_n, \varepsilon_n$ and δ_n given in the previous propositions.

(i) If $0 < p < \alpha_n$ then

$$\frac{(1-q^n)^2}{p} < B_1(n,p) < \frac{1-q^n}{p},$$
$$\frac{1-q^n}{np} < b_1(n,p) < \frac{1}{np}.$$

(ii) If $\alpha_n < p < \beta_n$ then

$$\frac{1-q^n}{p} < B_1(n,p) < \frac{1}{p},$$
$$\frac{1}{np} < b_1(n,p) < \frac{1}{np(1-q^n)}.$$

(iii) For $\epsilon_n < p < \varepsilon_n$ and $\varphi_n(\delta_n)$ given by (28),

$$\left(\frac{n}{n-1}\right)\left(\frac{1-q^n}{p}\right) < B_1(n,p) < \left(\frac{n}{n-1}\right)\left(\frac{1-q^n}{p}\right) - \varphi_n(\delta_n),$$
$$\frac{1}{(n-1)p} < b_1(n,p) < \frac{1}{(n-1)p} - \frac{\varphi_n(\delta_n)}{n(1-q^n)}.$$

(iv) If $\varepsilon_n < p < 1$ hen

$$\begin{array}{rcl} \left(\frac{n}{n-1}\right)\left(\frac{1-q^n}{p}\right)-\frac{1}{n-1} & < B_1(n,p) < & \left(\frac{n}{n-1}\right)\left(\frac{1-q^n}{p}\right), \\ \frac{1}{(n-1)p}-\frac{1}{n(n-1)(1-q^n)} & < b_1(n,p) < & \frac{1}{(n-1)p}. \end{array}$$

Proof. Apply Corollary 3.1 and Propositions 3.2, 3.3 and 3.6. ■

When n is large we can say that (i) and (ii) of Corollary 3.5 do not hold since in these cases β_n is almost null. Hence we can say that (iii) holds for almost the whole interval $]0,1/2[$ and (iv) holds for almost the whole interval $]1/2,1[$.

4 Conclusion

The previous sections give a number of interesting properties and bounds for the k^{th} inverse moment $b_k(n,p)$ of a positive binomial variate and for its associated Bernstein polynomial $B_k(n,p)$. In particular, Corollary 3.5 gives simple bounds for the first inverse moment $b_1(n,p)$ and its associated $B_1(n,p)$.

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