SOME ASPECTS IN N-DIMENSIONAL
ALMOST PERIODIC FUNCTIONS III

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Abstract

The properties of almost periodical functions and some new results have been published in [CA1], [CA2] and [CA3] In this paper we show some new definitions in order to analyze some singularities. For this functions we find some uniqueness sets in \( \mathbb{R} \) and \( \mathbb{R}^n \). The paper finishes analizing the relation of this functions and the function \( \text{sinc} \).

Keywords: Almost periodic functions, structure theorem, Radon transform.

Resumen

Las propiedades de las funciones cuasiperiódicas y algunos resultados nuevos se han presentado en [CA1], [CA2] y [CA3]. En este artículo variamos un poco la definición para incluir cierto tipo de singularidades y encontramos para estas funciones algunos conjuntos numerables de unicidad en \( \mathbb{R} \) y en \( \mathbb{R}^n \). El artículo termina analizando la relación entre estas funciones y la función \( \text{sinc} \).

Palabras clave: Funciones cuasiperiódicas, teorema de estructura, transformada de Radon.

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1 Some notations and reminders

Elementary properties of some sets of almost periodic functions have been published in [Ca], [CO], [A-P], [BO], [COR] This paper is a natural continuation of [CA1], [CA2] and [CA3]. We keep the basic notations and results.

Let us summarize some important results:

\( f : \mathbb{R}^N \to \mathbb{R} \) is an almost periodic function if \( \forall \varepsilon > 0 \) there is a \( N \)-dimensional vector \( L \) whose entries are positive and satisfies that \( \forall y \) in \( \mathbb{R}^N \) there is an \( T \) in the \( N \)-dimensional box \([y, y+L]\) (component wise) such that \( |f(x+T) - f(x)| < \varepsilon \) for all \( x \) in \( \mathbb{R}^N \).

Let \( x \in \mathbb{R}^N \), \( x[i] \) denotes the \( i \)-th component of \( x \). We write \( x > 0 \) if \( x[i] > 0 \), \( i = 1, \ldots, N \).

If \( x, y \) are in \( \mathbb{R}^N \) we write:

\[
|x - y| := \begin{pmatrix}
\vdots \\
|x[N] - y[N]| 
\end{pmatrix}
\]

In the case of the usual functions \( \sin, \cos, \exp, \text{sinc} \), we write: \( \sin : \mathbb{R}^N \to \mathbb{R} \) as

\[
\sin \begin{pmatrix}
x_1 \\
\vdots \\
x_N 
\end{pmatrix} := \sin(x_1) * \ldots * \sin(x_N)
\]

and the same definition holds for the other functions. In general we extend in the multiplicative way any finite family of functions.

A set \( E \subset \mathbb{R}^N \) is called relatively dense (r.d) if there is an \( L \in \mathbb{R}^N, L > 0 \) such that for all \( a \in \mathbb{R}^N \), \([a, a+L] \cap E \neq \emptyset\).

There are many examples of r.d sets, for instance:

- \( \mathbb{Z} \) and \( p\mathbb{Z} \), which that \( p \in \mathbb{R} \) and \( p \notin \mathbb{Z} \), are r.d in \( \mathbb{R} \).
- \( \mathbb{Z}^N, p_1\mathbb{Z} \times \ldots \times p_N\mathbb{Z}, p_i \notin \mathbb{Z}, i = 1, \ldots, N \) are r.d in \( \mathbb{R}^N \).
- If \( A \) is an r.d set in \( \mathbb{R}^N \) and \( B \) is an r.d set in \( \mathbb{R}^M \) then \( A \times B \) is an r.d set in \( \mathbb{R}^{N+M} \).
- If \( A \) is an r.d set in \( \mathbb{R}^N \) and \( \pi_i : \mathbb{R}^N \to \mathbb{R} \) is the \( i \)-th projection then \( \pi_i[A] \) is an r.d set in \( \mathbb{R} \).
- If \( f : \mathbb{R}^N \to \mathbb{R}^N \) is an isometry then \( f[A] \) is an r.d set for any \( A \) r.d set in \( \mathbb{R}^N \).
- Let \( G \) in \( \mathbb{R}^N \) a discrete non trivial additive subgroup then \( G \) is r.d. also \( a + G \) is r.d. for all \( a \) in \( \mathbb{R}^N \).

\( C_b(\mathbb{R}^N, \mathbb{R}) \) denotes the set of all bounded functions from \( \mathbb{R}^N \to \mathbb{R} \) endowed with the norm \( \| \cdot \|_\infty \)

\( f[x+m] \) denotes the function \( x \to f[x+m], m \) fixed.
We use the following definition:

Let \( f : \mathbb{R}^N \to \mathbb{R} \) be an almost periodic function; \( f \) is said to have Bochner compact range (BCR) if for any \( N \)-dimensional sequence \( (x_n)_{n \in \mathbb{N}} \) there is a subsequence \( (x_{n_k})_{k \in \mathbb{N}} \) and \( x_0 \in \mathbb{R}^N \) such that \( f[x_0 + x_{n_k}] \to f[x_0] \) uniformly when \( k \to \infty \).

We proved in those papers results like:

- Let \( f : \mathbb{R}^N \to \mathbb{R} \) be a continuous function, \( f \) is almost periodic iff \( A = \{ f[x_0 + y] \mid y \in \mathbb{R}^N \} \) is relatively compact in \( C(\mathbb{R}^N, \| \cdot \|_\infty) \).
- \( f \) is almost periodic iff for any sequence \( (y_n)_{n \in \mathbb{N}} \) there is a subsequence \( (y_{n_k})_{k \in \mathbb{N}} \) and a function \( g : \mathbb{R}^N \to \mathbb{R} \) such that \( f[x_0 + y_{n_k}] \to g \) in \( C(\mathbb{R}^N, \| \cdot \|_\infty) \).
- Let \( f : \mathbb{R}^N \to \mathbb{R} \) be a uniformly continuous bounded function, \( (y_n)_{n \in \mathbb{N}} \subset \mathbb{R}^N \) be a sequence such that \( f[x_0 + y_n] \to g[x_0] \) uniformly, and let \( (x_n)_{n \in \mathbb{N}} \subset \mathbb{R}^N \) be a sequence such that \( x_n \to x_0 \). Then \( f[x_0 + y_n + x_n] \to g[x_0 + x_0] \).
- Let \( f : \mathbb{R}^N \to \mathbb{R} \) be a continuous bounded function, and let \( E \subset \mathbb{R}^N \), \( E \) r.d and \( \bigcup_{y \in E} \{ f[x_0 + y] \} \) relatively compact in \( C_b(\mathbb{R}^N, \| \cdot \|_\infty) \). Then \( f \) is uniformly continuous.
- (Haraux condition) Let \( f : \mathbb{R}^N \to \mathbb{R} \) be a continuous bounded function, \( E \subset \mathbb{R}^N \), \( E \) r.d and \( \bigcup_{y \in E} \{ f[x_0 + y] \} \) relatively compact in \( C_b(\mathbb{R}^N, \| \cdot \|_\infty) \), then \( f \) is almost periodic.
- Let \( f : \mathbb{R}^N \to \mathbb{R} \) be an almost periodic function that attains its maximum and minimum. Then for any sequence \( (x_n)_{n \in \mathbb{N}} \) there is a subsequence \( (x_{n_k})_{k \in \mathbb{N}} \) and \( x_0 \in \mathbb{R}^N \) such that \( f'[x_0 + x_{n_k}] \to f[x_0 + x_0] \) uniformly.
- Let \( f : \mathbb{R} \to \mathbb{R} \) be an almost periodic function, \( f \) is periodic if and only if \( f \) has Bochner compact range.

2 Periodic and almost periodic functions and its relations to some sets

It is well known that any non trivial additive subgroup \( G \) of \( \mathbb{R}^N \) such that for all \( x > 0 \), there exists \( g \in G \) with \( 0 < g < x \) (lexicographic) is dense in \( \mathbb{R}^N \). From that result it follows immediately that \( \{n + m \cdot r\} \) is dense in \( \mathbb{R} \) with \( n, m \) integers and \( r \) irrational. Without difficulties it is easy to prove the same result in \( \mathbb{R}^N \) with \( n, m \) in \( \mathbb{Z}^N \) and \( r \) irrational for \( i = 1, \ldots, N \), \( m \cdot r \) denotes the componentwise multiplication.

Interesting though is that from the above results it follows that:

- \( \{ \sin(n), n \in \mathbb{Z} \} \) and \( \{ \cos(n), n \in \mathbb{Z} \} \) are dense in \([-1, 1]\).
- \( \{ |\sin(n)|, n \in \mathbb{Z} \} \) and \( \{ |\cos(n)|, n \in \mathbb{Z} \} \) are dense in \([0, 1]\).
- \( \{ \sin(n), n \in G \} \) and \( \{ \cos(n), n \in G \} \) are dense in \([-1, 1]\), where \( G \) is any non trivial additive subgroup of \( \mathbb{R} \) such that for all \( x > 0 \), there is \( g \in G \) with \( 0 < g < x \).
The above statements can be formulated in $\mathbb{R}^N$, for example: \{sin(n), n \in \mathbb{Z}^N\} is dense in $[-1, 1]$.

**Definition 1** Let $G$ be any discreet non trivial additive group of $\mathbb{R}^N$. $L \subset \mathbb{R}^N$ is called a lattice — determined by $G$ — if $L = G$ or there exists $a \in \mathbb{R}^N$ with $L = a + G$.

It is easy to prove that any $n$-dimensional lattice is r.d.

In $\mathbb{R}$ a lattice $G$ has the form: $G = a + p\mathbb{Z}$, for $a, p$ in $\mathbb{R}$.

Let $f, g : \mathbb{R} \to \mathbb{R}$ be two periodic, non trivial, continuous functions, then $f/g$ is a continuous function except for a lattice $L$, $L = \{x \in \mathbb{R}/g(x) = 0\}$.

If $f, g$ have measurable periods $T_1, T_2$, then $f/g$ is periodic–measurable means $T_1/T_2 \in \mathbb{Q}$.

If $f, g$ have no measurable periods then $f/g$ is almost almost periodic (a.a.p). Here, non measurable means $T_1/T_2 /\in \mathbb{Q}$.

Let $A_p := \{g : \mathbb{R} \to \mathbb{R}, g \text{ continuous of period } p\}$.

**Theorem 1** If $p$ in $\mathbb{R}$ is an irrational number then $\mathbb{Z}$ is a uniqueness set for $A_p$.

**Proof:** $B = \{n + m \ast p/n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Then $f(x = n + m \ast p) = f(n)$ for all $n, m \in \mathbb{Z}$.

**Theorem 2** Let $f \in A_p$, with a uniqueness set $E$, then $f(x_\ast + z) \in A_p$ for all $z \in \mathbb{R}$ with the same uniqueness set $E$.

As a matter of fact sometimes if $f \in A_p$, $f$ an odd function, there is $z \in \mathbb{R}$ with $f(x_\ast + z)$ an even function.

Some examples are:

- $\sin(x_\ast)$ and $z = \pi/2$;
- $\sum_{k=0}^{p} a_k \sin((2k + 1)x)$ and $z = \pi/2, a_k \in \mathbb{R}, k = 0, \ldots, p$.
- For the odd function: $\sin(x_\ast) + \sin(2x_\ast) + \sin(3x_\ast) + \sin(4x_\ast)$ there is not such a $z$.

Some graphics illustrate this situation in Figures 1, 2 and 3.

**Theorem 3** If we take in consideration in $A_p$ only the even functions we obtain that $\mathbb{N}_0$ is a uniqueness set for this class of functions.

As examples we have:

- $\{\sin(n), n \in \mathbb{N}_0\}$ is dense in $[-1, 1]$.
- $\{\cos(n), n \in \mathbb{N}_0\}$ is dense in $[-1, 1]$.
- $\{|\sin(n)|, n \in \mathbb{N}_0\}$ is dense in $[0, 1]$.
- $\{|\cos(n)|, n \in \mathbb{N}_0\}$ is dense in $[0, 1]$. 
In the case \( p \in \mathbb{Q} \) we get:

**Theorem 4** If \( p \in \mathbb{R} \) is a rational number then \( \mathbb{Z}r \), \( r \) irrational, is a uniqueness set for \( A_p \).

\( \mathbb{Z} \) and \( \mathbb{Z}r \) are lattices. We may summarizes the result as: let \( f \) be a continuous function of period \( p \) then there is a lattice \( L \) which is a uniqueness set for \( A_p \).

This statement can be extended to the set of functions: \( B_p := \{ f/g | f, g \in A_p \} \). There are discontinuous functions on this set.

We introduce now the sets:

\[
AP_p := \{ f : \mathbb{R} \to \mathbb{R} | f \text{ almost periodic } \}
\]

and the set of a.a. functions \( BB_p \),

\[
BB_p := \{ f/g | f, g \in AP_p \}.
\]

Actually, those sets are vector spaces over \( \mathbb{R} \).
For instance we get: \( \{ \tan(n), n \in \mathbb{N}_0 \} \) is dense in \( \mathbb{R} \).

In the \( n \)-dimensional case there are several definitions of the concept of periodic function, but we work with the \( R \)-periodic concept: \( f : \mathbb{R}^N \to \mathbb{R} \) is an \( R \)-periodic function if there are \( N \) linearly independent vectors \( e_k, k = 1, \ldots, N \) such that: \( f(x + e_k) = f(x) \), \( \forall x \in \mathbb{R}^N \). The vectors \( e_k, k = 1, \ldots, N \) are called periods of \( f \).

We get that if \( f \) is \( R \)-periodic and all the \( e_k \) in the definition are irrational then \( \sum_{k=1}^{N} \mathbb{Z}e_k \) is an uniqueness set for the set of functions: \( A_{e_1, \ldots, e_N} := \{ f : \mathbb{R}^N \to \mathbb{R} \) is a continuous \( R \)-periodic function, with periods \( e_k, k = 1, \ldots, N \} \) and for \( B_{e_1, \ldots, e_N} := \{ f/g, f, g \in A_{e_1, \ldots, e_N} \} \); of course there are discontinuous functions on this set.

We have an immediate generalization of Theorem 2.

**Theorem 5** Let \( f \in A_{e_1, \ldots, e_N} \) with a uniqueness set \( E \), then \( f(x_+ + z) \in A_{e_1, \ldots, e_N} \) for all \( z \in \mathbb{R}^N \) with the same uniqueness set \( E \).

**Theorem 6** Let \( f \in A_{e_1, \ldots, e_N} \) then there exists a lattice \( L \) such that \( L \) is a uniqueness set of \( A_{e_1, \ldots, e_N} \).

3 The relation between sinc and \( A_p, B_p, AP_p, \) and \( BB_p \)

**Theorem 7** Let \( L \) be a numerable uniqueness lattice of a function \( f \) in \( A_p \) or \( AP_p \), \( L = \mathbb{Z}h \). Then \( \sum_{k \in L} f(kh) \sin(\frac{\pi}{N}(x - k)) \) is convergent toward \( f \). When \( f \in A_p \) this convergence is uniform. When \( f \in AP_p \) this convergence is uniform when restricted to compact sets. Over \( \mathbb{R}^N \) it holds the same result.

**Proof:** A detailed proof will appear elsewhere.

In an schematic way we proceed as follows: We associate to \( f \) a function \( f_c \in C_c(\mathbb{R}) \) and apply the Fourier band limited theory and Wiener-Paley like theorem.

A point wise proof in one variable is: Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous periodic function of period \( \pi \), let us consider the case \( f \) even.

Let \( a_n(x_\pi) := f(n) \sin(\pi(x - n)) + f(-n) \sin(\pi(x + n)), n \in \mathbb{N} \), then \( a_n(x_\pi) = (-1)^n 2 \frac{f(n)}{\pi} \sin(\pi x) \frac{x}{x^2 - n^2} \) from this follows the convergence over compact sets of \( \sum_{n=0}^{\infty} a_n(x_\pi) \) toward a function \( g \). It follows immediately that \( g(n) = f(n) \) for all \( n \in \mathbb{Z} \) then \( f = g \).

In the odd case we have: \( a_n(x_\pi) := f(n) \sin(\pi(x - n)) + f(-n) \sin(\pi(x + n)), n \in \mathbb{N} \), then: \( a_n(x_\pi) = (-1)^n 2 \frac{f(n)}{\pi} \sin(\pi x) \frac{n}{x^2 - n^2} \) from this follows the point wise convergence.

In the general case of a continuous periodic function \( f \) of period \( \pi \) we get that: \( f(x_\pi) = f(\frac{x}{2}) + f(\frac{x}{2}) + f(\frac{x}{2}) - f(-\frac{x}{2}) + f(-\frac{x}{2}) - f(-\frac{x}{2}) \) is an even periodic function and \( f(x) - f(-x) \) is an odd periodic function, by using the preceding method we get the result. The choice of the period \( \pi \) is irrelevant, the same with respect to the choice of the lattice \( \mathbb{Z} \).

At this moment we do not know what happens to \( \sum_{k \in L = \mathbb{Z}h} f(kh) \sin(\frac{\pi}{N}(x - k)) \) when \( f \) belongs to \( B_p \) or \( BB_p \).

However, it is that a function \( f \) in \( BB_p \) has not necessarily the property that for any sequence \( (x_n) \in \mathbb{R} \) there is a subsequence \( (x_{n_k}) \) such that \( f(x_\pi + x_{n_k}) \to g \).
An easy counterexample is: \( f(x) := \frac{\sin(\sqrt{2}x)}{\sin(x)} \)

We define: \( x_1 = [2\pi], x_2 = [2 \cdot 2\pi] + 0.d_1 \ldots, x_n = [n \cdot 2\pi] + 0.d_1 \ldots d_{n-1} \), where 0.d_1 \ldots d_{n-1} denotes the \( n-1 \) decimal expansion of the number \( n \cdot 2\pi \).

4 Some graphical examples

Let us see the graphics in the interval \([-2\pi, 2\pi]\).

![Figure 4](image.png)

Figure 4: \( \sum_{k=-5}^{5} \frac{\sin(k) \cdot \sin(\pi \cdot (x - k))}{\pi \cdot (x - k)} \).

![Figure 5](image.png)

Figure 5. \( \sin(x) \).

![Figure 6](image.png)

Figure 6: \( \sum_{k=-5}^{5} \frac{\sin(k) \cdot \sin(\pi \cdot (x - k))}{\pi \cdot (x - k)} \).

![Figure 7](image.png)

Figure 7. \( \sum_{k=-10}^{10} \frac{\sin(k) \cdot \sin(\pi \cdot (x - k))}{\pi \cdot (x - k)} \).

See the case of the tangent in \((-\pi/2, \pi/2)\) in Figure 10.
Figure 8: \[ \sum_{k=-10}^{10} \frac{\sin(k) \cdot \sin(\pi \cdot (x - k))}{(\pi \cdot (x - k))}. \]

Figure 9: \[ \sum_{k=-5}^{5} \frac{\tan(k) \cdot \sin(\pi \cdot (x - k))}{(\pi \cdot (x - k))}. \]

Figure 10: \[ \tan(x). \]

Figure 11: \[ \sum_{k=-100}^{100} \frac{\tan(k) \cdot \sin(\pi \cdot (x - k))}{\pi \cdot (x - k)}. \]

References


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