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A note on Dirichlet and Fejér kernels

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Abstract

We exhibit a trigonometric identity wich implies a link between the kernels of Dirichlet and Fejér.

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Resumen

Exhibimos una identidad trigonométrica que implica una relación entre los núcleos de Dirichlet y de Fejér.

Mathematics Subject Classification: 42A20, 42A24, 43A50.

1 Introduction

In the original approach to Fourier series [1], it is convenient to consider the following partial sums for the interval $[-\pi, \pi]$:

$$f_n(y) = \frac{1}{2}a_0 + a_1 \cos y + \cdots + a_n \cos(ny) + b_1 \sin y + \ldots b_n \sin(ny) \tag{1.1}$$

assuming for a_r, b_r the values:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(mt)dt, \qquad b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(mt)dt, \tag{1.2}$$

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and investigate what happens if n increases to infinity. From (1.1) and (1.2) we obtain:

$$f_n(y) = \int_{-\pi}^{\pi} f(t) \underset{D}{K_n}(t-y)dt, \tag{1.3}$$

with the Dirichlet kernel [2]-[4]:

$$\underset{D}{K_n}(\theta) = \frac{1}{2\pi} \frac{\sin\,(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}} \tag{1.4}$$

Then the convergence:

$$\lim_{n\to\infty} f_n(y) = f(y),$$

has to be restricted to a definite class of functions $f(y)$ verifying the known Dirichlet conditions [2]-[4].

Fejér [5] invented a new method of summing the Fourier series by which he greatly extended the validity of the series. Using the arithmetic means of the partial sums (1.1), instead of the $f_n(y)$ themselves, he could sum series which were divergent. The only condition the function still has to satisfy is the natural restriction that $f(y)$ shall be absolutely integrable. Then, in the Fejér approach we construct the sequence:

$$g_1(y) = f_0(y), \qquad g_2(y) = \frac{1}{2}[f_0(y) + f_1(y)], \qquad g_3(y) = \frac{1}{3}[f_0(y) + f_1(y) + f_2(y)], \ldots,$$
$$g_n(y) = \frac{1}{n}[f_0(y) + f_1(y) + \cdots + f_{n-1}(y)], \tag{1.5}$$

accepting the expressions (1.1) and (1.2), therefore:

$$g_n(y) = \int_{-\pi}^{\pi} f(t) \underset{F}{K_n}(t-y)dt, \tag{1.6}$$

thus we see that Fejér's results come about by the fact that his method is related with the following Kernel [2]-[4],[5]:

$$\underset{F}{K_n}(\theta) = \frac{1}{2\pi n} \frac{\sin^2(n\frac{\theta}{2})}{\sin^2(\frac{\theta}{2})} \tag{1.7}$$

then a $f(y)$ absolutely integrable in $[-\pi, \pi]$ guarantess the convergence of $g_n(y)$ towards $f(y)$.

In the next section we exhibit a trigonometric identity which shows relationships between the kernels of Dirichlet and Fejér.

2 Links between Fejér and Dirichlet kernels

It is possible to demonstrate the following identity:

$$\frac{\sin(2n+1)\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \frac{1}{\sin^2\frac{\theta}{2}}\Big[\sin^2(n+1)\frac{\theta}{2} - \sin^2(n\frac{\theta}{2})\Big], \tag{2.1}$$

which in terms of (1.4) and (1.7) means:

$$\underset{D}{K_n}(\theta) = (n+1)\underset{F}{K_{n+1}}(\theta) - n\underset{F}{K_n}(\theta), \tag{2.2}$$

that is, the Fejér kernel generates to Dirichlet kernel. For example:

$$\begin{aligned} \underset{D}{K_0} &= \underset{F}{K_1}, \qquad \text{because} \qquad \underset{F}{K_0} = 0, \\ \underset{D}{K_1} &= 2\underset{F}{K_2} - \underset{F}{K_1} \qquad \therefore \qquad \underset{F}{K_2} = \frac{1}{2}(\underset{D}{K_0} + \underset{D}{K_1}), \\ \underset{D}{K_2} &= 3\underset{F}{K_3} - 2\underset{F}{K_2} \qquad \therefore \qquad \underset{F}{K_3} = \frac{1}{3}(\underset{D}{K_0} + \underset{D}{K_1} + \underset{D}{K_2}), \qquad \text{etc.} \end{aligned}$$

then the inverse relation to (2.2) adopts the form:

$$\underset{F}{K_{n+1}}(\theta) = \frac{1}{n+1}\sum_{r=0}^{n} \underset{D}{K_r}(\theta), \quad n = 0, 1, 2, \ldots \tag{2.3}$$

thus the Dirichlet kernel permits to construct the Fejér kernel.

On the other hand, the fourth-kind Chebyshev polynomials are given by [6],[7]:

$$W_n(x) = W_n(\cos\theta) \equiv 2\pi \underset{D}{K_n}(\theta) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}, \tag{2.4}$$

with $\theta\epsilon[0,\pi]$ and $x\epsilon[-1,1]$, therefore:

$$\begin{aligned} W_0 = 1, \qquad W_1 = 2x+1, \qquad W_2 = 4x^2+2x-1, \\ W_3 = 8x^3+4x^2-4x-1, \quad W_4 = 16x^4+8x^3-12x^2-4x+1, \quad \text{etc.} \end{aligned} \tag{2.5}$$

The relationship (2.3) is important because with it we can introduce new polynomials, in fact:

$$\tilde{W}_n(x) = \tilde{W}_n(\cos\theta) \equiv 2\pi(n+1)\underset{F}{K_{n+1}}(\theta) = \frac{\sin^2(n+1)\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}, \tag{2.6}$$

$$= \sum_{r=0}^{n} W_r(x), \tag{2.7}$$

that we name "fifth-kind Chebyshev polynomials", which are not explicitly in the literature. For example, (2.5) and (2.7) imply the expressions:

$$\begin{aligned} \tilde{W}_0 = 1, \qquad \tilde{W}_1 = 2x+2, \qquad \tilde{W}_2 = 4x^2+4x+1, \\ \tilde{W}_3 = 8x^3+8x^2, \qquad \tilde{W}_4 = 16x^4+16x^3-4x^2-4x+1, \qquad \text{etc.} \end{aligned} \tag{2.8}$$

and some of their properties are:

$$
\begin{gathered}
\tilde{W}_n(x) = (n+1)\sum_{r=1}^{n+1} \frac{2^r}{n+r+1}\begin{pmatrix} n+r+1 \\ 2r \end{pmatrix}(x-1)^{r-1}, \quad \tilde{W}_n(1) = (n+1)^2, \\
\tilde{W}_{n+1}(x) = 2 + 2x\tilde{W}_n(x) - \tilde{W}_{n-1}(x), \\
\sum_{r=0}^{\infty} \tilde{W}_r(x) z^r = \frac{1+z}{[(1-z)(1-2xz+z^2)]}, \qquad | z | < 1, \\
(x-1)\tilde{W}_n(x) = 2^n \sum_{r=0}^{n+1} \begin{pmatrix} n+1 \\ r \end{pmatrix} {}_2F_1(r-n-1, -n-\frac{1}{2}; -2n-1; 2) x^r - 1.
\end{gathered} \tag{2.9}
$$

In a forthcoming work we will study some related topics such as Rodrigues formula, orthonormality, interpolation properties, etc., for this new set of polynomials $\tilde{W}_n(x)$.

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