CONFORMALLY INVARIANT DIFFERENTIAL OPERATORS AND BILINEAR FUNCTIONALS IN SIX DIMENSIONS

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Abstract

We review how to construct the Paneitz operator in dimension four and the corresponding operator in dimension six, by constructing symmetric bilinear differential functionals that are conformally invariant.

Keywords: Conformal invariants, Paneitz operator, bilinear differential functionals.

Resumen

Revisamos como construir el operador de Paneitz en cuatro dimensiones y el correspondiente operador en seis dimensiones, mediante la construcción de funcionales diferenciales bilineales simétricos que son conformemente invariantes.

Palabras clave: Invariantes conformes, operador de Paneitz, funcionales bilineales diferenciales.

Mathematics Subject Classification: 53A30, 53-04.

1   Introduction

Using his quantized calculus, Connes [3] found a couple of conformal invariants on a four-dimensional oriented compact Riemannian manifold M without boundary. The first one is a bilinear differential functional of order 4, that we denote B_4 acting on $C^\infty(M)$. This bilinear differential functional is symmetric $B_4(f, h) = B_4(h, f)$, and conformally invariant in the sense that $\widehat{B_4}(f, h) = e^{-4\eta}B_4(f, h)$, for a conformal change of the metric $\widehat{g} = e^{2\eta}g$.

The second conformal invariant, P_4, the Paneitz operator [8], can be derived derived from B_4 by the relation

$$\int_M B_4(f,h)\,dx = \int_M f P_4(h)\,dx.$$

These two conformal invariants can be defined in any even-dimensional oriented compact Riemannian manifold without boundary by the relations

$$\mathrm{Wres}(k[F,f][F,h]) = \int_M k B_n(f,h)\,dx, \qquad \int_M B_n(f,h)\,dx = \int_M f P_n(h)\,dx,$$

with $k, f, h \in C^\infty(M)$. For details in the four-dimensional case see [3] and for the general case see [12]. Here Wres is the non-commutative residue. The operator F is a pseudo-differential operator of order 0 acting on the space of square integrable sections of middle dimension forms on M.

In this paper, we use `Ricci.m` [7], a package to do tensor computations to compute the conformal invariants B_4 and P_4 in the four-dimensional case, and the equivalent ones B_6 and P_6 in the six-dimensional conformally flat case. In detail we give a short introduction on how to use `Ricci.m` to do the computations required in this work. Next, we present our computation of B_4 and we prove Theorem 3.2 which characterizes any formally self-adjoint differential operator P with leading term Δ^2 and with no zeroth-order term which is furtermore conformal invariant as a constant multiple of the Paenitz operator. After that, we present our computation of B_6 in the conformally flat case and characterize in Theorem 3.5 all the symmetric bilinear differential functionals that are conformally invariant and that define a Hochschild 2-cocycle on the algebra $C^\infty(M)$ for a six-dimensional manifold M. At the end, we present Theorem 4.1 which characterizes all the differential operators in dimension six that are of type `GJMS` [5] as explained in Section 4.

In this work, R represents Riemann tensor and the Ricci tensor is represented by $\mathrm{Rc}_{ij} = R^k{}_{ikj}$. The scalar curvature by $\mathrm{Sc} = \mathrm{Rc}^i{}_i$. We use J to denote the normalized scalar curvature $2(n-1)J = \mathrm{Sc}$ in dimension n. If needed, we will "raise" and "lower" indices without explicit mention for example, $g_{mi}R^i{}_{jkl} = R_{mjkl}$. The relation between the Weyl tensor and the Riemann tensor is given by

$$W^i{}_{jkl} = R^i{}_{jkl} + \mathsf{P}_{jk}\delta_K{}^i{}_l - \mathsf{P}_{jl}\delta_K{}^i{}_k + \mathsf{P}^i{}_l g_{jk} - \mathsf{P}^i{}_k g_{jl}$$

where δ_K represents the Kronecker's delta tensor. Last $(n-2)\mathsf{P} = \mathrm{Rc} - Jg$.

2   Working in a conformal setting with `Ricci.m`

There are results supported on computations that are beyond reasonable hand manipulations. Introductions are presented in [1] with spin geometry and conformally invariant differential operators, and in [11] with gravitational field equations. Further examples are present in [6] with tractors, and in [9] and [10] with noise kernel for scalars fields in the Schwarzschild black hole and for thermal fields in flat space. This calculations require a package for tensor calculus with the capability of handling indices, covariant derivatives,

curvature identities, symmetries, dependence on an arbitrary dimension, etc., and a particular kind of philosophy. The idea is to play an interactive session with a computer aiming to produce a desired result, not as the output of a program, but as the conclusion of a series of procedures, each based on a decision of what sort of identities or properties of the objects involved to use next. We use `Ricci.m`, a package prepared by J. M. Lee [7]. In what remains of this section we give a short presentation on how to use `Ricci.m` for the computations ahead.

We choose to work with the convention on the Laplacian having positive spectrum. To declare it in `Ricci.m` we must use

$$\texttt{\$LaplaceBeltramiConvention = PositiveSpectrum}$$

The first step is to define the bundle we will be working on:

```
DefineBundle[tangent,n,g,i,j,k,MetricType->Riemannian,
RiemannConvention->FirstUp]
```

where `g` represents the Riemannian metric associated to the base manifold of dimension `n` of the bundle `tangent`, and `i,j,k` are indices associated to the bundle.

To be able to work with a conformal metric we must explicitly define a new metric `gz` which is conformal to the background metric `g`. To define the new metric we use

```
DefineTensor[gz,2,Symmetries -> Symmetric]
DefineRelation[gz,Exp[2*z*eta]*g]
```

The `z` keeps track of the order of the conformal deformation or variation. For example, to compute the scalar curvature `Sc` in the new metric `gz` we use `ScalarCurv[gz]` which after simplifications with `BasisExpand` and `TensorSimplify` produces

$$\mathrm{Sc}(gz) = \mathrm{Sc}(g)e^{-2z\eta} - 2(n-1)z\eta_{;i}{}^i e^{-2z\eta} - (n-2)(n-1)z^2\eta_{;i}\eta_{;}{}^i e^{-2z\eta}. \tag{1}$$

It is very common to write $\widehat{g}$ for $gz = e^{2z\eta}g$, $\widehat{Sc}$ for $\mathrm{Sc}(gz)$, and simply Sc for $\mathrm{Sc}(g)$. In this way, the first order conformal variation of Sc is $-2(n-1)z\eta_{;i}{}^i e^{-2z\eta}$ or simply $-2(n-1)\eta_{;i}{}^i e^{-2\eta}$ if the metric is conformally changed to $e^{2\eta}g$.

Before going any further is it important to make the following observation. If we set a metric g in M and consider another metric $\widehat{g}$ conformally related to g by the relation $\widehat{g} = e^{2z\eta}g$ where $\eta \in C^\infty(M)$ and z a constant, then the conformal variation of each expression is a polynomial in z whose coefficients are expressions in the metric and the conformal factor η (actually, this is an abuse of the language since the conformal factor is $e^{2z\eta}$). In this way, when studying conformal invariance, there is no need to study the whole conformal deformation. It is enough to study the conformal deformation up to order one in η as follows. The conformal deformation up to order one in η is given by $\frac{d}{dz}\big|_{z=0}$. If a natural tensor or a differential operator is invariant up to order one in η, i.e. if its conformal deformation up to order one is equal to zero, then by integration it follows that it is fully invariant, for details see [2].

The next step to keep working with a conformal metric is to define its inverse `Invgz`. For that we use:

```
DefineTensor[Invgz,2,Symmetries->Symmetric,Variance->Con]
DefineRelation[Invgz[U[i],U[j]],Exp[-2*z*eta]*g[U[i],U[j]]]
```

to declare it as a 2-contravariant symmetric tensor g^{-1} such that $(g^{-1})^{ij} = e^{-2z\eta}g^{ij}$ and

```
 DefineRule[gzInvgz,gz[L[i],L[j]]*Invgz[U[j],U[k]],Kronecker[[L[i],U[k]]]
```

to define its properties as an inverse of g.

To declare smooth functions we consider them as 0-tensors `f` and `h`. Once they are declared using `DefineTensor[f,0]` and `DefineTensor[h,0]` we can use the new metric to compute the conformal transformation of the different tensors we want to work with. For example, the conformal transformation of

$$f_{;ij}h_{;}{}^{ij} = g^{ii_1}g^{jj_1}f_{;ij}h_{;i_1j_1}$$
$$= \texttt{g[U[i],U[i1]]*g[U[j],U[j1]]*f[][L[i],L[j]]*h[][L[i1],L[j1]]}$$

is obtained with

```
Invgz[U[i],U[i1]]*Invgz[U[j],U[j1]]*Del[Del[f,Metric->gz],Metric->gz]
[L[i],L[j]]*Del[Del[h,Metric->gz],Metric->gz][L[i1],L[j1]]
```

3   Computing B_4 and B_6.

The bilinear differential form $B_n(f,h)$ is natural in the sense it can be written as a universal polynomial expression in the metric g, its inverse g^{-1}, the connection ∇, and the curvature R; using tensor products and contractions. Furthermore, by Lemma 3.2 [12], we know that $B_n(f,h)$ is a polynomial in the ingredients $\nabla^\alpha df$, $\nabla^\beta dh$, and $\nabla^\gamma R$ for multi-indices α, β, and γ, with

$$2\deg_R + \deg_\nabla = n, \text{ with } \deg_\nabla \geq 2.$$

Here for derivatives we count all of the covariant derivatives on R, f, and h, as well as the exterior derivatives, and any occurrence of W, Rc, P, Sc, or J is counted as an occurrence of R.

In the particular case $n = 4$, $\deg_R$ can be 0 or 1, hence B_4 can be written as

$$B_4(f,h) = p_4(df,dh) + p_{R,4}(df,dh)$$

where $p_{R,4}(df,dh)$ is a trilinear form on R, df, and dh. In the six-dimensional case, $\deg_R \in \{0,1,2\}$ thus

$$B_6(f,h) = p_6(df,dh) + p_{6,1}(df,dh) + p_{6,2}(df,dh),$$

with $p_{6,1} = p_{6,R} + p_{6,R'} + p_{6,R''}$, and

1. $p_{6,R}(df,dh)$ is polynomial on $\nabla^\alpha df$, $\nabla^\beta dh$, and R,

2. $p_{6,R'}(df, dh)$ is polynomial on $\nabla^\alpha df$, $\nabla^\beta dh$, and ∇R,

3. $p_{6,R''}(df, dh)$ is polynomial on $\nabla^\alpha df$, $\nabla^\beta dh$, and $\nabla\nabla R$, and

4. $p_{6,2}(df, dh)$ is polynomial on $\nabla^\alpha df$, $\nabla^\beta dh$, and RR.

The four-dimensional case was fully treated in [3], nevertheless we present our approach. By taking into consideration the symmetry of $B_4(f, h)$ we know

$$p_4(df, dh) \in \operatorname{span}\left\{(f_{;i}{}^i{}_j h_{;}{}^j + h_{;i}{}^i{}_j f_{;}{}^j),\ f_{;ij} h_{;}{}^{ij},\ f_{;i}{}^i h_{;j}{}^j\right\}.$$

There are no terms in $p_{R,4}(df, dh)$ involving the Weyl tensor. Indeed, any term of the form $f_{;*} h_{;*} W_{****}$ will force indices in W to repeat and hence to vanish since the Weyl tensor is trace-free. In this way, the only terms to consider are those with a factor P or J. That is to say

$$p_{R,4}(f, h) \in \operatorname{span}\left\{f_{;i} h_{;}{}^i R_{jk}{}^{jk},\ f_{;i} h_{;j} R^i{}_k{}^{jk}\right\} = \operatorname{span}\left\{f_{;i} h_{;}{}^i J,\ f_{;i} h_{;j} \mathsf{P}^{ij}\right\}.$$

In particular, in the four-dimensional case, to determine $B_4(f, h)$ is enough to work in the conformally flat case. Thus, in the four dimensional case, $B_4(f, h)$ is a linear combination of the form:

$$A(f_{;i}{}^i{}_j h_{;}{}^j + h_{;i}{}^i{}_j f_{;}{}^j) + B f_{;ij} h_{;}{}^{ij} + C f_{;i}{}^i h_{;j}{}^j + D f_{;i} h_{;}{}^i J + E f_{;i} h_{;j} \mathsf{P}^{ij}.$$

Its conformal variation, under a change of metric of the form $\widehat{g} = e^{2z\eta} g$ with η a smooth function on M, is given by:

$$(2C + B - 2A) z \eta_{;i}(f_{;j}{}^j h_{;}{}^i + f_{;}{}^i h_{;j}{}^j) + 2(A - B) z \eta_{;i}(f_{;}{}^i{}_j h_{;}{}^j + f_{;j} h_{;}{}^{ij})$$
$$- D z \eta_{;i}{}^i f_{;j} h_{;}{}^j + (4A - E) z \eta_{;ij} f_{;}{}^i h_{;}{}^j$$
$$+ (2B - D - E/2) z^2 \eta_{;i} \eta_{;}{}^i f_{;j} h_{;}{}^j + (2B - 8A + 4C + E) z^2 \eta_{;i} \eta_{;j} f_{;}{}^i h_{;}{}^j.$$

Hence to get rid of the variation we just need to consider $B = A = 2C$, $E = 4A$, and $D = 0$. Also note how the expressions with a z^2 factor vanish when the expressions with a z factor vanish. In this way, by taking $C = -1$

$$B_4(f, h) = -2(f_{;i}{}^i{}_j h_{;}{}^j + h_{;i}{}^i{}_j f_{;}{}^j) - 2 f_{;ij} h_{;}{}^{ij} - f_{;i}{}^i h_{;j}{}^j - 8 f_{;i} h_{;j} \mathsf{P}^{ij}$$
$$= -2(f_{;ij}{}^j h_{;}{}^i + h_{;ij}{}^j f_{;}{}^i) - 2 f_{;ij} h_{;}{}^{ij} - f_{;i}{}^i h_{;j}{}^j + 4 f_{;i} h_{;}{}^i J. \qquad (2)$$

By using the relations

$$f_{;i}{}^i = -\Delta f \qquad f_{;i} h_{;}{}^i = \langle df, dh \rangle \qquad f_{;ij} h_{;}{}^{ij} = \langle \nabla(df), \nabla(dh) \rangle,$$

and

$$f_{;ij}{}^j h_{;}{}^i + h_{;ij}{}^j f_{;}{}^i = -\Delta\langle df, dh \rangle - 2\langle \nabla(df), \nabla(dh) \rangle,$$

in the four-dimensional case, B_4 is given by the following

Theorem 3.1. *For a four-dimensional oriented compact Riemanian manifold M without boundary, any symmetric bilinear differential functional B acting on $C^\infty(M) \times C^\infty(M)$, conformal invariant in the sense that $\widehat{B}(f,h) = e^{-4\eta} B(f,h)$ for a conformal change of the metric $\widehat{g} = e^{2\eta} g$, and such that $B(f,h)$ does not have linear terms on f or h is a constant multiple of*

$$B_4(f,h) = 2\Delta(\langle df, dh \rangle) - \Delta(f)\Delta(h) + 2\langle \nabla(df), \nabla(dh) \rangle + 4\langle df, dh \rangle J. \qquad (3)$$

We use ∇ to represent the covariant derivative and the Laplacian $\Delta = -\nabla_j \nabla^j$. Furthermore,

$$\tau_4(f_0, f_1, f_2) := \int_M f_0 B_4(f,h)\, dx$$

defines a Hochschild 2-cocycle on the algebra of smooth functions on M.

The exclusion of linear terms on f or h in the statement is to avoid the presence of multiples of $fh|W|^2$. Compare with [3]. The cocycle property follows from an easy computation. What we have determined is actually B_4 up to a proportional factor. In [3], Connes use this expression for B_4 to obtain the Paneitz operator.

Theorem 3.2. *Let M be a four-dimensional oriented compact Riemanian manifold without boundary. Let P be any formally selfadjoint differential operator P with leading term Δ^2 and with no zeroth-order term which is furtermore conformal invariant in the sense that $\widehat{P}(f) = e^{-4\eta} P(f)$ for a conformal change of the metric $\widehat{g} = e^{2\eta} g$. Then P is a constant multiple of the Paenitz operator.*

Proof. Stokes' theorem applied to (2) leads to an expression for P_4, indeed using the metric to raise and lower indices and reordering covariant derivatives

$$\int_M B_4(f,h)\, dx = \int_M \left\{ f_{;i}{}^i{}_j{}^j - 2f_{;i}{}^i J + 2f_{;i} J_;{}^i + 4f_{;ij} \mathsf{P}^{ij} \right\} h\, dx.$$

Because this equality holds for every f and every h in $C^\infty(M)$ we obtain,

$$\begin{aligned}
P_4(f) &= f_{;i}{}^i{}_j{}^j - 2f_{;i}{}^i J + 4f_{;ij}\mathsf{P}^{ij} + 2f_{;i} J_;{}^i \\
&= \Delta^2(f) + 2\Delta(f)J + 4\langle \nabla df\, \mathsf{P} \,\rangle + 2\langle df, dJ \rangle,
\end{aligned} \qquad (4)$$

the Paneitz operator.

Now let P be another formally selfadjoint differential operator of order 4 on the four-dimensional manifold M such that P is of the form Δ^2 plus lower order terms and with no zeroth-order term. Define the symmetric bilinear differential functional B by the relation

$$B(f,h) := P(fh) - fP(h) - P(f)h.$$

If $\widehat{P}(f) = e^{-4\eta} P(f)$, for a conformal change of the metric $\widehat{g} = e^{2\eta} g$ then, the same holds for B, that is $\widehat{B}(f,h) = e^{-4\eta} B(f,h)$. Hence B must be a constant multiple of B_4 above. Now

$$\int_M f P_4(h)\, dx = \int_M B_4(f,h)\, dx = c \int_M B(f,h)\, dx = c \int_M f P(h)\, dx$$

where the relation holds for any smooth functions f and h on M. Hence $P = cP_4$ and we have determined as well any possible differential operator of such a kind in the four-dimensional case up to a constant multiple. $\square$

In the 6-dimensional case, symmetry and conformal invariance are not enough to fully described B_6 as we will find terms like $f_{;i}h_{;}{}^{i}W_{jklm}W^{jklm}$ and $f_{;i}h_{;j}W^{i}{}_{klm}W^{jklm}$ which are symmetric on f and h, and conformally invariant. We will need to use the Hochschild 2-cocycle property of $\mathrm{Wres}(f_0[F, f][F, h])$ on the algebra $C^\infty(M)$ to determine some of the components of B_6. Even such a property will not suffice to completely describe B_6 as we will find terms that are conformally invariant and satisfy such a property (see (10)).

The list of terms we consider in the six-dimensional conformally flat case is as follows:

$$p_6(df, dh) \in \mathrm{span}\{(f_{;i}h_{;j}{}^{j}{}_{k}{}^{ki} + h_{;i}f_{;j}{}^{j}{}_{k}{}^{ki}),\ (f_{;i}{}^{i}h_{;j}{}^{j}{}_{k}{}^{k} + h_{;i}{}^{i}f_{;j}{}^{j}{}_{k}{}^{k}),$$
$$(f_{;ij}h_{;k}{}^{kij} + h_{;ij}f_{;k}{}^{kij}),\ (f_{;ijk}h_{;}{}^{ijk}),\ (f_{;i}{}^{i}{}_{j}h_{;k}{}^{kj})\}$$

$$p_{6,R}(df, dh) \in \mathrm{span}\{(f_{;i}h_{;j}{}^{ji} + h_{;i}f_{;j}{}^{ji})J,\ (f_{;i}{}^{i}h_{;j}{}^{j})J,\ (f_{;ij}h_{;}{}^{ij})J,$$
$$(f_{;i}h_{;k}{}^{k}{}_{j} + h_{;i}f_{;k}{}^{k}{}_{j})\mathsf{P}^{ij},\ (f_{;i}h_{;}{}^{i}{}_{jk} + h_{;i}f_{;}{}^{i}{}_{jk})\mathsf{P}^{jk},$$
$$(f_{;ij}h_{;k}{}^{k} + h_{;ij}f_{;k}{}^{k})\mathsf{P}^{ij},\ (f_{;ij}h_{;}{}^{i}{}_{k})\mathsf{P}^{jk}\}$$

Next, for $p_{6,R'}$ we might consider

$$p_{6,R'} \in \mathrm{span}\{(f_{;i}h_{;j}{}^{j} + h_{;i}f_{;j}{}^{j})J_{;}{}^{i},\ (f_{;i}h_{;}{}^{i}{}_{j} + h_{;i}f_{;}{}^{i}{}_{j})J_{;}{}^{j},$$
$$(f_{;ij}h_{;k} + h_{;ij}f_{;k})\mathsf{P}^{ij}{}_{;}{}^{k},\ (f_{;ij}h_{;k} + h_{;ij}f_{;k})\mathsf{P}^{ik}{}_{;}{}^{j},$$
$$(f_{;ij}h_{;}{}^{i} + h_{;ij}f_{;}{}^{i})\mathsf{P}^{j}{}_{k}{}_{;}{}^{k},\ (f_{;ij}h_{;}{}^{i} + h_{;ij}f_{;}{}^{i})\mathsf{P}_{k}{}^{k}{}_{;}{}^{j}\}.$$

The relations

$$\mathsf{P}_{i}{}^{i} = J, \qquad \mathsf{P}_{ij}{}_{;}{}^{j} = J_{;i}, \qquad \mathsf{P}_{ij\,;k} = \mathsf{P}_{ik\,;j} - W_{iljk}{}_{;}{}^{l}/(n-3) \qquad (5)$$

imply that these terms are already considered in the conformally flat case and hence

$$p_{6,R'} \in \mathrm{span}\{(f_{;i}h_{;j}{}^{j} + h_{;i}f_{;j}{}^{j})J_{;}{}^{i},\ (f_{;i}h_{;}{}^{i}{}_{j} + h_{;i}f_{;}{}^{i}{}_{j})J_{;}{}^{j}, (f_{;ij}h_{;k} + h_{;ij}f_{;k})\mathsf{P}^{ij}{}_{;}{}^{k}\}.$$

For $p_{6,R''}$ we might consider

$$\mathrm{span}\{f_{;i}h_{;}{}^{i}J_{;j}{}^{j},\ f_{;i}h_{;j}J_{;}{}^{ij},\ f_{;i}h_{;j}\mathsf{P}^{ij}{}_{;k}{}^{k},\ f_{;i}h_{;}{}^{i}\mathsf{P}_{jk}{}_{;}{}^{jk},$$
$$(f_{;i}h_{;j} + h_{;i}f_{;j})\mathsf{P}_{ik}{}_{;}{}^{jk},\ (f_{;i}h_{;j} + h_{;i}f_{;j})\mathsf{P}^{ij}{}_{;k}{}^{k}\}.$$

The relation

$$\mathsf{P}_{ij\,;lk} - \mathsf{P}_{ij\,;kl} = \mathsf{P}_{j}{}^{m}(g_{lm}\mathsf{P}_{ik} - g_{km}\mathsf{P}_{il} - g_{il}\mathsf{P}_{km} + g_{ik}\mathsf{P}_{lm} + W_{imkl})$$
$$+ \mathsf{P}_{i}{}^{m}(g_{lm}\mathsf{P}_{jk} - g_{km}\mathsf{P}_{jl} - g_{jl}\mathsf{P}_{km} + g_{jk}\mathsf{P}_{lm} + W_{jmkl})$$

and (5) above imply

$$p_{6,R''} \in \mathrm{span}\{f_{;i}h_{;}{}^{i}J_{;j}{}^{j},\ f_{;i}h_{;j}J_{;}{}^{ij}\}.$$

Last,

$$p_{6,R^2} \in \mathrm{span}\big\{ f_{;i}h_{;}{}^{i}J^2,\ f_{;i}h_{;j}J\mathsf{P}^{ij},\ f_{;i}h_{;}{}^{i}\mathsf{P}_{jk}\mathsf{P}^{jk},\ f_{;i}h_{;j}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk}\big\}.$$

In this way, we know $B_6(f,h)$ in the conformally flat case is a linear combination of up to 21 terms. We use `Ricci.m` to compute the proper coefficients, the result reads:

Theorem 3.3. *In the 6-dimensional conformally flat case, an expression for B_6 is*

$$
\begin{aligned}
B_{6,confflat}(f,h) =\ & 2(f_{;i}{}^{i}{}_{j}{}^{j}{}_{k}h_{;}{}^{k} + h_{;i}{}^{i}{}_{j}{}^{j}{}_{k}f_{;}{}^{k}) + (f_{;i}{}^{i}{}_{j}{}^{j}h_{;k}{}^{k} + h_{;i}{}^{i}{}_{j}{}^{j}f_{;k}{}^{k}) \\
& + 4(f_{;i}{}^{i}{}_{jk}h_{;}{}^{jk} + h_{;i}{}^{i}{}_{jk}f_{;}{}^{jk}) + 4f_{;i}{}^{i}{}_{j}h_{;k}{}^{kj} + 8(f_{;ijk}h_{;}{}^{ijk})/3 \\
& - 4(f_{;i}{}^{i}{}_{j}h_{;}{}^{j} + h_{;i}{}^{i}{}_{j}f_{;}{}^{j})J - 4f_{;i}{}^{i}h_{;j}{}^{j}J \\
& + 80(f_{;i}h_{;j}{}^{j}{}_{k} + h_{;i}f_{;j}{}^{j}{}_{k})\mathsf{P}^{ik}/3 \\
& + 32(f_{;ijk}h_{;}{}^{i} + f_{;ijk}h_{;}{}^{i})\mathsf{P}^{jk}/3 + 64f_{;ij}h_{;}{}^{ik}\mathsf{P}^{jk} \\
& + 8(f_{;ij}h_{;}{}^{i} + h_{;ij}f_{;}{}^{i})J^{;j} + 32(f_{;ij}h_{;k} + h_{;ij}f_{;k})\mathsf{P}^{ij}{}^{;k} \\
& + 16f_{;i}h_{;j}J^{;ij} \\
& - 32f_{;i}h_{;j}J\mathsf{P}^{ij} - 80f_{;i}h_{;}{}^{i}\mathsf{P}_{jk}\mathsf{P}^{jk}/3 + (608f_{;i}h_{;j}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk})/3. \qquad (6)
\end{aligned}
$$

It is not difficult to verify that the above expression for B_6 in the conformally flat case, as a sum of explicitely symmetric terms is:

$$
\begin{aligned}
& B_{6,confflat}(f,h) \\
& = 2(\langle d\Delta^2 f, dh\rangle + \langle d\Delta^2 h, df\rangle) - (\Delta^2 f \Delta h + \Delta^2 h \Delta f) \\
& \quad -4(\langle \nabla d\Delta f, \nabla dh\rangle + \langle \nabla d\Delta h, \nabla df\rangle) + 4\langle d\Delta f, d\Delta h\rangle + 8\langle \nabla^2 df, \nabla^2 dh\rangle/3 \\
& \quad +4(\langle d\Delta f, dh\rangle + \langle df, d\Delta h\rangle)J - 4\Delta f \Delta h J \\
& \quad -80\langle df \otimes d\Delta h + dh \otimes d\Delta f, \mathsf{P}\rangle/3 + 32\langle \nabla^2 df, dh \otimes \mathsf{P}\rangle \\
& \quad +\langle \nabla^2 dh, df \otimes \mathsf{P}\rangle/3 + 64\langle \nabla d\langle df, dh\rangle, \mathsf{P}\rangle \\
& \quad -32(\langle \nabla^2 df, dh \otimes \mathsf{P}\rangle + \langle \nabla^2 dh, df \otimes \mathsf{P}\rangle) \\
& \quad +8(\langle \nabla df, dh \otimes dJ\rangle + \langle \nabla dh, df \otimes dJ\rangle) \\
& \quad +32(\langle \nabla df \otimes dh + \nabla dh \otimes df, \nabla\ V\rangle) + 16\langle df \otimes dh, \nabla dJ\rangle \\
& \quad -32\langle df \otimes dh, \mathsf{P}\rangle J - 80\langle df, dh\rangle\langle \mathsf{P}, \mathsf{P}\rangle/3 + (608\langle df \otimes dh, V^2\rangle)/3.
\end{aligned}
\qquad (7)
$$

Here we are using $\langle df \otimes dh, V^2\rangle$ to denote $f_{;i}h_{;j}\mathsf{P}^{i}{}_{k}\mathsf{P}^{jk}$.

Leaving for an instant the conformally flat case, in the general conformally curved case, the conformal variation of $B_6(f,h)$ in (6), up to order one in η is given by

$$-64z\eta_{;i}\big(f_{;jk}h_{;l} + f_{;l}h_{;kj}\big)W^{ijkl}/3 - 32z\eta_{;ij}f_{;k}h_{;l}W^{iljk} + 160z^2\eta_{;i}\eta_{;j}f_{;k}h_{;l}W^{ikjl}/3. \quad (8)$$

which vanishes in the conformally flat case, meaning that our expression is conformally invariant inside the conformally flat class of metrics on M. In the general conformally curved case, this variation will be useful in finding the extra terms we are missing, that is to say, those terms that vanish in the conformally flat case.

If we define using (6) the trilinear form on $C^\infty(M)$

$$\tau(f_0, f_1, f_2) := \int_M f_0 B_6(f_1, f_2)\, dx$$

then

$$(b\tau)(f_0, f_1, f_2, f_3)$$
$$= \int_M f_0 \Big(32 f_{1\,;\,i} f_{2\,;\,j} f_{3\,;\,k} W^{ik\ j\ l}_{\ \ l\ ;} + 32 \big(f_{1\,;\,ij} f_{3\,;\,l} - f_{1\,;\,l} f_{3\,;\,ij}\big) f_{2\,;\,k} W^{ikjl}/3\Big)\, dx \qquad (9)$$

which vanishes in the conformally flat case meaning that τ is a Hochschild 2-cocycle on the algebra of smooth functions in the conformally flat case.

We do not restrict ourselves anymore to the conformally flat case. Our objective now is to identify those terms that we must add to $p_{6,R}$, $p_{6,R'}$, $p_{6,R''}$, and $p_{6,2}$ (in (6)) to obtain the expression for B_6 in the general conformally curved case. In particular, we are looking for terms having the Weyl tensor, or its covariant derivatives as factors.

The first set of terms to be added will complete the expression for $p_{6,R}$. In this case there is just one possibility which is $f_{;ij} h_{;kl} W^{ikjl}$. Any other possibility is ruled out by the Bianchi identities. To complete the expression for $p_{6,R'}$ we only need to consider $(f_{;ij} h_{;k} + h_{;ij} f_{;k}) W^i_l{}^{jk}{}_{;}{}^l$. The only term to complete the expression for $p_{6,R''}$ is $f_{;i} h_{;j} W^i_k{}^j_l{}_{;}{}^{kl}$, symmetric on f and h. For $p_{6,2}$ we consider at this time only $f_{;i} h_{;j} \mathsf{P}_{kl} W^{ikjl}$. It happens that the other possible terms

$$f_{;i} h_{;}{}^i W_{jklm} W^{jklm} \quad \text{and} \quad f_{;i} h_{;j} W^i{}_{klm} W^{jklm}. \qquad (10)$$

are conformally invariant.

Up to this point, what we must add to $B_{6,\text{conf. flat}}$ is a linear combination of the form

$$A f_{;ij} h_{;kl} W^{ikjl} + B(f_{;ij} h_{;k} + h_{;ij} f_{;k}) W^i_l{}^{jk}{}_{;}{}^l + C f_{;i} h_{;j} W^i_k{}^j_l{}_{;}{}^{kl} + D f_{;i} h_{;j} \mathsf{P}_{kl} W^{ikjl}.$$

Its conformal variation up to order one in η is given by

$$(2A - 3B) z\eta_{;i}(f_{;jk} h_{;l} + h_{;jk} f_{;l}) W^{ijkl} + (3C - D) z\eta_{;ij} f_{;k} h_{;l} W^{iljk}$$
$$- (B + 2C) z\eta_{;i} f_{;j} h_{;k}(W^{ijk}{}_l + W^{ikj}{}_l)_{;}{}^l - (2A - 6B - 3C - D) z^2 \eta_{;i} \eta_{;j} f_{;k} h_{;l} W^{ikjl}.$$

By comparing it with (8), we deduce the conditions

$$B + 2C = 0, \quad 2A - 3B = 64/3, \quad 3C - D = 32, \qquad (11)$$

which means, conformally invariance and symmetry are not enough to find the right values for all the constants. So far, the term to be added to $B_6(f, h)$ is given by

$$(32/3 - 3C) f_{;ij} h_{;kl} W^{ikjl} - 2C(f_{;ij} h_{;k} + h_{;ij} f_{;k}) W^i_l{}^{jk}{}_{;}{}^l + C f_{;i} h_{;j} W^i_k{}^j_l{}_{;}{}^{kl}$$
$$+ 3C f_{;i} h_{;j} \mathsf{P}_{kl} W^{ikjl} + E f_{;i} h_{;}{}^i W_{jklm} W^{jklm} + F f_{;i} h_{;j} W^i{}_{klm} W^{jklm} \qquad (12)$$

where the last two terms come from (10).

In the 4-dimensional case, the Hochschild 2-cocycle property of $\mathrm{Wres}(f_0[F, f_1][F, f_2])$ was used merely to make sure the constants found had the right values. In the 6-dimensional case, as we shall see, this property will play a more important role in the non-conformally flat case. Nevertheless, the full description of B_6 escapes even these properties, requiring some more information to be used.

Proposition 3.4. *The trilinear form*

$$\tau'(f_0, f_1, f_2) :=$$

$$\int_M f_0 \Big\{ f_{1\,;\,i} f_{2\,;\,j} \big(C W^i{}_k{}^j{}_l{}_;^{kl} + D\mathsf{P}_{kl} W^{ikjl} + F W^i{}_{klm} W^{jklm} \big) + E f_{1\,;\,i} f_2{}_;^{\,i} W_{jklm} W^{jklm} \Big\} \, dx$$

defines a Hochschild 2-cocycle on the algebra $C^\infty(M)$ for any value of the constants C, D, E, and F, that is, it satisfies $(b\tau')(f_0, f_1, f_2, f_3) = 0$ for any $f_i \in C^\infty(M)$.

On the other hand, if we define

$$\tau''(f_0, f_1, f_2) := \int_M f_0 \Big\{ A f_{1\,;\,ij} f_{2\,;\,kl} W^{ikjl} + B(f_{1\,;\,ij} f_{2\,;\,k} + f_{2\,;\,ij} f_{1\,;\,k}) W^i{}_l{}^{jk}{}_;^{\,l} \Big\} \, dx$$

then

$$(b\tau'')(f_0, f_1, f_2, f_3) = \tag{13}$$
$$\int_M f_0 \Big(3B f_{1\,;\,i} f_{2\,;\,j} f_{3\,;\,k} W^{ik}{}_l{}^j{}_;^{\,l} + 2A(f_{1\,;\,ij} f_{3\,;\,l} - f_{1\,;\,l} f_{3\,;\,ij}) f_{2\,;\,k} W^{ikjl} \Big) \, dx.$$

To have that $\int_M f_0 B_6(f_1, f_2) \, dx$ is a Hochschild 2-cocycle on $C^\infty(M)$ we need $(9) + (13) = 0$, for any $f_i \in C^\infty(M)$. Thus $3B = -32$ and $3A = -16$, which agree with (11). We must also have $3C = 16$ and $D = -16$. Hence using (6), (10), and (12) we conclude the following expression for $B_6(f, h)$:

$$B_6(f, h) = B_{6,conf\,flat}(f, h) - 16 f_{;\,ij} h_{;\,kl} W^{ikjl}/3 - 32(f_{;\,ij} h_{;\,k} + h_{;\,ij} f_{;\,k}) W^i{}_l{}^{jk}{}_;^{\,l}/3$$
$$+ 16 f_{;\,i} h_{;\,j} W^i{}_k{}^j{}_l{}_;^{kl}/3 + 16 f_{;\,i} h_{;\,j} \mathsf{P}_{kl} W^{ikjl}$$
$$+ E f_{;\,i} h_{;}^{\,i} W_{jklm} W^{jklm} + F f_{;\,i} h_{;\,j} W^i{}_{klm} W^{jklm}, \tag{14}$$

where the last two terms are the needed ones to fully complete the expression for $p_{6,2}$ as in (10). Here it is important to keep in mind that one may have many different expressions for the same object, furthermore different expressions that yield the same invariant on conformally flat structures, may represent different invariants in more curved settings. In this particular case, any value of E or F will yield the same invariant in the conformally flat case.

Theorem 3.5. *The expression* (14) *gives a family of bilinear differential functionals acting on $C^\infty(M) \times C^\infty(M)$ (without order zero terms) of order 6 associated to a six-dimensional compact oriented Riemannian manifold M without boundary. Each of these differential functionals is symmetric on f and h, conformally invariant in the sense that $\widehat{B_6}(f, h) = e^{-6\eta} B_6(f, h)$, for a conformal change of the metric $\widehat{g} = e^{2\eta} g$, and defines a Hochschild 2-cocycle on the algebra $C^\infty(M)$ by the relation $\tau(f_0, f_1, f_2) = \int_M f_0 B_6(f_1, f_2) \, dx$.*

Finding the right values of E and F to obtain in the 6-dimensional conformally curved case the unique bilinear differential functional satisfying the relation

$$\mathrm{Wres}(f_0[F, f_1][F, f_2]) = \int_M f_0 B_6(f_1, f_2) \, dx$$

will be the topic of a different paper.

4   The analogue of the Paneitz operator in six dimensions

Stokes' theorem applied to (14) leads to the expression

$$\mathrm{Wres}([F, f][F, h]) = \int_M P_6(f) h \, dx$$

(see [12] for details). Because this equality holds for every f and every h in $C^\infty(M)$ we obtain an operator P_6 constructed in a canonical way from the Wodzicki residue $\mathrm{Wres}([F, f][F, h])$. In the flat case we obtain simply

$$P_{6\mathrm{flat}}(f) = 2\Delta^3(f)/3 = -2f_{;i}{}^i{}_j{}^j{}_k{}^k/3.$$

In the conformally flat case we have terms with $\nabla^a J \nabla^b V$ as a factor:

$$\begin{aligned}
P_{6\mathrm{conf.\,flat}}(f) &= P_{6\,\mathrm{flat}}(f) \\
&+ 4h_{;i}{}^i{}_j{}^j J - 32/3h_{;i}{}^i{}_{jk}\mathsf{P}^{jk} - 8/3h_{;i}{}^i{}_j J_{;}^j - 32/3h_{;ijk}\mathsf{P}^{ij}{}_{;}^k \\
&+ h_{;i}{}^i(4/3J_{;j}^j - 16/3J^2 - 32/3\mathsf{P}_{jk}\mathsf{P}^{jk}) \\
&+ h_{;ij}(64/3J_{;}^{ij} + 64/3J\mathsf{P}^{ij} + 64\mathsf{P}^i{}_k\mathsf{P}^{jk} - 32\mathsf{P}^{ij}{}_{;k}{}^k) \\
&+ h_{;i}(32/3JJ_{;}^i - 8/3J_{;j}^{ji} - 128/3J_{;j}\mathsf{P}^{ij} - 128/3\mathsf{P}_j{}^k\mathsf{P}^{ij}{}_{;k}).
\end{aligned}$$

In the general conformally curved case, we also those terms with a Weyl tensor as a factor:

$$\begin{aligned}
P_6(f) &= P_{6\,\mathrm{conf.\,flat}}(f) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (15) \\
&- 16/3h_{;ijkl}W^{iljk} - 16h_{;ijk}W^{ikj}{}_l{}_;{}^l - Eh_{;i}{}^i W_{jklm}W^{jklm} \\
&+ h_{;ij}(64/9W^i{}_k{}^j{}_l{}^{;kl} - (8/3 + F)W^i{}_{klm}W^{jklm} - 80/3\mathsf{P}_{kl}W^{ikjl}) \\
&+ h_{;i}(80/9\mathsf{P}_{jk}W^{ijk}{}_l{}_;{}^l - 104/3\mathsf{P}_{jk;l}W^{ijk} - 2EW_{jklm}W^{jklm}{}_;{}^i \\
&\quad - 2(44 + 3F)/3W^i{}_{jkl}W^{jkl}{}_m{}_;{}^m + 52/3W_{jklm}W^{ijkl}{}_;{}^m + (34 + 3F)W_{jklm}W^{ijlm}{}_;{}^k).
\end{aligned}$$

The GJMS operators [5] are invariant operators on conformal densities

$$P_{2k} : \mathcal{E}[-n/2 + k] \to \mathcal{E}[-n/2 - k]$$

with principal parts Δ^k, unless the dimension is even and $2k > n$. If n is even, the n-th order operator

$$P_n : \mathcal{E}[0] \to \mathcal{E}[-n]$$

is called *critical* GJMS operator. As noted in [5], $\mathcal{E}[0] = C^\infty(M)$ and $\mathcal{E}[-n]$ is the bundle of volume densities on M. In the 4-dimensional case, the Paneitz operator coincide with the *critical* GJMS operator. The GJMS operators P_{2k}, of Graham-Jenne-Mason-Sparling [5] by construction have the following properties.

i. P_{2k} exists for all k if n is odd, and if n is even exists for $1 \le k \le n/2$.

ii. P_{2k} is formally self-adjoint.

iii. P_{2k} is conformally invariant in the sense that

$$\widehat{P}_{2k} = e^{-(n/2+k)\eta} P_{2k} e^{(n/2-k)\eta}$$

for conformally related metrics $\widehat{g} = e^{2\eta}g$.

iv. P_{2k} has a polynomial expression in ∇ and R in which all coefficients are rational in the dimension n.

v. $P_{2k} = \Delta^k + \text{lot}$. (Here and below, lot $=$ "lower order terms".)

vi. P_{2k} has the form

$$\delta S_{2k} d + \left(\frac{n}{2} - k\right) Q_{2k},$$

where Q_{2k} is a local scalar invariant, and S_{2k} is an operator on 1-forms of the form

$$(d\delta)^{k-1} + \text{lot} \ \text{ or } \Delta^{k-1} + \text{lot}.$$

In this last expression, d and δ are the usual de Rham operators and Δ is the form Laplacian $d\delta + \delta d$. The original work [5] uses the "ambient metric construction" of [4] to prove their existence.

Theorem 4.1. *Any polynomial P_6 of the form* (15) *is conformally invariant and a critical* GJMS *operator (in the sense that it satisfies the properties i. to vi. above) in a six dimensional, compact, conformal manifold without boundary.*

A computation verifies the following particular six dimensional case of Theorem 1.2.vi. of [12]

Proposition 4.2. *The functional B_6 and the polynomial P_6 are related by*

$$P_6(fh) = P_6(f)h + fP_6(h) - 2B_6(f,h).$$

This relation opens the gate to study the relation between products of GJMS-operators of order different than the critical and bilinear differential functionals. Relation that we will explore in future work.

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