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AN APPROACH TO LADDER OPERATORS FOR THE TWO-DIMENSIONAL HARMONIC OSCILLATOR

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Abstract

We show that elementary properties of associated Laguerre polynomials generate the ladder operators for the harmonic oscillator radial wave function in two dimensions.

Keywords: Harmonic oscillator, ladder operators, Laguerre polynomials.

Resumen

Demostramos que propiedades elementales de polinomios de Laguerre generan operadores escalera para la función de onda del oscilador radial armónico en dos dimensiones.

Palabras clave: Oscilador armónico, operadores escalera, polinomios de Laguerre.

Mathematics Subject Classification: 33C45; 34B30.

1 Introduction

Let $\psi_{MN}(\varrho,\varphi)$ be the wave function of the two-dimensional harmonic oscillator $(2DHO)$ in polar coordinates with natural units($\hbar = 1$ and Mass=1), which can be expressed in its radial and angular parts as [1]:

$$\psi_{MN} = R_{|M|N}(\varrho)e^{iM\varphi}, \qquad N = 0, 1, 2, \dots \tag{1}$$

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with $M = -N, -N+2, \ldots, N-2, N$ where

$$R_{MN} = C_{mN}\varrho^m e^{-\frac{\varrho^2}{2}} L^m_{\frac{N-m}{2}}(\varrho^2), \quad C_{mN} = \left[\frac{(\frac{N-m}{2})!}{\pi\,(\frac{N+m}{2})!}\right]^{\frac{1}{2}}, \quad m \geq 0 \tag{2}$$

and L^p_q denoting the associated Laguerre polynomials [2].

Here we shall determine the ladder operators $\hat{O}^{\pm}_{mN}$, for the radial wave function(2), such that:

$$R_{m\pm 2,N} = \hat{O}^{\pm}_{mN} R_{mN}, \tag{3}$$

employing only elementary properties of L^p_q, that is, it is possible to construct $\hat{O}^{\pm}_{mN}$ without the use of specific techniques as the factorization method [3-6].

2 Ladder operators for the 2DHO

From (2) we have that:

$$R_{m+2,N} = \frac{C_{mN}\ \varrho^{m+2}}{\left[(\frac{N-m}{2})(\frac{N+m}{2}+1)\right]^{\frac{1}{2}}}\ e^{-\frac{\varrho^2}{2}} L^{m+2}_{\frac{N-m}{2}-1}(\varrho^2), \tag{4}$$

but if we remember the known relations [2]:

$$\frac{d}{d\mu}L^{\alpha}_{\beta}(\mu) = -L^{\alpha+1}_{\beta-1}(\mu), \qquad L^{\beta+1}_{\gamma}(\mu) = \left(1 - \frac{d}{d\mu}\right)L^{\beta}_{\gamma}(\mu), \tag{5}$$

then (4) may be written as:

$$R_{m+2,N} = -\frac{C_{mN}\varrho^{m+1}}{\sqrt{(N-m)(N+m+2)}}\ e^{-\frac{\varrho^2}{2}}\ \frac{d}{d\varrho}\left(1 - \frac{1}{2\varrho}\frac{d}{d\varrho}\right)L^m_{\frac{N-m}{2}}(\varrho^2), \tag{6}$$

From the associated Laguerre equation[2] it is immediate the expression:

$$\frac{d^2}{d\varrho^2}L^m_{\frac{N-m}{2}}(\varrho^2) = \left(2\varrho - \frac{1+2m}{\varrho}\right)\frac{d}{d\varrho}L^m_{\frac{N-m}{2}}(\varrho^2) - 2(N-m)L^m_{\frac{N-m}{2}}(\varrho^2), \tag{7}$$

and thus (6) adopts the form:

$$R_{m+2,N} = -\frac{C_{mN}\ \varrho^m e^{-\frac{\varrho^2}{2}}}{\sqrt{(N-m)(N+m+2)}}\left[\frac{(1+m)}{\varrho}\frac{d}{d\varrho}L^m_{\frac{N-m}{2}}(\varrho^2) + (N-m)L^m_{\frac{N-m}{2}}(\varrho^2)\right], \tag{8}$$

but (2) implies that $L^m_{\frac{N-m}{2}}(\varrho^2) = \frac{\varrho^{-m}}{C_{mN}}\ e^{\frac{\varrho^2}{2}} R_{mN}$, then (8) leads to (3) with:

$$\hat{O}^{+}_{mN} = \frac{m+1}{\sqrt{(N-m)(N+m+2)}}\left(\frac{m}{\varrho^2} - \frac{N+1}{m+1} - \frac{1}{\varrho}\frac{d}{d\varrho}\right)R_mN. \tag{9}$$

A similar process permits to deduce the other ladder operator:

$$\hat{O}^{-}_{mN} = \frac{m-1}{\sqrt{(N+m)(N-m+2)}}\Big(\frac{m}{\varrho^2} - \frac{N+1}{m-1} + \frac{1}{\varrho}\frac{d}{d\varrho}\Big)R_m N. \tag{10}$$

The radial wave function for the Coulomb potential is in terms of L^p_q, to see [7,8], the method here showed is applicable to obtain its ladder operators, which represents an alternative procedure to another techniques [9,10].

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