Revista de Matemática: Teoría y Aplicaciones 2008 **15**(2) : 109–129
cimpa – ucr issn: 1409-2433

A NEW METHOD FOR THE ANALYSIS OF SIGNALS: THE SQUARE-WAVE METHOD

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Recibido/Received: 26 Mar 2008 — Aceptado/Accepted: 11 Jul 2008

Abstract

The "Square-Wave Method" (SWM) presented here is a new method for the systematic analysis of signals – either locally or globally – depending on only one variable (time). The SWM is based on a technique (previously described elsewhere) for the representation of this type of signals using a sum of trains of square waves.

The SWM is applied here to several analytically characterized signals and to an audio signal.

Keywords: signal analysis, trains of square waves, functions of time, representation of functions

Resumen

Se presenta un nuevo método —el "Método de ondas cuadradas"— para el análisis sistemático —tanto de carácter local como de carácter global— de señales dependientes de una sola variable: el tiempo. Este método está basado en una técnica previamente descripta para la representación de señales de dicho tipo mediante una suma de trenes de ondas cuadradas.

El método se ha aplicado a varias señales caracterizadas analíticamente y a una señal de audio.

Palabras clave: análisis de señales, trenes de ondas cuadradas, funciones del tiempo, representación de funciones

Mathematics Subject Classification: 94A12

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1 Introduction

The objective of this paper is to present a new method for the systematic analysis of signals, either locally or globally, depending on only one variable – time. Given the significant role of square waves in this method, it will be referred to here as the "Square-Wave Method" (SWM).

The technique for the representation of signals on which the SWM is based cannot be considered a particular case of the representation of signals with the Fourier series or any other series of orthogonal functions. The trains of square waves whose sum is an approximation of a signal characterized in a certain interval are not elements belonging to a set of orthogonal functions in that interval.

What is the state of the art regarding the SWM, the topic discussed in this article? Extremely incipient. Through this paper the intention is to describe the new method sufficiently clearly and precisely to allow any reader interested in applying it to do so.

In view of the fact that the SWM is being presented here, it is not possible, to the authors' best knowledge, to locate references concerning that method. (Reference [2] concerns a technique which is the basis for the SWM, and a brief review of that technique is provided in section 2 below.)

It is not the objective of this article to make a comparative analysis between this new method and Fourier's classic method. Therefore, although a vast amount of technical literature is available on topics such as Fourier series, the Fourier transform, the Fourier fast transform and wavelets, it is not pertinent to include references to them here.

2 Brief review of a previously published technique for the representation of signals, as a basis for the SWM

To facilitate the comprehension of this article for those who are not familiar with the technique for the representation of functions presented in [2], its main features will be described in this section.

For the purposes of this paper, the unit of time – such as seconds (s), milliseconds (ms) – will not be specified in the sections on the treatment of analytically characterized signals. Thus the value of a given interval of time Δt will be specified as $\Delta t = 3$. (It is implicit here that we are dealing with 3 time units.) However, in section 5, regarding the treatment of a signal which is not characterized analytically but rather generated by a physical process, the unit of time *is* indicated.

Let the following function of time $f(t)$ be characterized in the interval $0 \leq t < 3$.

$$f(t) = \begin{cases} 4 & \text{if} \quad 0 \leq t < 1 \\ -5 & \text{if} \quad 1 \leq t < 2 \\ 7 & \text{if} \quad 2 \leq t < 3 \end{cases} \tag{1}$$

In the intervals in which this function is continuous, it can be represented by the sum of three trains of square waves that will be called S_1, S_2 and S_3. T_1 (the period of S_1)

is equal to 6. (That is, $\frac{T_1}{2}$ is equal to the lapse of time for which the function has been characterized.) T_2 (the period of S_2) is equal to 4, and T_3 (the period of S_3) is equal to 2. To determine the amplitudes of S_1, S_2 and S_3, the following system of linear algebraic equations has been solved:

$$\begin{aligned} A_1 + A_2 + A_3 &= 4 \\ A_1 + A_2 - A_3 &= -5 \\ A_1 - A_2 + A_3 &= 7 \end{aligned} \tag{2}$$

$|A_1|$, $|A_2|$ and $|A_3|$ are the amplitudes of S_1, S_2 and S_3, respectively. If this system of equations is solved, the following values are obtained for the unknowns:

$$A_1 = 1; \qquad A_2 = -\frac{3}{2} \qquad \text{and} \qquad A_3 = \frac{9}{2} \tag{3}$$

Trains S_1, S_2 and S_3 of square waves are shown below in figure 1, in the interval $0 \leq t < 12$.

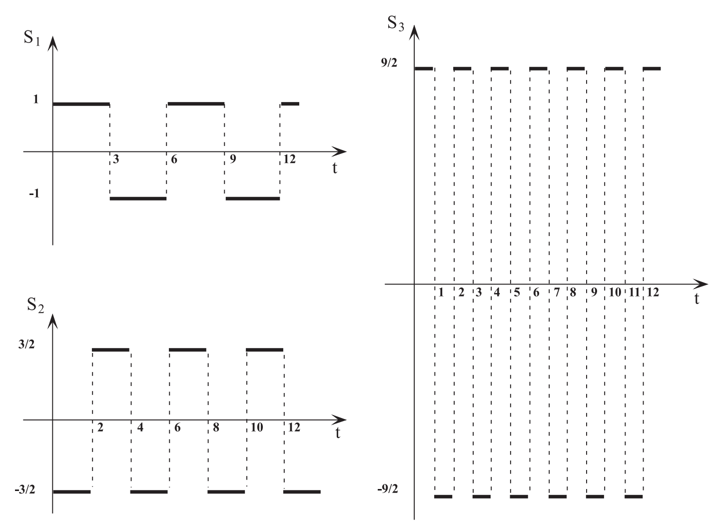

Figure 1: Trains S_1, S_2 and S_3 of square waves for the interval $0 \leqslant t < 12$

Upon adding S_1, S_2 and S_3, a periodic function with a period of $T = 12$ is obtained. In the interval $0 \leq t < 3$, it coincides with the function $f(t)$. This is illustrated in figure 2.

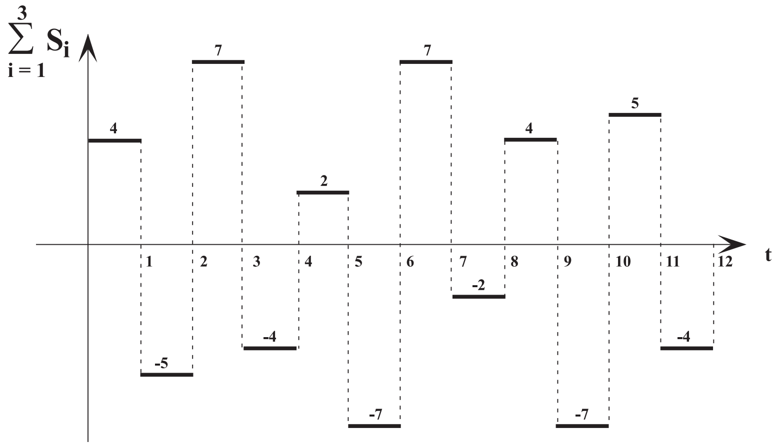

Figure 2: Sum of S_1, S_2 and S_3 in the interval $0 \leqslant t < 12$

This approach may be generalized for any number n of equal intervals in which a decision is made to divide the lapse of time Δt for which the function to be analyzed is characterized. The signal can be represented in the specified lapse of time, by the sum of n trains of square waves. The periods corresponding to the different trains $S_1, S_2, \ldots, S_n$ of square waves are as follows:

$$\begin{aligned} T_1 &= 2\Delta t\left(\frac{n}{n}\right) = 2\Delta t \\ T_2 &= 2\Delta t\left(\frac{n-1}{n}\right) \\ T_3 &= 2\Delta t\left(\frac{n-2}{n}\right) \\ &\vdots \\ T_n &= 2\Delta t\left(\frac{1}{n}\right) \end{aligned} \tag{4}$$

In other words:

$$T_i = 2\Delta t\left(\frac{n-i+1}{n}\right); \qquad i = 1, 2, \ldots, n \tag{5}$$

Of course, the frequencies of these trains of waves will be the inverse of the corresponding periods:

$$f_i = \frac{1}{2\Delta t}\left(\frac{n}{n-i+1}\right); \qquad i = 1, 2, \ldots, n \tag{6}$$

For example, attention will be given below to the system of equations resulting if $n = 7$:

$$
\begin{aligned}
A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 &= C_1 \\
A_1 + A_2 + A_3 + A_4 + A_5 + A_6 - A_7 &= C_2 \\
A_1 + A_2 + A_3 + A_4 + A_5 - A_6 + A_7 &= C_3 \\
A_1 + A_2 + A_3 + A_4 - A_5 - A_6 - A_7 &= C_4 \\
A_1 + A_2 + A_3 - A_4 - A_5 + A_6 + A_7 &= C_5 \\
A_1 + A_2 - A_3 - A_4 - A_5 + A_6 - A_7 &= C_6 \\
A_1 - A_2 - A_3 - A_4 + A_5 - A_6 + A_7 &= C_7
\end{aligned}
\tag{7}
$$

In the above equations, $|A_1|$, $|A_2|$, ... and $|A_7|$ are the amplitudes corresponding respectively to trains S_1, S_2, ... and S_7. C_1, C_2, ... and C_7 are the values of $f(t)$ corresponding to the midpoints of the seven subintervals considered here of the time interval Δt in which the function treated by the technique described above has been characterized.

It has been supposed that the values of $f(t)$ at these midpoints are computable. What should be done if one of the midpoints – t_i – turns out to be precisely a point of discontinuity of the signal $f(t)$? In other words, what value should be attributed to $f(t)$, in this case, at that point t_i? One possible criterion – based on results obtained by the Fourier series – is the following: the value given to $f(t_i)$ is the average of the left and right lateral limits, which is

$$f(t_i) = \frac{1}{2}\left(\lim_{t \to t_i-} f(t) + \lim_{t \to t_i+} f(t)\right) \tag{8}$$

Of course, the time interval for which the function to be treated is specified may be any whatsoever. (There is no reason for one of the extremes to be, necessarily, the origin of the time axis of the system of coordinates used.)

Let the following function of time $g(t)$ be characterized in the interval $1 \le t < 4$.

$$g(t) = \begin{cases} 1 & \text{if} \quad 1 \le t < \frac{3}{2} \\ -2 & \text{if} \quad \frac{3}{2} \le t < \frac{7}{3} \\ e^t & \text{if} \quad \frac{7}{3} \le t < 4 \end{cases} \tag{9}$$

Consider that the interval $1 \le t < 4$ is subdivided into seven subintervals with a length of 3/7. Here the function $g(t)$ will be approximated by the step function $h(t)$ which, in each of the seven subintervals considered, is constant and is such that its value corresponds to that of $g(t)$ at the midpoint of the corresponding subinterval. Then the amplitudes corresponding to the trains of square waves S_1, S_2, ... and S_7, whose sum is approximated by the function $g(t)$ in the interval $1 \le t < 4$, will be determined. This is equivalent to expressing the step function $h(t)$ as a sum of seven trains of square waves. To determine the amplitudes of these trains of square waves, the following system of equations should be solved:

$$
\begin{aligned}
A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 &= 1 \\
A_1 + A_2 + A_3 + A_4 + A_5 + A_6 - A_7 &= -2 \\
A_1 + A_2 + A_3 + A_4 + A_5 - A_6 + A_7 &= -2 \\
A_1 + A_2 + A_3 + A_4 - A_5 - A_6 - A_7 &= e^{5/2} \\
A_1 + A_2 + A_3 - A_4 - A_5 + A_6 + A_7 &= e^{41/14} \\
A_1 + A_2 - A_3 - A_4 - A_5 + A_6 - A_7 &= e^{47/14} \\
A_1 - A_2 - A_3 - A_4 + A_5 - A_6 + A_7 &= e^{53/14}
\end{aligned}
\tag{10}
$$

Figure 3 represents the approximation to $g(t)$ by the sum of the seven trains of square waves S_1, S_2, ... and S_7 in the interval $1 \leq t < 4$.

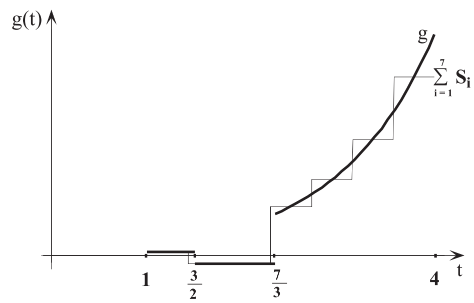

Figure 3: Approximation of g(t) using seven trains of square waves

For this case, the result is $g(t) \simeq \sum_{i=1}^{7} S_i$, with each S_i being a train of square waves.

The periods corresponding to S_1, S_2, ... and S_7 are as follows:

$$T_1 = 6, \quad T_2 = \frac{36}{7}, \quad T_3 = \frac{30}{7}, \quad T_4 = \frac{24}{7}, \quad T_5 = \frac{18}{7}, \quad T_6 = \frac{12}{7} \quad \text{and} \quad T_7 = \frac{6}{7}. \tag{11}$$

Of course, the larger the number of trains of square waves added, the closer the approximation of a given function. Thus, for example, in figures 4 and 5 the approximations obtained for that function are shown, when adding 17 and 119 trains of square waves respectively. In figure 5, with the degree of resolution used, it is impossible to distinguish between the approximation obtained and the given function.

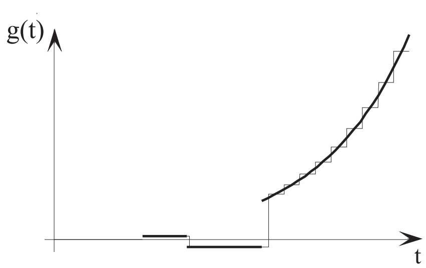

Figure 4: Approximation of function g(t) using 17 square waves

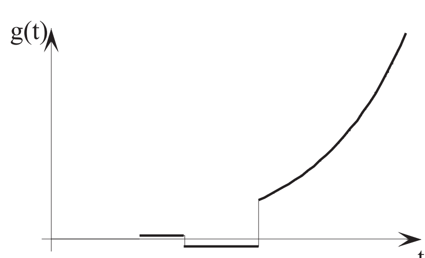

Figure 5: Approximation of function g(t) using 119 square waves

When using the above technique, the signals may be approximated in their regions of continuity as precisely as desired.

3 Description of the method used

The method used for the analysis of signals is described in this section, and the corresponding results are presented in sections 4 and 5.

Suppose that one wants to analyze an interval of a signal of which there are N "samples". If the signal considered has been characterized analytically, the values of these "samples" are computed (using the method described in section 2) for a sequence of instants. However, if the signal has been generated by a physical process, these values are taken from a file containing a sequence of them. Of course, in this case, the values of the "samples" have been obtained by measurements and not by computation. It may be admitted that the subintervals found between any two consecutive "samples" whatsoever are the same. If the signal to be analyzed has been characterized analytically, these N values can be calculated.

Let us admit that the signal displays a certain structure such that parts of it corresponding to the sequences of M "samples" (with $M \leq N$), or comprised of a number of "samples" near to M (such as $M-1$ or $M+2$, for example), are particularly interesting. In this case it would be useful to map, onto the time axis, a "window" with a length of M – that is, one composed of M "samples" – with which the sequence of N "samples" mentioned above will be swept. The first position of the "window" will include "samples" 1, 2, ... and M. The second position of the "window" will include "samples" 2, 3, ... and $M+1$, and so on accordingly, in such a way that the last position of the "window" will include "samples" $N-M+1, N-M+2, \ldots$ and N. The total number of positions of the "window" will be equal to $N-M+1$.

It will be convenient for M to be equal to an odd number so that the "window" will have a middle "sample"; therefore, if a "window" includes 17 "samples", the middle "sample" will be the ninth.

Let us admit that the "samples" of the sequence of N "samples" considered have been numbered from 1 to N. For the first position of the "window", the middle "sample" will have the number $\dfrac{M-1}{2}+1$; for the second position of the "window", the middle "sample" will have the number $\dfrac{M-1}{2}+2$; and for the third position of the "window", the middle "sample" will have the number $\dfrac{M-1}{2}+3$, and so on successively. In general, for the kth position of the "window", the middle "sample" will have the number $\dfrac{M-1}{2}+k$; in particular, for the last position of the "window", the middle "sample" will have the number $\dfrac{M-1}{2}+N-M+1 = N-\dfrac{M}{2}+\dfrac{1}{2}$.

For each one of the positions of the "window" mentioned, the type of analysis described in section 2 will be carried out, based on the use of trains of square waves. Thus if the window has a length equal to M, then for the corresponding signal analysis process, M

trains of square waves will be used: S_1, S_2, ... and S_M.

A description is provided below of the procedure used to obtain a graph in which it can be clearly seen how, for the signal analyzed, a certain value corresponding to S_1 (the value of A_1 mentioned in section 2) depends on the location of the window with a length of M.

The axis of the abscissas will be time.

Consider the window located in its first position. The type of analysis described in section 2 will be applied to the part of the signal corresponding to that position of the window. Hence, a certain value A_1 will be determined. The first point to be plotted on the graph is one whose abscissa is the midpoint of the interval corresponding to the first position of the window and whose ordinate is that value A_1.

Now consider the window located in its second position. The type of analysis described in section 2 will be applied to the part of the signal corresponding to that position of the window. Thus, a certain value A_1 will be determined. The second point to be plotted on the graph is one whose abscissa is the midpoint of the interval corresponding to the second position of the window and whose ordinate is that value A_1. Because the window has been displaced, there is no reason for that second value graphed to be the same as the first value graphed.

Consider the window located in its third position. The type of analysis described in section 2 will be applied to the part of the signal corresponding to that position of the window. Hence, a certain value A_1 will be determined. The third point to be plotted on the graph is one whose abscissa is the midpoint of the interval corresponding to the third position of the window and whose ordinate is the value A_1. Because the window has been displaced, there is no reason for that third value graphed to be the same as either of the values graphed previously.

And so on accordingly.

It may be considered that the sequence of values obtained by this procedure corresponds to a certain function of time A_1^*.

An analogous procedure may be followed for the trains of square waves S_2, S_3, ... and S_M. It may be considered that each of the sequences of values obtained for those trains of waves correspond to the functions of time A_2^*, A_3^*, ... and A_M^* respectively.

4 Application of the SWM to analytically characterized signals

4.1 Example 1

Consider the following function:

$$\begin{aligned} g_1(t) = B \sin 2\pi f_1 t + C \sin 2\pi f_2 t&, \\ B = 6 \qquad\qquad C &= 2 \\ f_1 = 1 \qquad\qquad f_2 &= 4 \end{aligned} \tag{12}$$

The signal $g_1(t)$ has been represented in figure 6.

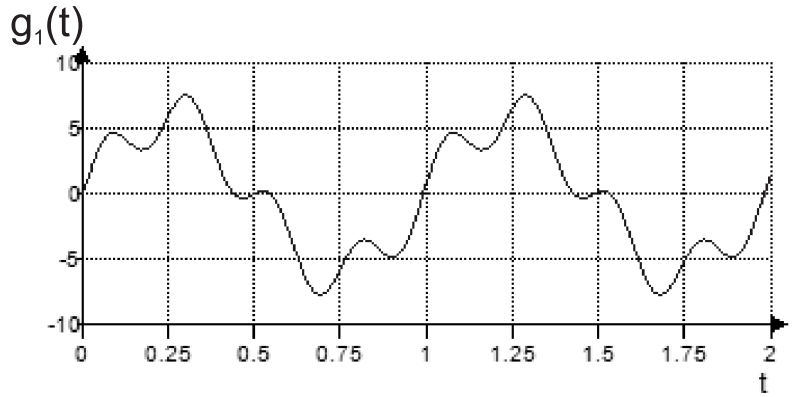

Figure 6

When applying the SWM to this function with $M = 25$ and $N = 2000$, the results shown in figures 7a, 7b, 7c and 7d are obtained. (The graphs corresponding to the functions S_i^*, $i = 4, 5, \ldots, 24$, have been omitted.)

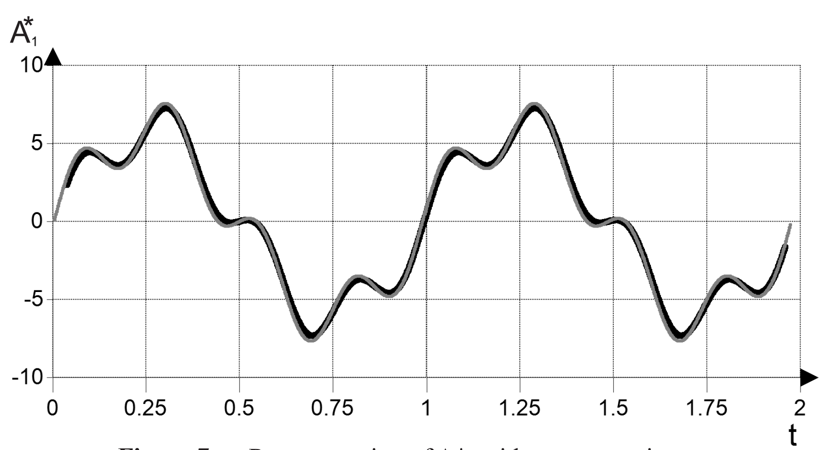

Figure 7a: Representation of A^*_1 with respect to time (Original curve $g_1(t)$ represented with thin line)

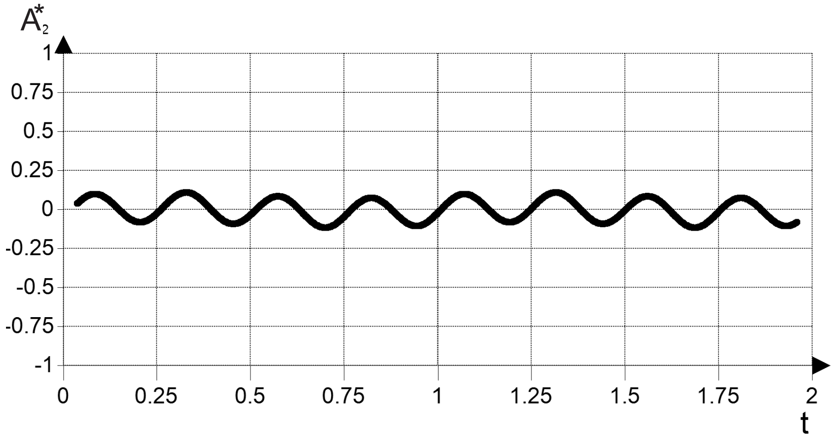

Figure 7b: Representation of A*$_2$ with respect to time

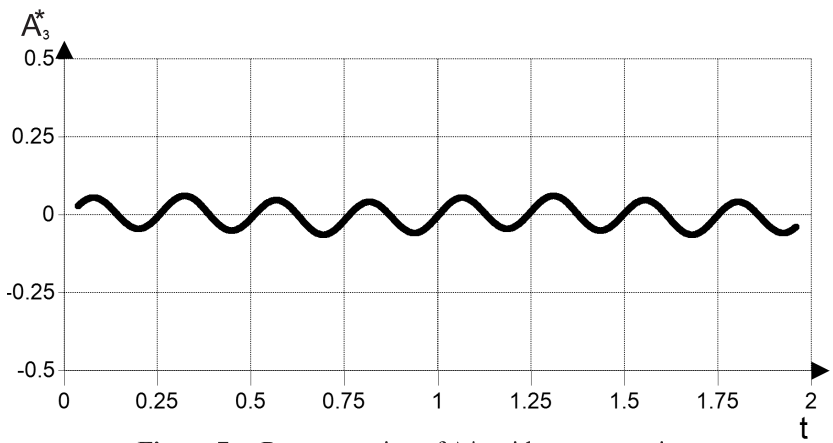

Figure 7c: Representation of A*$_3$ with respect to time

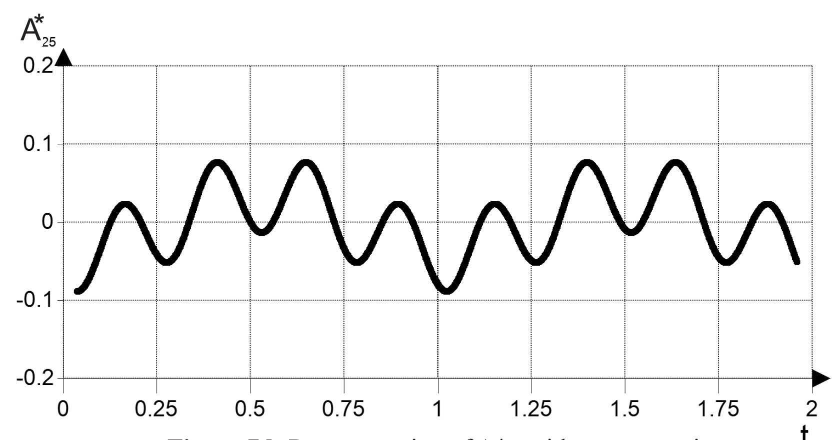

Figure 7d: Representation of A*$_{25}$ with respect to time

When applying the SWM to $g_1(t)$ with $M = 155$ and $N = 2000$, the results shown in figures 8a, 8b, 8c and 8d are obtained. (The graphs corresponding to the functions S_i^*, $i = 4, 5, \ldots, 154$, have been omitted.)

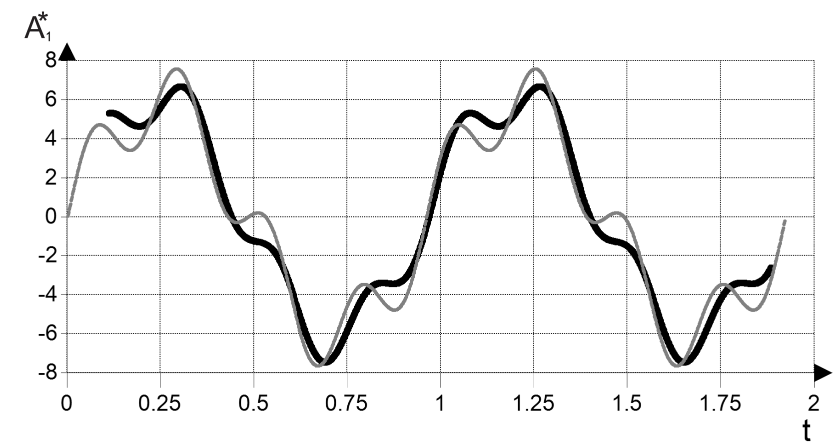

Figure 8a: Representation of A^*_1 with respect to time (Original curve $g_1(t)$ represented with thin line)

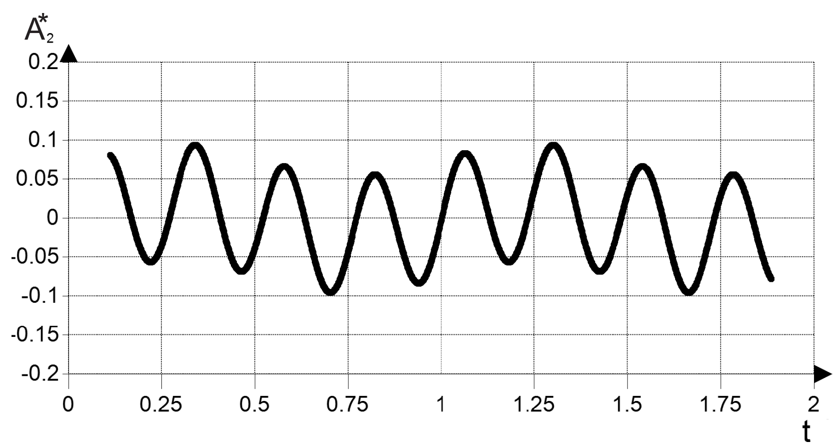

Figure 8b: Representation of A^*_2 with respect to time

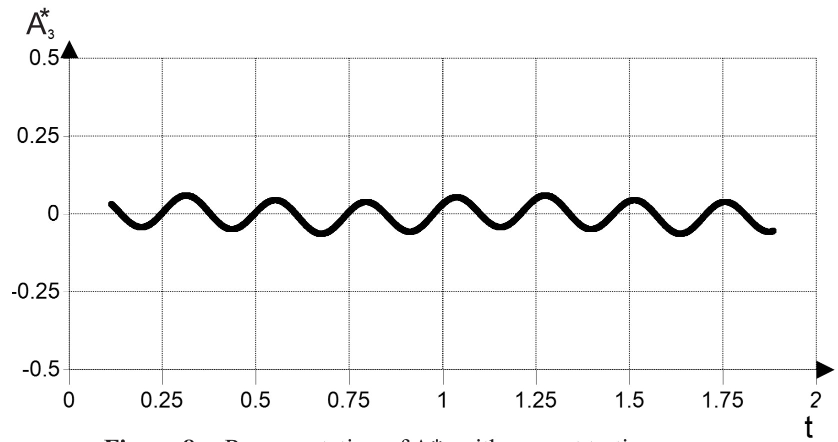

Figure 8c: Representation of A^*_3 with respect to time

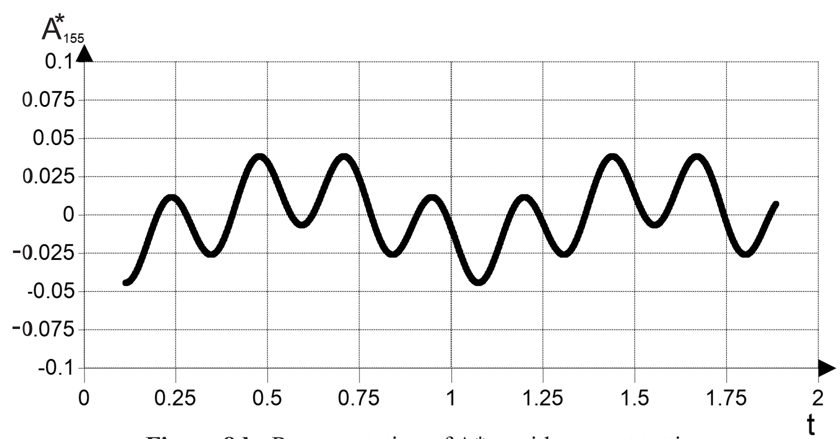

Figure 8d: Representation of A^*_{155} with respect to time

4.2 Example 2

Consider the following function:

$$
\begin{aligned}
g_2(t) = B \sin 2\pi f_1 t \times C \sin 2\pi f_2 t,& \\
B = 6 \qquad\qquad & C = 2 \\
f_1 = 1 \qquad\qquad & f_2 = 4
\end{aligned}
\tag{13}
$$

The signal $g_2(t)$ has been represented in figure 9.

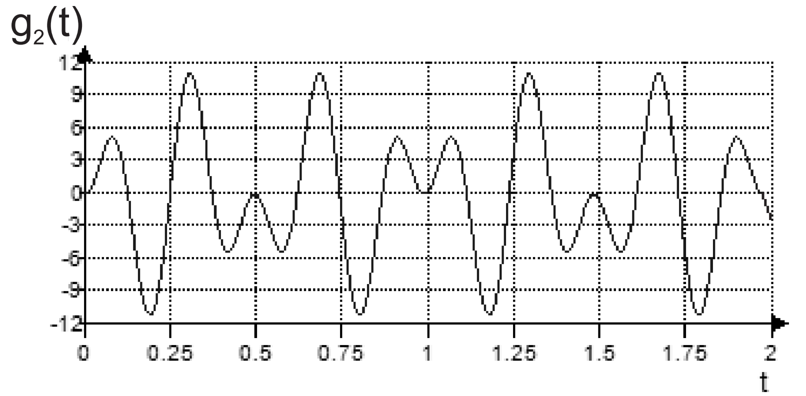

Figure 9

When applying the SWM to this function with $M = 25$ and $N = 2000$, the results shown in figures 10a, 10b, 10c and 10d are obtained. (The graphs corresponding to the functions S_i^*, $i = 4, 5, \ldots, 24$, have been omitted.)

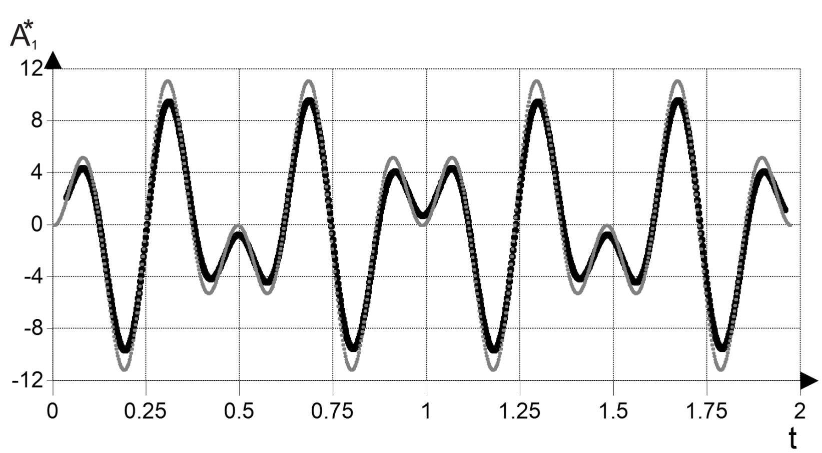

Figure 10a: Representation of A^*_1 with respect to time (Original curve $g_2(t)$ represented with thin line)

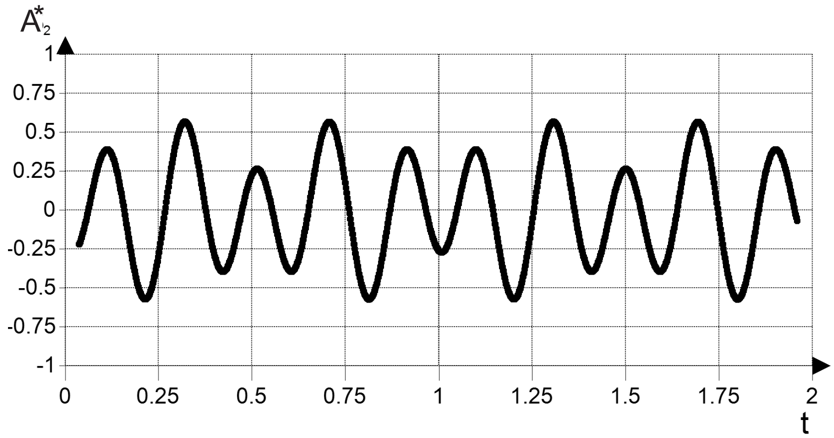

Figure 10b: Representation of A^*_2 with respect to time

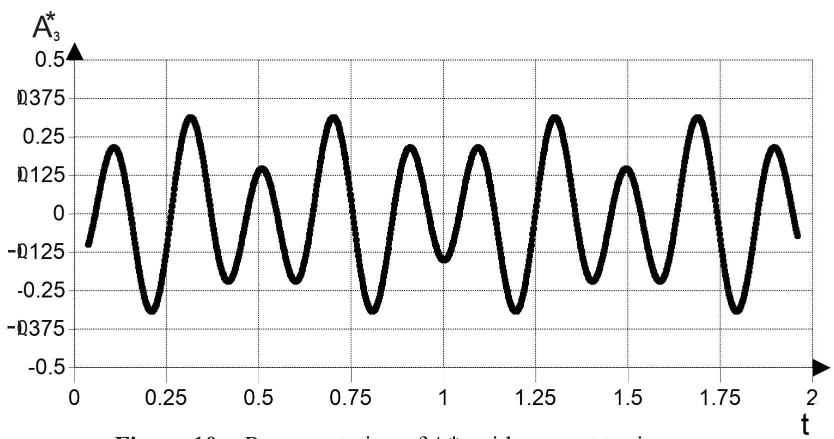

Figure 10c: Representation of A^*_3 with respect to time

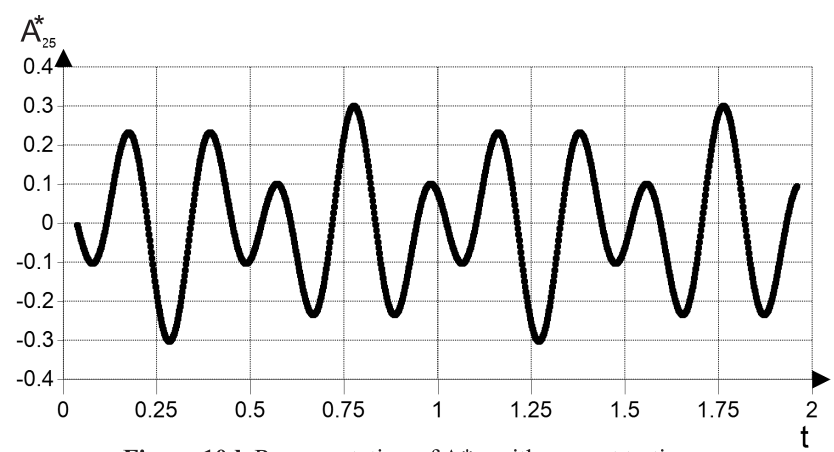

Figure 10d: Representation of A^*_{25} with respect to time

When applying the SWM this function with $M = 155$ and $N = 2000$, the results shown in figures 11a, 11b, 11c and 11d are obtained. (The graphs corresponding to the functions

S_i^*, $i = 4, 5, \ldots, 154$, have been omitted.)

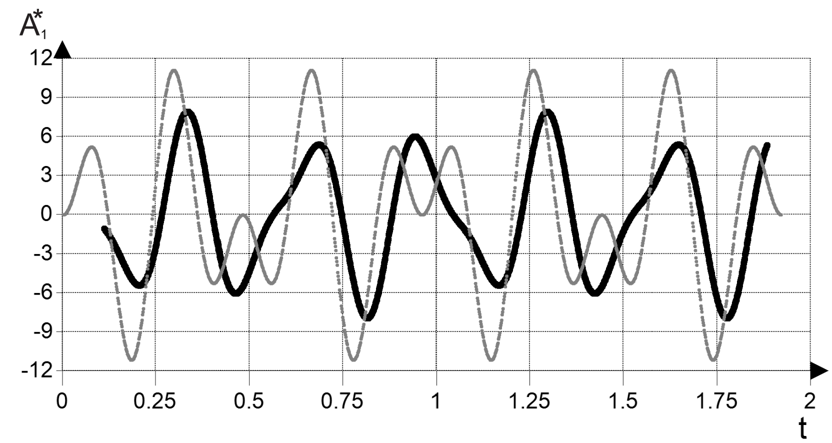

Figure 11a: Representation of A^*_1 with respect to time (Original curve $g_2(t)$ represented with thin line)

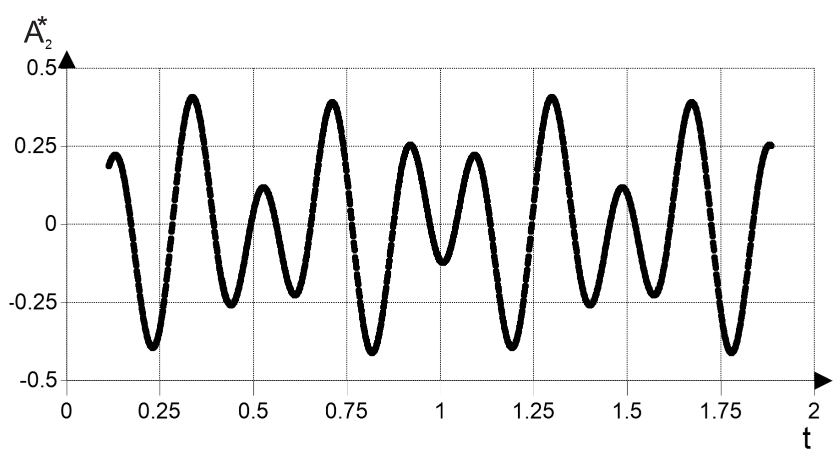

Figure 11b: Representation of A^*_2 with respect to time

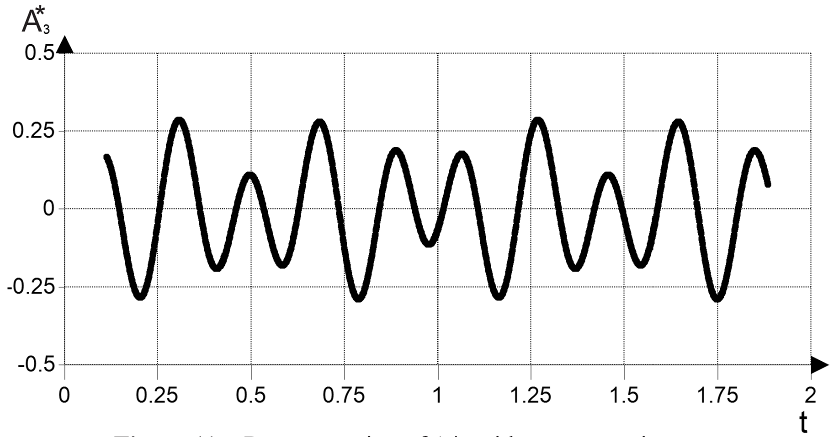

Figure 11c: Representation of A^*_3 with respect to time

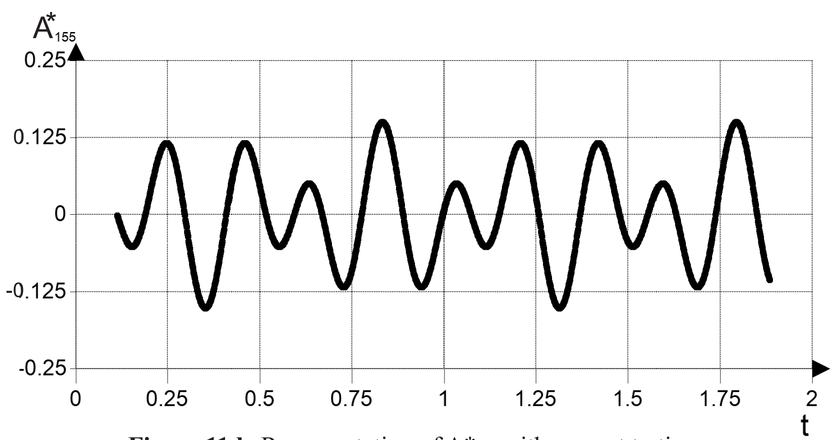

Figure 11d: Representation of A^*_{155} with respect to time

5 Application of the SWM to the analysis of an audio signal

Consider the audio signal $A(t)$ corresponding to the sound obtained by pressing a piano key – middle C. In this case, the frequency of the audio signal is equal to 268Hz. The sampling frequency is 44.1KHz. In figure 12, this audio signal has been represented in the time interval found between 0 and 10^{-1} s.

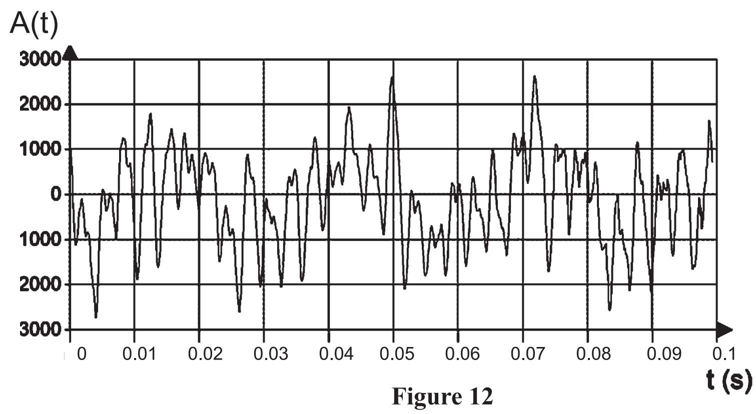

Figure 12

When the SWM is applied to this signal with $M = 25$ and $N = 4192$, the results shown in figures 13a, 13b, 13c and 13d are obtained. (The graphs corresponding to the functions S_i^*, $i = 4, 5, \ldots, 24$, have been omitted.)

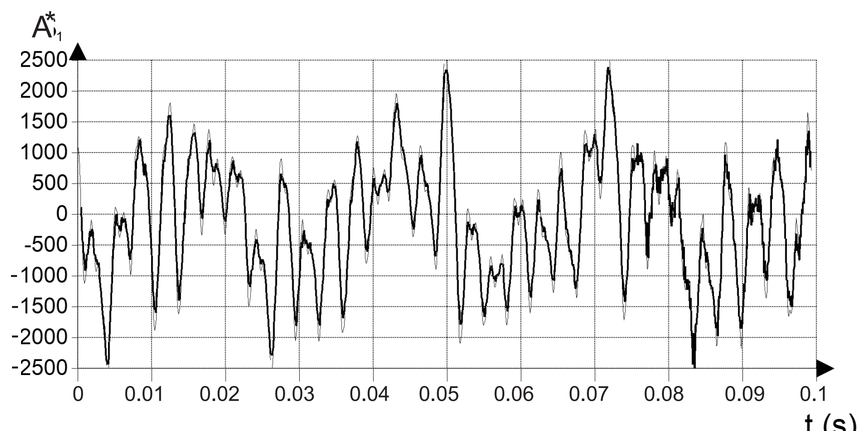

Figure 13a: Representation of A^*_1 with respect to time
(Original curve A(t) represented with thin line)

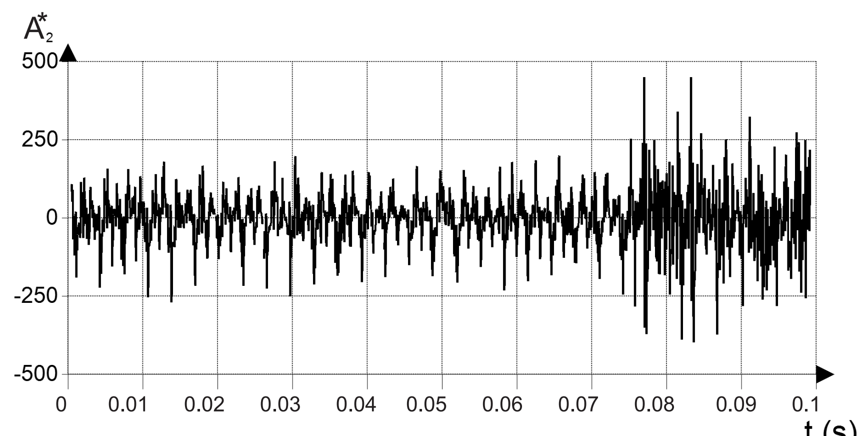

Figure 13b: Representation of A^*_2 with respect to time

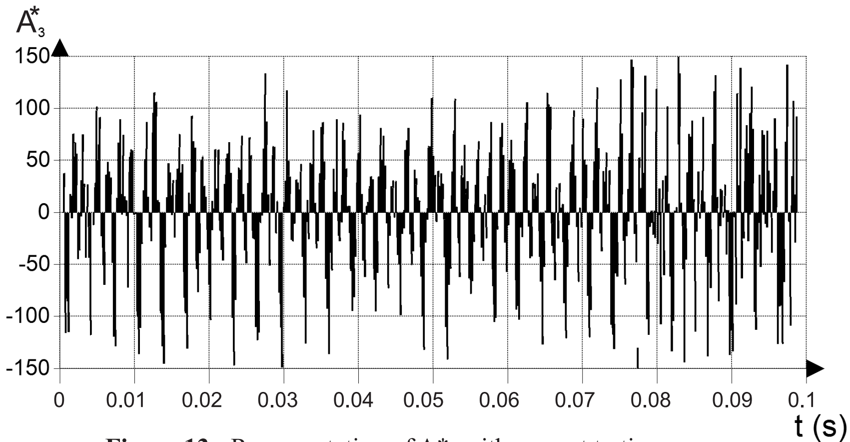

Figure 13c: Representation of A^*_3 with respect to time

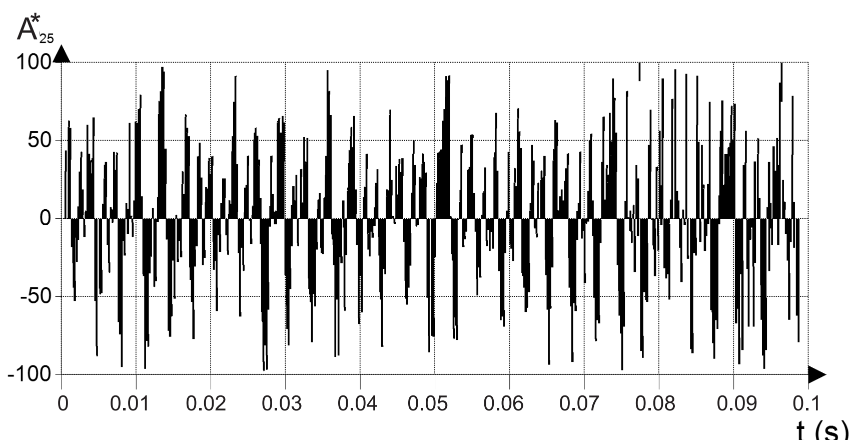

Figure 13d: Representation of A^*_{25} with respect to time

When the SWM is applied to this same audio signal with $M = 617$ and $N = 4192$, the results shown in figures 14a, 14b, 14c and 14 d are obtained. (The graphs corresponding to the functions S^*_i, $i = 4, 5, \ldots, 616$, have been omitted.)

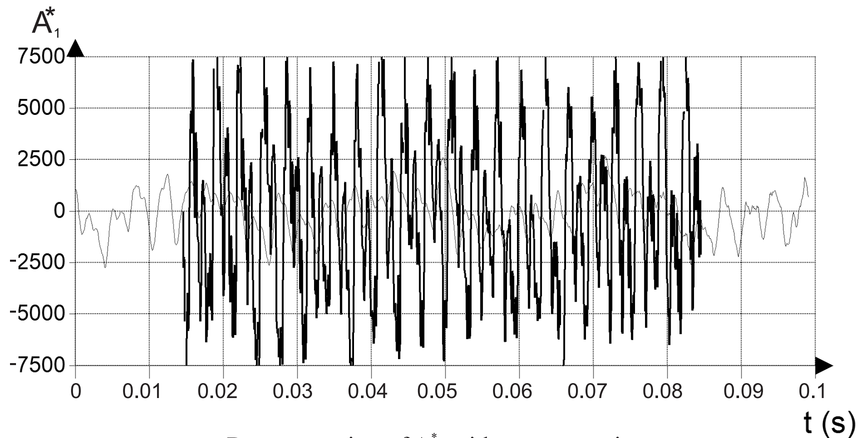

Figure 14a: Representation of A^{*}_{1} with respect to time
(Original curve A(t) represented with thin line)

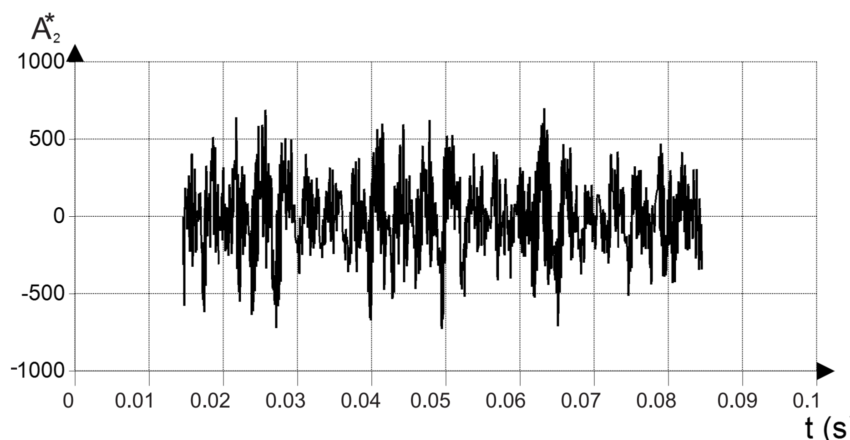

Figure 14b: Representation of A^{*}_{2} with respect to time

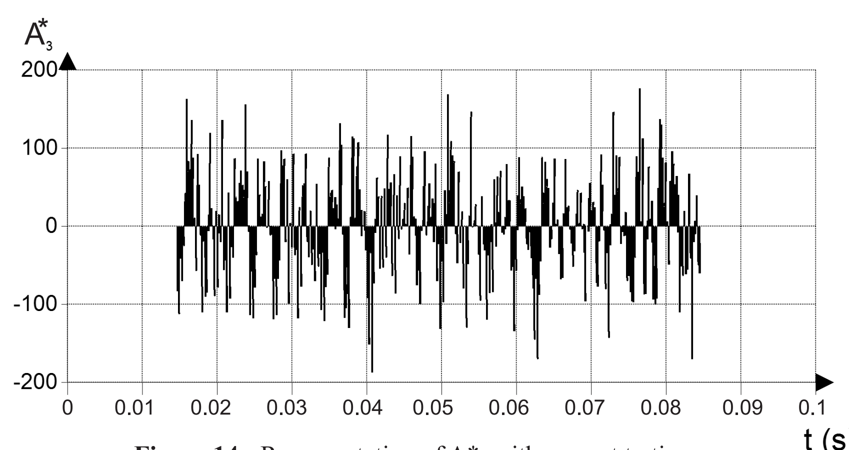

Figure 14c: Representation of A^{*}_{3} with respect to time

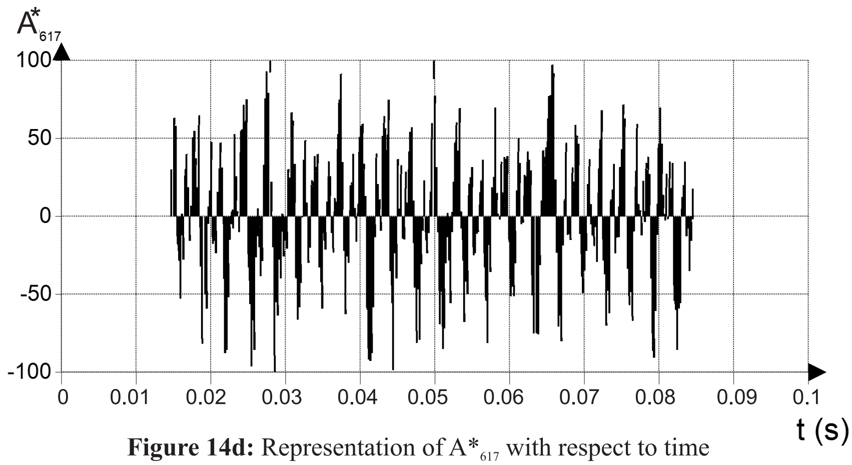

Figure 14d: Representation of A^*_{617} with respect to time

6 Conclusions and prospects

No limitations have been found for the applicability of the SWM to functions of time, in time intervals in which these functions satisfy the Dirichlet conditions [1].

It should be stressed that the computations required for the application of the SWM are relatively simple. They only involve solving systems of linear algebraic equations.

In future papers the following topics will be discussed:

I. Dependency on the $\frac{M}{N}$ relation in the results obtained by applying the SWM;

II. Generalization of the SWM for the analysis of signals depending on more than one variable.

It should be emphasized that with the introduction of the SWM no attempt is made, in the least, to take the place of the extremely important field of mathematics generated from Fourier's seminal findings. Nevertheless, the fact that outstanding conceptual and technical resources exist for signal analysis does not mean that the possibility of developing a new method for that purpose should be dismissed.

As mentioned in the introduction of this article, it is not our objective at this time to compare the SWM with the Fourier approach. (For that to be meaningful, this new method must first be presented, and later developed at least minimally.) However, attention should probably be given to two preliminary questions which some readers of this article may ask:

a. Is it more complicated and tedious to use the SWM than it is to resort to the Fourier approach? No, it is actually simpler. Thus, for example, it is easier to compute each A_i $(i = 1, 2, 3, \dots)$ by solving systems of linear algebraic equations than it is to calculate Fourier coefficients.

b. Are the results obtained by applying the SWM less exact and precise than those calculated using the Fourier approach? No, in both cases, the analysis may be as exact and as precise as desired, if enough computing time is devoted to the task.

References

Note: To the authors' best knowledge, the technique of representing signals by sums of trains of square waves and the method of signal analysis described here are both original and no scientific literature is available on these topics. For this reason, only two references have been included below.

[1] Boas, M. L. (2005) *Mathematical Methods in the Physical Sciences* (3rd ed.). John Wiley and Sons, Hoboken, NJ.

[2] Skliar, O.; Medina, V.; Láscaris, T. (2000) "Análisis digital. Una nueva técnica para la representación de funciones acotadas que, en un intervalo finito, satisfacen las condiciones de Dirichlet", *Revista de Matemática: Teoría y Aplicaciones* **7**(1-2): 175–184.