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N-DIMENSIONAL ALMOST PERIODIC FUNCTIONS II

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Abstract

In this paper we continue the research begun in [CA-2]. Some new results are shown and proven, like the structure theorem for n-dimensional almost periodic functions by using the Bochner Transform. Also, the Haraux [Har] condition in the n-dimensional case, and some topological theorems similar to Bochner and Ascoli theorems. Furthermore, we answer a question formulated by Prof. Fischer [Fis], and we study an average theorem for integrals of almost periodic functions.

Keywords: Almost periodic functions, structure theorem, Radon transform.

Resumen

El presente trabajo se continúan las investigaciones iniciadas en [CA-2]. Demostramos algunos nuevos resultados, como el teorema de estructura para funciones cuasi-periódicas n-dimensionales usando la Transformada de Bochner. También, se prueba un condición de Haraux [Har] en el caso n-dimensional, y algunos teoremas topológicos similares a los teoremas de Bochner y Ascoli. Además, se responde a una una conjetura del Profesor Fischer [Fis], y se estudia un un teorema promedio para integrales cuasi-periódicas.

Palabras clave: Funciones cuasi-periódicas, teorema de estructura, transformada de Radon.

Mathematics Subject Classification: 42A75,43A60,35A22,46F12

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Introduction

Harold Bohr's theory of almost-periodic functions [Bo] was studied in connection with differential equations and other theories, for example Riesz and Nagy presented some applications for compact operators and Banach algebras. S. Bochner [Bo] presents some generalizations of Bohr's definition. The definition for functions with values in abstract spaces is useful in the study of differential equations, Fourier series analysis and Fourier Transform.

Our definition for functions of several variables will be useful in the study of some open problems related to Bochner Transforms. This definition is very similar to Bohr's definition for functions of one variable.

1 Some notations and definitions

In [CA-2] we call a continuous function $f:\mathbb{R}^N\to\mathbb{R}$ almost periodic if $\forall\varepsilon>0$ there is an N-dimensional vector L whose entries are positive and satisfies that $\forall y\in\mathbb{R}^N$ there is a T in the N-dimensional box $[y,y+L]$ (componentwise) such that $|f[x+T]-f[x]|<\varepsilon$ for all $x\in\mathbb{R}^N$. Let $x\in\mathbb{R}^N, x[[i]]$ denotes the i-th component of x. We write $x>0$ if $x[[i]]>0, i=1,\dots,N$.
If $x\in\mathbb{R}^N, x>0, [y,y+x]:=[y[[1]],y[[1]]+x[[1]]]\times\cdots\times[y[[N]],y[[N]]+x[[N]]]$.
If $x,y\in\mathbb{R}^N$ we write $|x-y|:=\begin{pmatrix}|x[[1]]-y[[1]]|\\ \vdots\\ |x[[N]]-y[[N]]|\end{pmatrix}$.

A set $E\subset\mathbb{R}^N$ is called relatively dense (r.d.) if there is an $L\in\mathbb{R}^N, L>0$ such that for all $a\in\mathbb{R}^N, [a,a+L]\cap E\neq\emptyset$.

There are many examples of r.d sets, for instance:

- $\mathbb{Z}, p\mathbb{Z}, p\in\mathbb{R}, p\notin\mathbb{Z}$ are r.d. in $\mathbb{R}$.
- $\mathbb{Z}^N, p_1\mathbb{Z}.\times\dots p_N\mathbb{Z}, p_i\notin\mathbb{Z}, i=1,\dots,N$ are r.d. in $\mathbb{R}^N$.
- If A is an r.d. set in $\mathbb{R}^N$ and B is an r.d. set in $\mathbb{R}^M$ then $A\times B$ is an r.d. set in $\mathbb{R}^{N+M}$.
- If A is an r.d. set in $\mathbb{R}^N$ and $\pi_i:\mathbb{R}^N,\to\mathbb{R}$ is the i-th projection then $\pi_i[A]$ is an r.d. set in $\mathbb{R}$.
- If $f:\mathbb{R}^N\to\mathbb{R}$ is an isometry then $f[A]$ is an r.d. set for any A r.d. set in $\mathbb{R}^N$.

$C_b(\mathbb{R}^N,\mathbb{R})$ denotes the set of all bounded functions from $\mathbb{R}^N\to\mathbb{R}$ endowed with the $||\cdot||_\infty$ norm $f[x_-+m]$ denotes the function $x\to f[x+m]$.

2 Bochner like theorems

Theorem 2.1 *Let $f:\mathbb{R}^N\to\mathbb{R}$ be a continuous function f is almost periodic if $A=\{,f[x_-\pm y],y\in\mathbb{R}^N\}$ is relatively compact in $C(\mathbb{R}^N,\|\cdot\|_\infty)$.*

Proof. If A is a relatively compact set in $C(\mathbb{R}^N, \|\cdot\|_\infty)$ then A is totally bounded . Let $\epsilon > 0$ arbitrary but fix, then there are $y_i \in \mathbb{R}^N$, $i \in \{1, \ldots, n_0\}$, $f_i = f[x \pm y_i] \in A$ with $A \subset \cup_i^{n_0} B(f_i, \epsilon)$.
From the fact that each f_i is uniformly continuous $i \in \{1, \ldots, n_0\}$ it follows that there is a $\delta > 0$ with $\|x' - x''\|_\infty < \delta$ implies $|f_i[x'] - f_i[x'']| < \epsilon$.
Without loss of generality, we can choose the $y_i, i \in \{1, \ldots, n_0\}$ such that $\|y_i - y_j\|_\infty < \delta$. We get that for any $T \in \mathbb{R}^N$ there are $i_0, i_1 \in \{1, \ldots, n_0\}$ such that:

$$\|f[x_- + T] - f[x_-]\|_\infty \leq \|f[x_- + T] - f_{i_0}\|_\infty + \|f_{i_0} - f_{i_1}\|_\infty + \|f[x_-] - f_{i_1}\|_\infty \leq 3\epsilon$$

which means f is almost periodic.
Reciprocally: If f is a almost periodic function then $f[x_- + y]$ is bounded for all $y \in \mathbb{R}^N$. Furthermore $A = \{f[x_- + y], y \in \mathbb{R}^N\}$ is equicontinuous [CA-2], then A is relatively compact. ■

Theorem 2.2 *f is almost periodic if for any sequence $(y_n)_{n\in}$ there is a subsequence $(y_{n_k})_{k\in}$ and a function $g : \mathbb{R}^N \to \mathbb{R}$ such that $f[x_- + y_{n_k}] \to g$ in $C(\mathbb{R}^N, \|\cdot\|_\infty)$*

Proof. A direct consequence of theorem 2.1. ■

Lemma 2.3 *The Bochner transformation of a function f defined as $B[f][y] := f[x_- + y]$ is a continuous operator from $AP := \{f : \mathbb{R}^N \to \mathbb{R}, f$ almost periodic$\}$ into AP.*

3 Haraux conditions

In this section, we study Haraux conditions [Har] for almost periodic functions.

Lemma 3.1 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a uniform continuous bounded function. Let $(y_n)_{n\in} \subset \mathbb{R}^N$ be a sequence such that $f[x_- + y_n] \to g[x_-]$ uniformly, and let $(x_n)_{n\in} \subset \mathbb{R}^N$ be a sequence such that $x_n \to x_0$. Then $f[x_- + y_n + x_n] \to g[x_- + x_0]$.*

Proof. It is a direct consequence of the inequality:

$$\|f[x_- +] - g[x_- + x_0]]Vert \leq \|f[x_- + x_n + y_n] - f[x_- + x_0 + y_n]\| + \|f[x_- + x_0 + y_n] - g[x_- + x_0]\|.$$

■

Lemma 3.2 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous bounded function, and let $E \subset \mathbb{R}^N$ r.d. and $\cup_{y\in E}\{f[x_- + y]\}$ relatively compact in $C_b(\mathbb{R}^N, \|\cdot\|_\infty)$. Then f is uniformly continuous .*

Proof. Let's suppose f is not uniformly continuous, then there is an $\epsilon > 0$ and two sequences $(x_n), (y_n)$ in $\mathbb{R}^N$ with $\|x_n - y_n\|_\infty < \frac{1}{n}$ for all n in $\mathbb{N}$ and $|f[|x_n] - f[y_n]| \geq \epsilon$. Let $L > 0$ in $\mathbb{R}^N$ with the property that for all $x \in \mathbb{R}^N, [x, x + L] \cap E$ is not empty. Then there are two sequences: $(x_n^1) \subset E$ and $(x_n^2) \subset [-L, L]$ with $x_n = x_n^1 + x_n^2$.
Then $y_n = x_n^1 + x_n^2 + (y_n - x_n)$. $x_n^2 + (y_n - x_n)$admits a convergent subsequence because $x_n^2 + (y_n - x_n) \in [-2L, 2L]$ for almost all n in $\mathbb{N}$.

We denote all the subsequences with the sequence's name.
Then there are subsequences with: $f[x_- + x_n^1] \to g[x_-] inf[x_- + x_n^1 + x_n^2] \to g[x_- + x_0]$ in $C_b(\mathbb{R}^N, \|\cdot\|_\infty)$. (A subsequence of x_n^2 converges to $x_0 \in [-L, L]$) $f[x_- + y_n] \to g[x_- + x_0] \in C_b(\mathbb{R}^N, \|\cdot\|_\infty)$ then: $f[x_n] - f[y_n] \to 0$, which is a contradiction. ■

Theorem 3.3 (Haraux Theorem) *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous bounded function. Let $E \subset \mathbb{R}^N$, E r.d and $\cup_{y \in E}\{f[x_- + y]\}$ relatively compact in $C_b(\mathbb{R}^N, \|\cdot\|_\infty)$, then f almost periodic.*

Proof. It follows directly form theorem 2.2 and lemmas 3.1, 3.2. ■

4 A consequence of the Haraux conditions

Definition 4.1 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be an almost periodic function; f is said to have a Bochner compact range (BCR) if for any N-dimensional sequence $(x_n)_{n\in}$ there is a subsequence $(x_{n_k})_{k\in}$ and $x_0 \in \mathbb{R}^N$ such that $f[x_- + x_{n_k}] \to f[x_- + x_0]$ uniformly when $k \to \infty$.*

It is clear that $f[x_-] := \sin[x] + \sin[\sqrt{2}x]$ does not have Bochner compact range.

Lemma 4.2 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be an almost periodic function that attains its maximum and minimum. Then for any sequence $(x_n)_{n\in}$ there is a subsequence $(x_{n_k})_{k\in}$ and $x_0 \in \mathbb{R}^N$ such that $f[x_- + x_{n_k}] \to f[x_- + x_0]$ uniformly.*

Proof. Case 1. The sequence $(x_n)_{n\in}$ converges to x_0. Since f is uniformly continuous, given $\varepsilon > 0$ there is a $\delta > 0$ such that $\|x_n - x_0\| < \delta$ implies $|f[x_n] - f[x_0]| < \varepsilon$ for all $x \in \mathbb{R}^N$. Then $|f[x + x_n] - f[x + x_0]| < \varepsilon$. It follows that $f[x_- + x_n] \to f[x_- + x_0]$ uniformly.

Case 2. The sequence $(x_n)_{n\in}$ as well as all its subsequences converge in norm to infinity. There is a subsequence of $(x_n)_{n\in}$ which we denote $(x_{n_k})_{k\in}$ such that $f[x_- + x_{n_k}] \to g[x_-]$ uniformly. From the fact that $m = f[x_1] \leq f[x_{n_k}] \leq M = f[x_2]$ we get $m \leq g[0] \leq M$.
Then there is an $x_0 \in \mathbb{R}^N$ with $g[0] = f[\ x_0]$.
Let $A := \{x \in \mathbb{R}^N$ such that $f[x + x_{n_k}] \to f[x + x_0]\}$. A is not empty because $0 \in A$.
Let us prove that $A = \overline{A}$. Let $z \in \overline{A}$. Then there exists a sequence $(y_n)_{n\in} \subset \mathbb{A}$ with $y_n \to z$. If z does not belong to A there is an $\varepsilon > 0$ with $|f[y_n + x_{n_k}] - f[x_0 + z]| > \varepsilon$ for an infinite set of indexes k.
But $|f[y_n + x_{n_k}] - f[x_0 + z]| \leq |f[y_n + x_{n_k}] - f[y_n + x_0]| + |f[y_n + x_0] - f[x_0 + z] < 2\varepsilon$ for an infinite set of indexes n, which is a contradiction.
Let us prove now that A is an open set. If A is not open, $\exists z_0 \in A$ and a sequence $(y_n)_{n\in} \subset \mathbb{R}^N$ such that $y_n \notin A$ and $y_n \in B(z_0, \frac{1}{n})$, then

$$\begin{aligned} |f[y_m + x_{n_k}] - f[y_m + x_0]| \leq\ & |f[y_m + x_{n_k}] - f[x_{n_k} + z_0]| + |f[x_{n_k} + z_0] - f[z_0 + x_0]| \\ & + |f[z_0 + x_0] - f[y_m + x_0]|. \end{aligned}$$

Each of these three terms converge to 0. So we get a contradiction and A is open.
Therefore $A = \mathbb{R}^N$ and $f[x_- + x_{n_k}] \to f[x_- + x_0]$ uniformly. ■

Theorem 4.3 *Let $f:\mathbb{R}^N \to \mathbb{R}$ be an almost periodic function; f has a Bochner compact range if and only if f attains its maximum and minimum.*

Proof. Lemma1 implies one direction.
Let f have a BCR then f is a bounded function.
Let $m = \inf\{f[x], x \in \mathbb{R}^N\}$, $M = \sup\{f[x], x \in \mathbb{R}^N\}$.
Since $m, M \in \overline{f[\mathbb{R}^N]}$, there are sequences $(x_n), (y_n) \subset \mathbb{R}^N$ with $f[x_n] \to m$ and $f[y_n] \to M$.
Then $f[x_- + x_{n_k}] \to f[x_- + x_0]$ uniformly and consequently $f[x_{n_k}] \to f[x_0] = m$, therefore $m \in f[\mathbb{R}^N]$.
The same can be done for M. ■

Theorem 4.4 *Let $f:\mathbb{R} \to \mathbb{R}$ be an almost periodic function. f is periodic if and only if f has Bochner compact range.*

Proof. If f periodic then f attains its maximum and minimum so f has Bochner compact range.
If f has BCR, let $(T_n)_{n\in}$ be a sequence of almost periods with the property:

$$|f[x+T_n] - f[x]| < \frac{1}{n} \forall x \in \mathbb{R}, n \in \mathbb{N}.$$

Then there is a $T \in \mathbb{R}$ and a subsequence $(T_{n_k})_{k\in}$ with $f[x_- + T_{n_k}] \to f[x_- + T]$ uniformly. Hence:

$$\begin{aligned} |f[x+T] - f[x]| &\leq |f[x+T] - f[x+T+T_{n_k}]| + |f[x+T+T_{n_k}] - f[x+T_{n_k}]| \\ &\quad + |f[x+T_{n_k}] - f[x]| \\ &\leq \frac{1}{n_k} + \varepsilon + \frac{1}{n_k}. \end{aligned}$$

From which it follows that T is a period of f. ■

This theorem answers positively the conjecture of Professor Fischer [Fis]. Theorem 4.4 has a natural generalization on $\mathbb{R}^N$.

Theorem 4.5 *Let $f:\mathbb{R}^N \to \mathbb{R}$ be an almost periodic, then function f is *periodic if and only if f has Bochner compact range.*

Proof. If f is *periodic then $f[\mathbb{R}^N]$ is a compact set.
Let $z \in \mathbb{R}^N$ be any fixed direction.
Let $(T_n)_{n\in}$ a sequence of almost periods with the property:

$$|f[x+z+T_n] - f[x+z]| < \frac{1}{n}, \forall x \in \mathbb{R}^N,\ T_n \forall n \in \mathbb{N}.$$

Then there is an $T \in \mathbb{R}^N$ and a subsequence $(T_{n_k})_{k\in}$ with $f[x_- + z + T_{n_k}] \to f[x_- + z + T]$ uniformly.
Then T is a period of the function $f[x+z]$ which proves the result. ■

Theorem 4.6 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a continuous bounded function. Let $E \subset \mathbb{R}^N$, E r.d. and $\cup_{y \in E}\{f[x\_ + y]\}$ compact in $C_b(\mathbb{R}^N, \| \cdot \|_\infty)$ then f *periodic.*

Proof. Because of theorem 3.3 f is almost periodic and then f is bounded.
Let $M = \sup\{f[x], x \in \mathbb{R}^N\}$; $m = \inf\{f[x], x \in \mathbb{R}^N\}$.
Then there are sequences $(x_n), (y_n) \subset \mathbb{R}^N$ with $f[x_n] \to m$ and $f[y_n] \to M$.
Then there are sequences $(x_n^1) \subset E, (x_n^2) \subset [-L, L]$ such that $x^{\cdot}_n = x_n^1 + x_n^2$.
Then $f[x\_ + x_n^1] \to f[x\_ + x_0], x_0 \in E$.
(x_n^2) admits a convergent subsequence in $[-L.L]$ which we denote (x_n^2). Let $x_1 \in [-L, L]$ be its limit. Then $f[x\_ + x^{\cdot}_n] \to f[x\_ + x_0 + x_1]$, then $M = f[x_0 + x_1]$ and hence f attains its maximum.
The same reasoning is valid for m. So we get f periodic because of theorems 4.3 and 4.4. ■

5 Structure Theorem

In [CA-2] we give a proof of the structure theorem. Here we presente another one using the Bochner Transform.

Theorem 5.1 *Let $f, g : \mathbb{R}^N \to \mathbb{R}$ be two almost periodic functions, then $f \pm g$ is almost periodic.*

Proof. Let $(x^{\cdot}_n) \subset \mathbb{R}^N$ be an arbitrary sequence. Then there are subsequences $(x^{\cdot}_n)$ of $(x^{\cdot}_n)$ such that $f[x\_ + x^{\cdot}_n] \to g[x\_]$ and $g[x\_ + x^{\cdot}_n] \to h[x\_]$ uniformly then $f[x\_ + x^{\cdot}_n] + g[x\_ + x^{\cdot}_n] \to g[x\_] + h[x\_]$ uniformly. Then $f + g$ is almost periodic. ■

6 Some consequences of the structure theorem

Lemma 6.1 *Let $f, g : \mathbb{R}^N \to \mathbb{R}$ be two almost periodic functions then $f \cdot g$, af, $|f|$, $\sup(f, g)$, $\inf(f, g)$ are almost periodic, $a \in \mathbb{R}$ a given constant.*

Proof. Immediate consequence of theorem 5.1. ■

7 Some integral formulae

Theorem 7.1 *Let $f : \mathbb{R}^N \to \mathbb{R}$ be an almost periodic then there exists the limit $\lim_{T\to\infty} \frac{1}{2^N T^N} \int_{[-T,T]^N} f[x\_ + t]dt$ and it is an almost periodic function.*

Proof. Let $g[x\_] = \frac{1}{2^N T^N} \int_{[-T,T]^N} f[x + t]dt, N, T$ fix, it is easy to see that g is an almost periodic function.
Because of theorem 6.1 f^+, f^-, are almost periodic functions. Then without any loss of generality we consider $f \geq 0$.
Then $h[x\_, T\_] = \frac{1}{2^N T^N} \int_{[-T,T]^N} f[x + t]dt = f[x + x_T], x_T \in [-T, T]^N$.
Let$(T_n) \subset \mathbb{R}$ an strict increasing positive sequence with $\lim T_n = \infty$.
Then $f[x\_ + x_{T_n}]$ admits a convergent subsequence and the theorem is proved. ■

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