Revista de Matemática: Teoría y Aplicaciones 1996 **3**(1) : 11–20
cimpa – ucr issn: 1409-2433

ON SPACE-TIME PROPERTIES OF SOLUTIONS FOR NONLINEAR EVOLUTIONARY EQUATIONS WITH RANDOM INITIAL DATA*

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Abstract

We consider space-time properties of periodic solutions of nonlinear wave equations, nonlinear Schrödinger equations and KdV-type equations with initial data from the support of the Gibbs' measure. For the wave and Schrödinger equations we establish the best Hölder exponents. We also discuss KdV-type equations which are more difficult due to a presence of the derivative in the nonlinearity.

Keywords: space-time properties; Hölder exponents; nonlinear evolutionary equations.

Resumen

Consideramos las propiedades en espacio tiempo de las soluciones periódicas de ecuaciones de onda no lineales, ecuaciones no lineales de Schrödinger y ecuaciones de tipo KdV con datos iniciales del soporte de la medida de Gibbs. Para las ecuaciones de onda y de Schrödinger establecemos los mejores exponentes de Hölder. También discutimos las ecuaciones de tipo KdV, que son más difíciles debido a la presencia de la derivada en la no linealidad.

Palabras clave: propiedades espacio-tiempo; exponentes de Hölder; ecuaciones no lineales evolutivas.

Mathematics Subject Classification: 35B65, 82C05.

*The author would like to thank IHES where the paper was written for hospitality. The work is supported by NSF grant DMS-9501002.

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1 Nonlinear wave equations

Consider the ID nonlinear wave equation

$$Q_{tt} - Q_{xx} + f(Q) = 0$$

with periodic boundary conditions $Q(0,t) = Q(2\pi, t)$. The equation can be written in the Hamiltonian form

$$\begin{aligned} Q_t &= \{Q, H\}, \\ P_t &= \{P, H\}, \end{aligned}$$

with $H(Q,P) = \int_0^{2\pi} \left[\frac{P^2}{2} + \frac{Q^2}{2} + F(Q)\right], F' = f$, and a classical bracket

$$\{A, B\} = \int_0^{2\pi} \left[\frac{\partial A}{\partial Q(x)} \frac{\partial B}{\partial P(x)} - \frac{\partial A}{\partial P(x)} \frac{\partial B}{\partial Q(x)}\right] dx.$$

An invariant Gibbs' state $e^{-H} d^\infty Q d^\infty P$ in the space[1] of pairs $(Q,P) \in H^0 \times H^{-1}$ was constructed in [6] under the assumption that $f(Q)$ is an odd locally Lipschitz function such that $f(Q) \geq kQ$, for some $k > 0$ and big Q. To simplify the proof we impose an additional condition on the growth of f at infinity: $f(Q) \leq C(\epsilon) e^{\epsilon Q^2}$, for any $\epsilon > 0$. The Gibbs' state is a product measure: the Q component is $e^{-\int Q_x^2/2} d^\infty Q$, a circular Brownian motion with uniformly distributed initial position multiplied by the Radon-Nikodym factor $e^{-\int F(Q)}$ and "white noise" measure $e^{-\int P^2/2} d^\infty P$ on the P component.

Using variation of parameters we can write the original differential equation in the integral form

$$\begin{aligned} Q(x,t) &= \frac{\sin\sqrt{-\partial_x^2} t}{\sqrt{-\partial_x^2}} P_0(x) + \cos\sqrt{-\partial_x^2} t Q_0(x) - \int_0^t ds \frac{1}{2} \int_{x-(t-s)}^{x+(t-s)} dy f(Q(y,s)) \\ &= Q_W(x,t) + N(x,t), \end{aligned}$$

where (Q_0, P_0) are the initial data. The term $Q_W(x,t)$ corresponding to the linear wave equation satisfies the Hölder condition

$$|Q_W(x_1,t_1) - Q_W(x_2,t_2)| \leq K(|x_1 - x_2|^{\beta_1} + |t_1 - t_2|^{\beta_2}), \tag{1}$$

with $0 < \beta_1, \beta_2 < \frac{1}{2}$ and some random constant K, $EK^2 < \infty$, which depends on the β's. The bound $\frac{1}{2}$ is optimal in a sense that we can not have (1) with some $\beta_1 > \frac{1}{2}$, or $\beta_2 > \frac{1}{2}$ and random K, $EK^2 < \infty$. The nonlinear part $N(x,t)$ is a differentiable function of x and t. The derivatives $\partial_x N(x,t)$ and $\partial_t N(x,t)$ satisfy (1) with the same exponents not exceeding $\frac{1}{2}$. These simply means that the local structure of the field $Q(x,t)$ is completely determined by the term $Q_W(x,t)$ corresponding to the linear wave equation. We split the proof of these facts in three different steps.

[1] H^s is a standard Sobolev's space, i.e. $Q(x) \in H^s$ if $(1 - \Delta^2)^{s/2} Q(x) \in L^2[0, 2\pi]$.

Step 1

In the proof of the statement concerning (1) we use Kolmogoroff's criteria of continuity, see [4].

Theorem 1 (A. N. Kolmogoroff) *Let $Q(x,t)$, $(x,t) \in D$ be a random field with real or complex values and D is a compact domain in $\mathbb{R}^2$. Assume that there exist positive constants γ, C, α_1 and α_2 with $\alpha_1^{-1} + \alpha_2^{-1} < 1$ satisfying*

$$E|Q(x_1,t_1) - Q(x_2,t_2)|^\gamma \le C[(x_1 - x_2)^{\alpha_1} + (t_1 - t_2)^{\alpha_2}]$$

for every $(x_i, t_i) \in D$. Then $Q[x,t)$ has a continuous modification.

Let β_1 and β_2 be arbitrary positive numbers less than $\alpha_1 c_0$ or $\alpha_2 c_0$ respectively, $c_0 \equiv (1 - \alpha_1^{-1} - \alpha_2^{-1})/\gamma$. Then there exists a positive random variable K with $EK^\gamma < \infty$ such that

$$|Q(x_1,t_1) - Q(x_2,t_2)| \le K[(x_1 - x_2)^{\beta_1} + (t_1 - t_2)^{\beta_2}].$$

Consider the wave equation $Q_{tt} - Q_{xx} = O$ with Gaussian initial data such that Q is $e^{-\int Q_x^2/2} d^\infty Q$, and P is $e^{-\int P^2/2} d^\infty P$ restricted to the submanifold $\hat{Q}(0) = \hat{P}(0) = 0$.

The Fourier coefficients are independent complex isotropic Gaussian variables such that $\overline{\hat{Q}(n)} = \hat{Q}(-n)$, $\overline{\hat{P}(n)} = \hat{P}(-n)$ and $E|\hat{Q}(n)|2 = n^{-2}$, $E|\hat{Q}(n)|^2 = 1$. Using rotation invariance of the measure and its invariance under the flow

$$\begin{aligned}
E|Q(x,t_1) - Q(x,t_2)|^2 &= E|Q(0,h) - Q(0,0)|^2 \qquad \text{(where } h = t_2 - t_1\text{)} \\
&= E\left|\sum_{k\ne 0} (\cos kh - 1)\hat{Q}(k)\right| + E\left|\sum_{k\ne 0} \frac{\sin kh}{k}\hat{P}(k)\right|^2 \\
&= 2\sum_{k>0} \frac{(\cos kh - 1)^2}{k^2} + 2\sum_{k>0} \frac{\sin^2 kh}{k^2}.
\end{aligned}$$

The first term can be overestimated as

$$\le c_1 \sum_{0<k\le h^{-1}} \frac{(kh)^4}{k^2} + c_2 \sum_{h^{-1}<k} \frac{1}{k^2} \le c_3 h^4 h^{-3} + c_4 h \le c_5 h.$$

The same estimate holds for the second term. Using the Gaussian character of the field[2] $Q_W(x,t)$:

$$E|Q_W(x,t_1) - Q_W(x,t_2)|^{2n} \le c_n |t_1 - t_2|^n.$$

Likewise

$$E|Q_W(x_1,t) - Q_W(x_2,t)|^{2n} \le c_n |t_1 - t_2|^n.$$

Now apply Kolmogoroff's criteria and pass to the limit with $n \to \infty$.

[2] $Ex^{2n} = \frac{(2n)!}{2^n n!}(Ex^2)^n$ if x is a Gaussian variable with zero mean.

Step 2

Optimality of the Hölder exponent $\frac{1}{2}$ for the space increment is a classical result. We present an elementary proof of this fact which works in other cases as well.

Note that $E|Q(\bullet,t)|_s^2 = \sum E|\widehat{Q}(n,t)^2(1+n^2)^s < \infty$ if and only if $s < \frac{1}{2}$. The following fact[3] implies the rest. Let $Q(x)$, $x \in [0, 2\pi]$ be a rotationally-invariant Gaussian process such that

$$|Q(x_1) - Q(x_2)| < K(x_1 - x_2)^\beta$$

with some $K, EK^2 < \infty$. Then $E|Q|_s^2 < \infty$, for all $s < \beta$.

From the assumptions made, we get:

$$E\int_0^{2\pi} |Q(x+h) - Q(x)|^2 dx \leq EK^2 h^{2\beta},$$

$$Q(x + \frac{\pi h}{4}) - Q(x - \frac{\pi h}{4}) = 2i \sum_{n \neq 0} \widehat{Q}(n) e^{inx} \sin \frac{\pi h n}{4}.$$

Parsevall's identity implies

$$E\int_0^{2\pi} |Q(x + \frac{\pi h}{4}) - Q(x - \frac{\pi h}{4})|^2 dx = 4 \sum_{n\neq 0} E|\widehat{Q}(n)|^2 \sin^2 \frac{\pi h n}{4}.$$

Therefore

$$\sum_{n\neq 0} E|\widehat{Q}(n)|^2 \sin^2 \frac{\pi h n}{4} \leq c_1 h^{2\beta}$$

and

$$\sum_{\frac{1}{h} \leq n < \frac{2}{h}} E|\widehat{Q}(n)|^2 \leq c_2 h^{2\beta}.$$

The substitution $h \to h/2^r$ yields

$$\sum_{\frac{2^r}{h} \leq n < \frac{2^{r+1}}{h}} E|\widehat{Q}(n)|^2 \leq c_3 \frac{h^{2\beta}}{4^{\beta r}}.$$

Finally

$$\sum_{\frac{1}{h}\leq n} E|\widehat{Q}(n)|^2 \leq \sum_{r=0}^{\infty} \sum_{\frac{2^r}{h} \leq n < \frac{2^{r+1}}{h}} E|\widehat{Q}(n)|^2 \leq c_4 h^{2\beta},$$

and[4]

$$\sum_{k \leq |n|} E|\hat{Q}(n)|^2 \leq c_4 \frac{1}{k^{2\beta}}.$$

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[3] This is stochastic version of the classical embeding theorem, [7].

[4] In the proof of this estimate we borrowed the idea from [1, section 82].

It implies for positive n:

$$S(n) \equiv \sum_{n\le k} |\hat{Q}(k)|^2 \le c_4 \frac{1}{n^{2\beta}}.$$

By Abel's summation formula for positive M and N, we have

$$\begin{aligned}\sum_{M}^{N} E|\hat{Q}(n)|^2(1+n^2)^s &= S(M)(1+M^2)^s - S(N+1)(1+N^2)^s \\ &\quad + \sum_{M+1}^{N} S(n)[(1+n^2)^s - (1+(n-1)^2)^s] \\ &\le S(M)(1+M^2)^s + \sum_{M+1}^{N} S(n)[(1+n^2)^s - (1+(n-1)^2)^s].\end{aligned}$$

For big n

$$(1+n^2)^s - (1+(n-1)^2)^s = (1+n^2)^s[\frac{2s}{n} + O(\frac{1}{n^2})].$$

This together with the estimate for $S(n)$ implies

$$\sum_{M}^{+\infty} E|\widehat{Q}(n)|^2(1+n^2)^s < \infty, \text{ for } s < \beta.$$

Negative indexes are handled in the same way. The proof is finished. ■

For any fixed x, $Q_W(x,\bullet)$ is a $2\pi-$periodic rotationally invariant Gaussian process such that $E|\hat{Q}_W(x,n)|^2 = n^{-2}$. The same arguments used above show optimality of the exponent in the time increment.

Step 3

First, we estimate Hölder exponents for a solution of the nonlinear equation. Let $h = t_1 - t_2$, using invariance of the measure

$$\begin{aligned}E|Q(x_1,t) - Q(x_2,t)|^{2n} &= E|Q_0(x_1) - Q_0(x_2)|^{2n} \le c_n(x_1-x_2)^n \\ E|Q(x,t_1) - Q(x,t_2)|^{2n} &= E|Q(x,h) - Q_0(x)|^{2n} \\ &\le c_n E|\sin\frac{\sqrt{-\partial_x^2}h}{\sqrt{-\partial_x^2}}P_0(x) + \cos\sqrt{-\partial_x^2}hQ_0(x) - Q_0(x)|^{2n} \\ &\quad + c_n E|\int_0^h ds \frac{1}{2}\int_{x-(h-s)}^{x+(h-s)} dy\, f(Q(y,s))|^{2n}.\end{aligned}$$

To estimate the first term replace the measure $e^{-\int F} \times e^{-\int Q_x^2/2} d^\infty Q$ by $Ce^{-k\int Q^2} \times e^{-\int Q_x^2/2} d^\infty Q$ with some big C and proceed like in Step 1. To estimate the second term use Hölder's inequality and $E|f(Q)|^{2n} < \infty$ for every n. Eventually

$$E|Q(x,t_1) - Q(x,t_2)|^{2n} \le c_n(t_1-t_2)^n.$$

Kolmogoroff's criteria implies that $Q(x,t)$ satisfies (1) with the same Hölder exponents not exceeding 1/2. The last statement concerning derivatives $\partial_x N(x,t) - \partial_t N(x,t)$ follows from the explicit formulas

$$\begin{aligned}
\partial_x N(x,t) &= -\frac{1}{2}\int_0^h ds[f(Q(x+(h-s),s)) - f(Q(x-(h-s),s))], \\
\partial_t N(x,t) &= -\frac{1}{2}\int_0^h ds[f(Q(x+(h-s),s)) + f(Q(x-(h-s),s))],
\end{aligned}$$

and locally Lipschitz character of f. The proof is completed. ∎

Nonlinear Schrödinger equations

The next point if the discussion is 1D nonlinear Schrödinger equation

$$i\psi_t = -\psi_{xx} + f(|\psi|^2)\psi,$$

where $\psi(x,t)$ is a complex function $\psi = Q + iP$ which satisfies periodic boundary conditions $\psi(0,t) = \psi(2\pi,t)$. It can be written in the Hamiltonian form

$$\psi_t = \{\psi, H\}$$

with the Hamiltonian $H = \frac{1}{2}\int_0^{2\pi} |\psi_x|^2 + F(|\psi|^2)dx$, $F' = f$ and a bracket

$$\{A,B\} = 2i\int_0^{2\pi}\left[\frac{\partial A}{\overline{\psi(x)}}\frac{\partial B}{\psi(x)} - \frac{\partial A}{\psi(x)}\frac{\partial B}{\overline{\psi}(x)}\right]dx.$$

An invariant Gibbs' state $e^{-H}d^\infty\psi d^\infty\overline{\psi}$ was constructed in [3, 5] under the assumption that $F \geq 0$ is an even polynomial. The Gibbs' state is a product of two independent circular Brownian motions on Q and P whose components are coupled together by the nonlinear factor $e^{-\int F(Q^2+P^2)}$.

Written in the integral form the equation is

$$\begin{aligned}
\psi(x,t) &= e^{i\partial_x^2 t}\psi_0(x) - i\int_0^t e^{i\partial_x^2(t-s)} f(|\psi|^2)\psi(x,s)ds \\
&= \psi_S(x,t) + N(x,t),
\end{aligned}$$

where $\psi_0(x)$ is initial data. The solution of the free Schrödinger equation satisfies

$$|\psi_S(x_1,t_1) - \psi_S(x_2,t_2)| \leq K\left(|x_1-x_2|^{\beta_1} + |t_1-t_2|^{\beta_2}\right) \tag{2}$$

with $0 < \beta_1 < \frac{1}{2}, 0 < \beta_2 < \frac{1}{4}$, and random constant K, $EK^2 < \infty$, which depend on β's. The exponents $\frac{1}{2}$, $\frac{1}{4}$ are optimal. The same can be said about $\psi(x,t)$, a solution of NLS itself. The proof of this statements is similar to the corresponding one for the nonlinear wave equation.

The nonlinear term $N(x,t)$ seems to be smoother then $\psi_S(x,t)$. This implies that the microstructure of the field $\psi(x,t)$ is determined by the linear term $\psi_S(x,t)$, but the proof is not known. Presumably, Hölder exponents for $N(x,t)$ depend on arithmetical properties of the coefficients of the polynomial F. There is no uniform smoothing as one can see from the following example.

Example

Let $\Gamma(x,t) \equiv \sum_{n\neq 0} e^{inx} e^{-i(n^2+n^\alpha)t}\widehat{\psi}_0(n)$, arbitrary $\alpha \geq 0$ and $\widehat{\psi}_0(n)$ are independent complex isotropic Gaussian variables, $E|\widehat{\psi}_0(n)|^2 = \frac{1}{1+n^2}$. The Gaussian field $\Gamma(x,t)$, $x \in [0,2\pi]$, $s \in \mathbb{R}^1$ is stationary in time and rotationally invariant; $\Gamma(\bullet,t)$ is a complex Ornstein-Uhlenbeck process with zero mean for any t. By straightforward computation

$$\begin{aligned} N(x,t) &= -i\int_0^t e^{i\partial_x^2(t-s)}\Gamma(x,s)ds \\ &= -i\int_0^t \sum_{n\neq 0} e^{inx} e^{-in^2(t-s)} e^{-i(n^2+n^\alpha)s}\widehat{\psi}_0(n)ds \\ &= -i\sum_{n\neq 0} e^{inx} e^{-in^2t}\widehat{\psi}_0(n)\frac{e^{-in^\alpha t}-1}{-in^\alpha}. \end{aligned}$$

We see that $N(\bullet,t)$ gains α Sobolev's exponents in comparison with $\Gamma(\bullet,t)$.

1.1 KdV-type equations

The last topic of the discussion to KdV-type equations

$$Q_t = -Q_{xxx} + (f(Q))_x$$

with periodic boundary conditions $Q(0,t) = Q(2\pi,t)$. The equation can be written in the Hamiltonian form

$$Q_t = \{Q,H\},$$

with the Hamiltonian $H = \int_0^{2\pi} \frac{Q_x^2}{2} + F(Q)dx$, $F' = f$ and a bracket

$$\{A,B\} = \int_0^{2\pi} \frac{\partial A}{\partial Q(x)}\partial_x \frac{\partial A}{\partial Q(x)}dx.$$

An invariant Gibbs state $e^{-H}d^\infty Q$ was constructed in [3] for particular nonlinearities $F(Q) = Q^3/3$ (KdV) and $F(Q) = Q^4/4$ (modified KdV). The measure is a circular Brownian motion $e^{-\int Q_x^2/2}d^\infty Q$ multiplied by the nonlinear term $e^{-\int F(Q)}$.

The equation can be written in the integral form

$$\begin{aligned} Q(x,t) &= e^{-\partial_x^3 t}Q_0(x) + \partial_x\int_0^t e^{-\partial_x^3(t-s)}f(Q(x,s))ds \\ &= Q_A(x,t) + U[f](x,t). \end{aligned}$$

According to J. Bourgain (private communication) the solution $Q(x,t)$ will be continuous in space-time. The solution of the linear Airy equation satisfies

$$|Q_A(x_1,t_1) - Q_A(x_2,t_2)| \leq \left(|x_1-x_2|^{\beta_1} + |t_1-t_2|^{\beta_2}\right) \tag{3}$$

with the optimal bounds $0<\beta_{<}\frac{1}{2}$, $0<\beta_2<\frac{1}{6}$, and some random constant K, $EK^2<\infty$, which depend on β's. Nothing is known about smoothness of the nonlinear term $U[f](x,t)$. To get some idea consider the KdV equation. In symbolic form

$$\begin{aligned} Q &= Q_A+U[Q^2] \\ &= Q_A+U[(Q_A+U[Q^2])^2] \\ &= Q_A+U[Q_A^2]+U[2Q_AU[Q^2]+U^2[Q^2]] \\ &= Q_A+U[Q_A^2]+\ldots \end{aligned}$$

Now look at $U[Q_A^2]$, the first term in the "approximation". We will prove

$$E(U[Q_A^2](x_1,t)-U[Q_A^2](x_2,t))^2 \le C|x_1-x_2|, \tag{4}$$

$$E(U[Q_A^2](x_1,t)-U[Q_A^2](x_2,t))^4 \le C(\epsilon)|x_1-x_2|^{(2-\epsilon)}, \tag{5}$$

for any $\epsilon>0$. This indicates that $U[Q_A^2](\bullet,t)$ is x-continuous due to (5) by Kolmogoroff and similar to the Brownian motion because of (4). It is possible that in this case the local structure of the field $Q(x,t)$ depends on the nonlinear term $N(x,t)$.

To prove (4) and (5) we need Wick's theorem, see [2].

Theorem 2 (Wick) *Let $\xi_1,\xi_2,\ldots,\xi_{2n}$ are real or complex Gaussian variables with zero mean, then*

$$E\xi_1\times\cdots\times\xi_{2n}=\frac{1}{2^n n!}\sum_{\mu}E\xi_{\mu_1}\xi_{\mu_2}\times\cdots\times E\xi_{\mu_{2n-1}}\xi_{\mu_{2n}},$$

where summation is taken over the permutation group of $2n$ elements.

Let $Q_A(x,t)=\sum_{n\neq 0}e^{inx}e^{in^3t}\widehat{Q}_0(n)$ where $\widehat{Q}_0(n)$ is a Gaussian complex isotropic variable, $\overline{\widehat{Q}_0(n)}=\hat{Q}_0(-n)$, $E|\widehat{Q}_0(n)|^2=\frac{1}{1+n^2}$. Then

$$\begin{aligned} U[Q_A^2](x,t) &= \partial_x\int_0^t e^{-\partial_x^3(t-s)}Q_A^2(x,s)ds \\ &= \sum_{n\neq 0}e^{inx}\sum_{\substack{n_1+n_2=n\\ n_i\neq 0}}\hat{Q}_0(n_1)\hat{Q}_0(n_2)\frac{e^{i(n_1^3+n_2^3)t}-e^{in^3t}}{n_1^3+n_2^3-n^3}. \end{aligned}$$

Using the arithmetical fact $n_1^3+n_2^3-n^3=-3n\,n_1n_2$ we obtain

$$U[Q_A^2](x,t)=\sum_{n\neq 0}e^{inx}\sum_{\substack{n_1+n_2=n\\ n_i\neq 0}}\hat{Q}_0(n_1)\hat{Q}_0(n_2)M(n_1,n_2,t),$$

where

$$M(n_1,n_2,t)=\frac{e^{i(n_1^3+n_2^3)t}-e^{in^3t}}{-3n_1n_2}.$$

Note $|M(n_1,n_2,t)|\le 2$ if $n_1n_2\neq 0$.

First, we prove that $E(U[Q_A^2](x,t))^2$ is finite. Using rotational invariance of the measure

$$\begin{aligned} E(U[Q_A^2](x,t))^2 &= E(U[Q_A^2](0,t))^2 \\ &= \sum_{n_1,n_2\neq 0} \sum_{\substack{p_1+p_2=n_1\\ p_3+p_4=n_2\\ p_i\neq 0}} E\hat{Q}_0(p_1)\hat{Q}_0(p_2)\hat{Q}_0(p_3)\hat{Q}_0(p_4)M(p_1,p_2)M(p_3,p_4). \end{aligned}$$

By Wick's rule

$$E\hat{Q}_0(p_1)\hat{Q}_0(p_2)\hat{Q}_0(p_3)\hat{Q}_0(p_4) = \frac{1}{2^2 2!}\sum_{\mu} E\hat{Q}_0(p_{\mu_1})\hat{Q}_0(p_{\mu_2}) \times E\hat{Q}_0(p_{\mu_3})\hat{Q}_0(p_{\mu_4}).$$

The sum venishes unless $n_1 = -n_2$ and $p_1 = -p_3$, $p_2 = -p_4$ or $p_1 = -p_4$, $p_2 = -p_3$. Therefore

$$\begin{aligned} &E\hat{Q}_0(p_1)\hat{Q}_0(p_2)\hat{Q}_0(p_3)\hat{Q}_0(p_4) \\ &= E|\hat{Q}_0(p_1)|^2 E|\hat{Q}_0(p_2)|^2 = \begin{cases} \frac{1}{1+p_1^2}\frac{1}{1+p_2^2}, & \text{if } p_1 \neq p_2 \\ 2\frac{1}{1+p_1^2}\frac{1}{1+p_2^2}, & \text{if } p_1 = p_2 \end{cases} \end{aligned}$$

and

$$\begin{aligned} E(U[Q_A^2](x,t))^2 &\leq 2\sum_{n\neq 0}\sum_{\substack{p_1+p_2=n\\ p_i\neq 0}} \frac{1}{1+p_1^2}\frac{1}{1+p_2^2}|M(p_1,p_2)|^2 \\ &\leq 8\sum_{n\neq 0}\sum_{\substack{p_1+p_2=n\\ p_i\neq 0}} \frac{1}{1+p_1^2}\frac{1}{1+p_2^2} \leq c_1\sum_{n\neq 0}\frac{1}{n^2} < \infty. \end{aligned}$$

In the last estimate we used

$$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\frac{1}{1+(n-x)^2}dx = \frac{2\pi}{n^2+4}.$$

The estimate for the second moment of the increment is similar.

$$\begin{aligned} &E(U[Q_A^2](h,t) - U[Q_A^2](0,t))^2 = \sum_{n_1,n_2\neq 0}(e^{in_1h}-1)(e^{in_2h}-1)\times \\ &\times \sum_{\substack{p_1+p_2=n_1\\ p_3+p_4=n_2\\ p_1\neq 0}} E\hat{Q}_0(p_1)\hat{Q}_0(p_2)\hat{Q}_0(p_3)\hat{Q}_0(p_4)M(p_1,p_2)M(p_3,p_4) \\ &\leq 2\sum_{n\neq 0}(e^{inh}-1)(-e^{inh}-1)\sum_{\substack{p_1+p_2=n\\ p_i\neq 0}} \frac{1}{1+p_1^2}\frac{1}{1+p_2^2}|M(p_1,p_2)|^2. \end{aligned}$$

Finally

$$E(U[Q_A^2](h,t) - U[Q_A^2](0,t))^2 \leq c_2\sum_{|n|<h^{-1}}\frac{|e^{ihn}-1|}{n^2} + c_3\sum_{h^{-1}\leq|n|}\frac{1}{n^2} \leq c_4 h.$$

The proof of (5) can be obtained by the same methods.azw

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