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CODIMENSION 1 DISTRIBUTIONS ON THREE DIMENSIONAL HYPERSURFACES

DISTRIBUCIONES DE CODIMENSIÓN 1 EN HIPERSUPERFICIES TRIDIMENSIONALES

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Abstract

We show that codimension 1 distributions with at most isolated singularities on threefold hypersurfaces $X_d \subset \mathbb{P}^4$ of degree d provide interesting examples of stable rank 2 reflexive sheaves. When $d \leq 5$, these sheaves can be regarded as smooth points within an irreducible component of the moduli space of stable reflexive sheaves. Our second goal goes in the reverse direction: we start from a well-known family of stable locally free sheaves and provide examples of codimension 1 distributions of local complete intersection type on X_d.

Keywords: holomorphic distributions, stable sheaves, moduli spaces, isolated singularities.

Resumen

Mostramos que las distribuciones de codimensión 1 con a lo más singularidades aisladas en hipersuperficies $X_d \subset \mathbb{P}^4$ de dimensión 3 y grado d proporcionan ejemplos interesantes de haces reflexivos estables de rango 2. Cuando $d \leq 5$, estos haces se pueden considerar como puntos suaves dentro de una componente irreducible del espacio de móduli de los haces reflexivos estables. Nuestro segundo objetivo va en dirección inversa: partimos de una familia conocida de haces estables localmente libres y proporcionamos ejemplos de distribuciones de codimensión 1 del tipo intersección completa local en X_d.

Palabras clave: distribuciones holomorfas, haces estables, espacios de móduli, singularidades aisladas.

Mathematics Subject Classification: 14D20, 14J60; Secondary 14D22, 14F06, 13D02

Contents

1 Introduction

The study of holomorphic foliations on complex manifolds is a classical topic of research that goes back to the end of the 19^{th} century, though the qualitative study of polynomial differential equations by Poincaré, Darboux, and Painlevé, and currently with ramifications to complex geometry and algebraic geometry.

In algebraic geometric terms, a *codimension 1 distribution* $\mathscr{D}$ *on a smooth projective variety* X is a short exact sequence of the form

$$\mathscr{D} \ : \ 0 \to T_{\mathscr{D}} \to TX \xrightarrow{\omega} \mathscr{I}_Z \otimes L \to 0, \tag{1}$$

where $T_{\mathscr{D}}$ is a reflexive sheaf called the *tangent sheaf* of $\mathscr{D}$, and $Z \subset X$ is a subscheme of codimension at least 2 called the *singular scheme* of $\mathscr{D}$, and L is a line bundle. Note that $\mathrm{rk}(T_{\mathscr{D}}) = \dim X - 1$. A full theory of algebraic distributions on smooth projective varieties is outlined in [1].

Since $\mathrm{Hom}(TX, \mathscr{I}_Z \otimes L) \subset H^0(\Omega^1_X \otimes L)$, the morphism ω can be regarded as a twisted 1-form on X_d, and $Z = (\omega)_0$ is just the set of points where ω vanishes.

In this paper, we focus on codimension 1 distributions on three dimensional hypersurfaces. So let $X_d \subset \mathbb{P}^4$ be a smooth hypersurface of degree d with $\mathrm{Pic}(X_d) = \mathbb{Z} \cdot H$. Let $\mathcal{O}_{X_d}(1)$ denote the ample generator of $\mathrm{Pic}(X_d)$, with $c_1(\mathcal{O}_{X_d}(1)) = H$; given a sheaf F on X_d we set $F(r) := F \otimes \mathcal{O}_{X_d}(1)^{\otimes r}$, as usual. Since $h^0(\Omega^1_{X_d}(t)) = 0$ for $t < 2$, we can rewrite the exact sequence in (1) as follows

$$\mathscr{D} \ : \ 0 \to T_{\mathscr{D}} \to TX_d \xrightarrow{\omega} \mathscr{I}_Z(r+2) \to 0. \tag{2}$$

Here, r is a non negative integer called the *degree* of $\mathscr{D}$.

In general, the singular scheme Z of a codimension 1 distribution on X_d may contain both 1- and 0-dimensional irreducible components. The two extreme cases are of particular interest; $\mathscr{D}$ is said to be *generic* when $\dim Z = 0$, and $\mathscr{D}$ is said to be *local complete intersection* when Z has pure dimension 1. The former nomenclature is motivated by the fact that there exists an open subset $U \subset \mathbb{P}H^0(\Omega^1_X(r + \rho_X))$ such that every $\omega \in U$ induces a generic codimension

1 distribution on X_d; the latter nomenclature comes from the fact that, in this case, Z is a local complete intersection as a subscheme of X_d.

In [2, Theorem 1], the authors proved that, under the conditions above, $T_{\mathscr{D}}$ is a stable rank 2 reflexive sheaf. Moreover, it was shown that there exists a non-singular, rational, irreducible component of the moduli space of stable rank 2 reflexive sheaves on $\mathbb{P}^3$ whose general point corresponds to the tangent sheaf of a generic distribution. The first main goal of the present paper is to generalize this result by replacing the three dimensional projective space by three dimensional hypersurfaces, and therefore provide an application of our ideas to the construction of interesting rank 2 reflexive sheaves on threefold hypersurfaces. To be precise, we prove the following results.

Main Theorem. *1. The moduli space of stable rank 2 sheaves on X_2 with Chern classes*

$$(c_1, c_2, c_3) = \begin{cases} (0, (3k^2+6k+4)H^2, (8k^3+24k^2+26k+10)H^3), & k \geq 1, \\ (-H, (3k^2+3k+2)H^2, (8k^3+12k^2+8k+2)H^3), & k \geq 0, \end{cases}$$

possesses an irreducible component which contains, as a closed subset, the tangent sheaves of a generic codimension 1 distribution on X_2.

2. *The moduli space of stable rank 2 sheaves on a smooth cubic $X_3 \subset \mathbb{P}^4$ with Chern classes*

$$(c_1, c_2, c_3) = \begin{cases} (0, (3k^2+4k+4)H^2, (8k^3+16k^2+16k+10)H^3), & k \geq 1, \\ (-H, (3k^2+7k+7)H^2, (8k^3+28k^2+38k+23)H^3), & k \geq 0, \end{cases}$$

possesses an irreducible component which contains, as a closed subset, the tangent sheaves of a generic codimension 1 distribution on X_3.

3. *The moduli space of stable rank 2 sheaves on a smooth quartic $X_4 \subset \mathbb{P}^4$ with Chern classes:*

$$(c_1, c_2, c_3) = \begin{cases} (0, (3k^2+8k+11)H^2, (8k^3+32k^2+54k+50)H^3), & k \geq 0, \\ (-H, (3k^2+5k+8)H^2, (8k^3+20k^2+28k+30)H^3), & k \geq 0, \end{cases}$$

possesses an irreducible component which contains, as a closed subset, the tangent sheaves of a generic codimension 1 distribution on X_4.

4. *The moduli space of stable rank 2 sheaves on a smooth quintic $X_5 \subset \mathbb{P}^4$ with Chern classes*

$$(c_1, c_2, c_3) = \begin{cases} (0, (3k^2+6k+13)H^2, (8k^3+24k^2+44k+68)H^3), & k \geq 0, \\ (-H, (3k^2+9k+17)H^2, (8k^3+36k^2+74k+97)H^3), & k \geq 0, \end{cases}$$

possesses an irreducible component which contains, as a closed subset, the tangent sheaves of a generic codimension 1 distribution on X_5.

Each of the four items above require slightly different calculations, and their proof takes Sections 5 through 8. We emphasize that not every point of the irreducible components constructed in the proof of the main theorem corresponds to the tangent sheaf of a generic distribution.

Our second goal, discussed in Section 9, goes in the reverse direction: we start from a well-known family of locally free sheaves and to provide examples of codimension 1 distribution of local complete intersection type on X_d. We briefly discuss distributions whose tangent sheaf splits as a sum of line bundles. In addition, we present an explicit family of codimension 1 distribution of local complete intersection type on X_d whose tangent sheaf is a stable locally free sheaf.

2 Stability and moduli spaces

Following the notation outlined in the introduction, let E be a torsion-free sheaf on X_d; recall that the slope of E with respect to $\mathcal{O}_{X_d}(1)$ is defined by

$$\mu(E) := \frac{c_1(E).H^{n-1}}{\mathrm{rk}(E)}.$$

We say that E is *stable* with respect to $\mathcal{O}_{X_d}(1)$ if, for every coherent subsheaf $0 \neq F \hookrightarrow E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$, we have that $\mu(F) < \mu(E)$. Moreover, E is stable if and only if the dual sheaf $E^\vee$ is stable as well.

Since $\Omega^1_{X_d}$ is stable (see [10, Corollary 1.5]), and hence the tangent bundle TX_d is also stable, [2, Theorem 1] guarantees that the tangent sheaf $T_{\mathscr{D}}$ of a generic codimension 1 distribution $\mathscr{D}$ is always a stable rank 2 reflexive sheaf on X_d. Note that $T_{\mathscr{D}}^\vee$ is also stable, because it is the dual of the stable sheaf.

Note that, since $\dim Z = 0$, we have that $\mathcal{E}xt^1(\mathscr{I}_Z, \mathcal{O}_{X_d}) = 0$ and the dualization of the exact sequence (2) leads to the following short exact sequence

$$0 \to \mathcal{O}_{X_d}(-2-r) \to \Omega^1_{X_d} \to T^\vee_{\mathscr{D}} \to 0; \tag{3}$$

in other words, the tangent sheaf of a generic distribution can be described as a quotient of $\Omega^1_{X_d}$.

The theory of sheaf stability arise in problems involving the classification of sheaves on a projective variety. The moduli spaces are solution spaces for these problems.

Roughly speaking, the moduli space of stable reflexive sheaves on a smooth projective variety X is a scheme whose points are in natural bijection to isomorphism classes of stable reflexive sheaves on X. This correspondence is given in terms of representable functors.

Let X be a smooth, irreducible projective variety of dimension n over $\mathbb{C}$ and let H be an ample divisor on X. For a fixed polynomial $P \in \mathbb{Q}[z]$, we consider the contravariant moduli functor

$$\begin{aligned} \mathcal{M}^{H,P}_X(-) : \mathfrak{Sch} &\to \mathfrak{Sets} \\ S &\mapsto \mathcal{M}^{H,P}_X(S), \end{aligned}$$

where

$$\begin{aligned} \mathcal{M}^{H,P}_X(S) = \{S- \text{ flat families } \mathcal{F} \to X \times S \text{ of reflexive sheaves on } X \text{ all} \\ \text{whose fibers are stable with respect to } H \text{ and have Hilbert} \\ \text{polynomial} P\}/\sim \end{aligned}$$

with,

$$\mathcal{F} \sim_S \mathcal{G} \Longleftrightarrow \mathcal{F} \simeq \mathcal{G} \otimes \pi_S^* \mathcal{L} \quad \text{for a line bundle } \mathcal{L} \to S,$$

being $\pi_S\colon X \times S \to S$ the natural projection. And if $f : S' \to S$ is a morphism in $\mathfrak{Sch}$, let $\mathcal{M}^{H,P}_X(f)(-)$ be the map obtained by pulling-back sheaves via $= f \times id_X$:

$$\begin{aligned} \mathcal{M}^{H,P}_X(f)(-) : \mathcal{M}^{H,P}_X(S) &\to \mathcal{M}^{H,P}_X(S') \\ [\mathcal{F}] &\mapsto [f_X^* \mathcal{F}]. \end{aligned}$$

In 1977, Maruyama proved the following theorem (see [7]).

Theorem 2.1. *The contravariant moduli functor $\mathcal{M}_X^{H,P}(-)$ has a coarse moduli scheme $M_X^{H,P}$ which is a separated scheme and locally of finite type over $\mathbb{C}$. In addition, $M_X^{H,P}$ decomposes into a disjoint union of schemes $M_X^{H,P}(r; c_1, \ldots, c_{\min(r,n)})$ where $n = \dim X$ and $M_X^{H,P}(r; c_1, \ldots, c_{\min(r,n)})$ is the moduli space of rank r stable with respect to H reflexive sheaves on X with Chern classes $(c_1, \ldots, c_{\min(r,n)})$ up to numerical equivalence.*

The next proposition gives us a bound to calculate the dimension of the Zariski tangent space of the moduli spaces of stable sheaves on a projective scheme X, see [5, Theorem 4.5.2].

Proposition 2.2. *Let X be a smooth, irreducible projective variety of dimension n and let E be a stable reflexive sheaf on X with Chern classes $c_i(E) = c_i \in H^{2i}(X, \mathbb{Z})$, representing a point $[E] \in M_X^{H,P}(r; c_1, \ldots, c_{\min(r,n)})$. Then the Zariski tangent space of $M_X^{H,P}(r; c_1, \ldots, c_{\min(r,n)})$ at $[E]$ is canonically given by*

$$T_{[E]}M_X^{H,P}(r; c_1, \ldots, c_{\min(r,n)}) \simeq \mathrm{Ext}^1(E, E).$$

If $\mathrm{Ext}^2(E, E) = 0$, then $M_X^{H,P}(r; c_1, \ldots, c_{\min(r,n)})$ is smooth at $[E]$. In general, there are bounds

$$\begin{aligned}\dim \mathrm{Ext}^1(E, E) &\geq \dim_{[E]} M_X^{H,P}(r; c_1, \ldots, c_{\min(r,n)}) \\ &\geq \dim \mathrm{Ext}^1(E, E) - \dim \mathrm{Ext}^2(E, E).\end{aligned}$$

Here are some useful formulas involving the Chern classes for a coherent sheaf F of rank r on X_d; for more details, see [4].

$$c_k(F(t)) = \sum_{i=0}^{k} \binom{r-k+i}{i} c_1(\mathcal{O}_{X_d}(t))^i c_{k-i}(F).$$

Moreover, for an exact sequence of coherent sheaves on X_d of the form

$$0 \to A \to B \to C \to 0,$$

we have that

$$c_k(B) = \sum_{i+j=k} c_i(A)c_j(C).$$

Finally, we recall that if E is a torsion-free sheaf of rank r over X_d, then there is a uniquely determined integer k_E such that $c_1(E(k_E)) \in \{0, -1, \ldots, -r+1\}$. We set $E_{\text{norm}} := E(k_E)$, and say that E is *normalized* if $E = E_{\text{norm}}$; compare with[8, Remark 1.2.6].

3 Family of the generic codimension 1 distributions

Here we construct a family of stable rank 2 reflexive sheaves F on X_d containing the tangent sheaves of the generic codimension 1 distributions on X_d.

Let $\mathcal{D}(r)$ denote the family of stable rank 2 reflexive sheaves on X_d given by the exact sequence in (3) and let $\mathcal{F}(r)$ be the family of the stable rank 2 reflexive sheaves F on X_d given by the exact sequence

$$0 \to \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d) \xrightarrow{\sigma} \Omega^1_{\mathbb{P}^4}|_{X_d} \to F \to 0. \tag{4}$$

Claim. $\mathcal{D}(r) \subset \mathcal{F}(r)$.

Indeed, given an exact sequence as in (3), we have the following commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}_{X_d}(-d) \arrow[d, "\varphi"] & & \\
0 \arrow[r] & \ker(\beta_2 \circ \beta_1) \arrow[r] \arrow[d] & \Omega^1_{\mathbb{P}^4}|_{X_d} \arrow[r, "\beta_2 \circ \beta_1"] \arrow[d, "\beta_1"] & T^\vee_{\mathscr{D}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \arrow[r] \arrow[d] & \Omega^1_{X_d} \arrow[r, "\beta_2"] \arrow[d] & T^\vee_{\mathscr{D}} \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{5}$$

where φ is given by the standard normal bundle sequence

$$0 \to \mathcal{O}_{X_d}(-d) \xrightarrow{\varphi} \Omega^1_{\mathbb{P}^4}|_{X_d} \to \Omega^1_{X_d} \to 0. \tag{6}$$

Now, applying the snake lemma we get the following commutative diagram:

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & & \\
& \mathcal{O}_{X_d}(-d) \arrow[r, equal] \arrow[d] & \mathcal{O}_{X_d}(-d) \arrow[d, "\varphi"] & & \\
0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d) \arrow[r] \arrow[d] & \Omega^1_{\mathbb{P}^4}|_{X_d} \arrow[r, "\beta_2 \circ \beta_1"] \arrow[d, "\beta_1"] & T_{\mathscr{D}}^{\vee} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \arrow[r] \arrow[d] & \Omega^1_{X_d} \arrow[r, "\beta_2"] \arrow[d] & T_{\mathscr{D}}^{\vee} \arrow[r] & 0 \\
& 0 & 0 & &
\end{tikzcd} \tag{7}$$

Note that $\ker(\beta_2 \circ \beta_1)$ must split because $\operatorname{Ext}^1(\mathcal{O}_{X_d}(-2-r), \mathcal{O}_{X_d}(-d)) \simeq H^1(\mathcal{O}_{X_d}(2+r-d)) = 0$, for all $r, d \in \mathbb{Z}$.

Therefore, we conclude that $\mathcal{D}(r) \subset \mathcal{F}(r)$; in other words, the tangent sheaf of a generic distribution always belongs to the family $\mathcal{F}(r)$.

We also have the following commutative diagram:

$$\begin{tikzcd}
& & 0 \arrow[d] & 0 \arrow[d] & \\
& & \mathcal{O}_{X_d}(-d) \arrow[r, equal] \arrow[d, "\varphi"] & \mathcal{O}_{X_d}(-d) \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \arrow[r, "\sigma_1"] \arrow[d, equal] & \Omega^1_{\mathbb{P}^4}|_{X_d} \arrow[r] \arrow[d] & T_{\sigma_1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \arrow[r, "\phi"] & \Omega^1_{X_d} \arrow[r] \arrow[d] & T_{\mathscr{D}}^{\vee} \arrow[r] \arrow[d] & 0 \\
& & 0 & 0 &
\end{tikzcd} \tag{8}$$

with $\sigma_1 \in H^0(\Omega^1_{\mathbb{P}^4}|_{X_d}(r+2))$ such that $T_{\sigma_1} := \operatorname{coker} \sigma_1$ is a rank 3 reflexive sheaf on X_d.

Given, in addition, a section $\sigma_2 \in H^0(\Omega^1_{\mathbb{P}^4}|_{X_d}(d))$, we consider the morphism

$$\sigma = \sigma_1 + \sigma_2 \, : \, \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d) \to \Omega^1_{\mathbb{P}^4}|_{X_d}, \tag{9}$$

and obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{O}_{X_d}(-d) \arrow[r, equal] \arrow[d] & \mathcal{O}_{X_d}(-d) \arrow[d, "\sigma_2"] & & \\
0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d) \arrow[r, "\sigma"] \arrow[d] & \Omega^1_{\mathbb{P}^4}|_{X_d} \arrow[r] \arrow[d] & T \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & T \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd} \tag{10}$$

where $G := \operatorname{coker} \sigma_2$ and $T := \operatorname{coker} \sigma$. For a generic choice of the section σ_1 and σ_2, both G and T are reflexive sheaves on X_d of ranks 3 and 2, respectively. Note that from the diagram (10), we have that $\mathcal{D}(r) \subsetneq \mathcal{F}(r)$.

So, in order to get the dimension of the family $\mathcal{F}(r)$, we must investigate when monomorphisms as displayed in (9) define isomorphic quotients. Before that, we proof two lemmas.

Below we present a brief study of the cohomology groups of the sheaves $\Omega^1_{\mathbb{P}^4}|_{X_d}$ and $\Omega^1_{X_d}$.

Lemma 3.1. *We have the following formulas:*

1) $h^i(\Omega^1_{\mathbb{P}^4}|_{X_d}(t)) = \begin{cases} 0, & \textit{for} \quad i=0, \quad t \le 1; \\ 0, & \textit{for} \quad i=1, \quad t \neq 0; \\ 1, & \textit{for} \quad i=1, t=0; \\ 0, & \textit{for} \quad i=2, \quad t \in \mathbb{Z}; \\ 0, & \textit{for} \quad i=3, \quad t \ge d-3. \end{cases}$

2) $h^i(\Omega^1_{X_d}(t)) = \begin{cases} 0, & \textit{for} \quad i=0, \quad t \le 1; \\ 0, & \textit{for} \quad i=1, \quad t \neq 0; \\ 1, & \textit{for} \quad i=1, t=0; \\ 0, & \textit{for} \quad i=2, \quad t \ge 2d-4; \\ 0, & \textit{for} \quad i=3, \quad t \ge d-3. \end{cases}$

Proof. The item $1)$ follows from the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^4}(-d) \xrightarrow{\cdot f} \Omega^1_{\mathbb{P}^4} \to \Omega^1_{\mathbb{P}^4}|_{X_d} \to 0 \tag{11}$$

twisted by $\mathcal{O}_{\mathbb{P}^4}(t)$, where $X_d = \{f = 0\} \subset \mathbb{P}^4$ is a hypersurface of degree d, and Bott's formula.

Item 2) follows from the standard normal bundle sequence in (6) twisted by $\mathcal{O}_{X_d}(t)$ and of item 1). □

Other information about the cohomology groups of the sheaves $\Omega^1_{\mathbb{P}^4}|_{X_d}$ and $\Omega^1_{X_d}$ used in the text were obtained from the exact sequences (6), (11) and the Euler exact sequence in $\mathbb{P}^4$ restricted to X_d.

Lemma 3.2. *The sheaf* $\Omega^1_{\mathbb{P}^4}|_{X_d}$ *is simple, i.e.* $\dim \mathrm{Hom}(\Omega^1_{\mathbb{P}^4}|_{X_d}, \Omega^1_{\mathbb{P}^4}|_{X_d}) = 1$.

Proof. Applying the functor $\mathrm{Hom}(\cdot, \Omega^1_{\mathbb{P}^4}|_{X_d})$ to the exact sequence in (11), we get the following sequence

$$0 \to \mathrm{Hom}(\Omega^1_{\mathbb{P}^4}|_{X_d}, \Omega^1_{\mathbb{P}^4}|_{X_d}) \to \mathrm{Hom}(\Omega^1_{\mathbb{P}^4}, \Omega^1_{\mathbb{P}^4}|_{X_d}) \to \cdots \tag{12}$$

Now, applying the functor $\mathrm{Hom}(\cdot, \Omega^1_{\mathbb{P}^4}|_{X_d})$ to the exact sequence

$$0 \to \Omega^1_{\mathbb{P}^4} \to \mathcal{O}_{\mathbb{P}^4}(-1)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^4} \to 0,$$

we get $\dim \mathrm{Hom}(\Omega^1_{\mathbb{P}^4}, \Omega^1_{\mathbb{P}^4}|_{X_d}) = 1$, as $H^0(\Omega^1_{\mathbb{P}^4}|_{X_d}(1)) = H^1(\Omega^1_{\mathbb{P}^4}|_{X_d}(1)) = 0$ and $h^1(\Omega^1_{\mathbb{P}^4}|_{X_d}) = 1$. Since, by the sequence (12),

$$1 \leq \dim \mathrm{Hom}(\Omega^1_{\mathbb{P}^4}|_X, \Omega^1_{\mathbb{P}^4}|_X) \leq \dim \mathrm{Hom}(\Omega^1_{\mathbb{P}^4}, \Omega^1_{\mathbb{P}^4}|_X),$$

we conclude that $\dim \mathrm{Hom}(\Omega^1_{\mathbb{P}^4}|_X, \Omega^1_{\mathbb{P}^4}|_X) = 1$, as desired. □

Lemma 3.3. *Let* $\sigma, \sigma' : \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d) \to \Omega^1_{\mathbb{P}^4}|_{X_d}$ *be two monomorphisms such that* $\operatorname{coker} \sigma := F$ *and* $\operatorname{coker} \sigma' := F'$ *are reflexive sheaves. We say that* F *and* F' *are isomorphic if and only if there exists an automorphism* $\psi \in \mathrm{Aut}(\mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d))$ *with* $\sigma' \circ \psi = \sigma$.

Proof. If $\sigma' \circ \psi = \sigma$., then it is easy to see that $\operatorname{coker} \sigma' \simeq \operatorname{coker} \sigma$. Conversely, suppose that

$$\sigma, \sigma' : \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d) \to \Omega^1_{\mathbb{P}^4}|_{X_d}$$

are monomorphisms, and assume that $g : F \to F'$ is an isomorphism between the quotients. Applying the functor $\mathrm{Hom}(\Omega^1_{\mathbb{P}^4}|_{X_d}, \cdot)$ to the exact sequence

$$0 \to \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d) \xrightarrow{\sigma'} \Omega^1_{\mathbb{P}^4}|_{X_d} \xrightarrow{p'} F' \to 0,$$

we get the isomorphism

$$\mathrm{Hom}(\Omega^1_{\mathbb{P}^4}|_{X_d}, \Omega^1_{\mathbb{P}^4}|_{X_d}) \simeq \mathrm{Hom}(\Omega^1_{\mathbb{P}^4}|_{X_d}, F'),$$

since

$$\begin{aligned}\operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_{X_d}, \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d)) \\ &\simeq H^0(\mathrm{T}\mathbb{P}^4|_{X_d}(-2-r)) \oplus H^0(\mathrm{T}\mathbb{P}^4|_{X_d}(-d)) \\ &= 0,\end{aligned}$$

and

$$\begin{aligned}\operatorname{Ext}^1(\Omega^1_{\mathbb{P}^4}|_{X_d}, \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d)) \\ &\simeq H^1(\mathrm{T}\mathbb{P}^4|_{X_d}(-2-r)) \oplus H^1(\mathrm{T}\mathbb{P}^4|_{X_d}(-d)) \\ &= 0.\end{aligned}$$

Thus, given $\xi \in \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_{X_d}, F^{'})$, there is a unique $\lambda \in \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_{X_d}, \Omega^1_{\mathbb{P}^4}|_{X_d})$ such that $p^{'} \circ \lambda = \xi$. Being $\Omega^1_{\mathbb{P}^4}|_{X_d}$ simple, by Lemma 3.2, it follows that $\lambda = c \cdot \mathbf{1}$.

Therefore, as $g \circ p \in \operatorname{Hom}(\Omega^1_{\mathbb{P}^4}|_{X_d}, F^{'})$, we get the following isomorphism between exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d) \arrow[r, "\sigma"] \arrow[d, "\psi"] & \Omega^1_{\mathbb{P}^4}|_{X_d} \arrow[r, "p"] \arrow[d, "c\cdot\mathbf{1}"] & F \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d) \arrow[r, "\sigma^{'}"] & \Omega^1_{\mathbb{P}^4}|_{X_d} \arrow[r, "p^{'}"] & F^{'} \arrow[r] & 0 \end{tikzcd}$$

After re-scaling ψ by $1/c$, we obtain an automorphism $\psi \in \operatorname{Aut}(\mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d))$ such that $\sigma^{'} \circ \left(\frac{1}{c} \cdot \psi\right) = \sigma$. □

As an immediate consequence of this lemma, we have that:

Proposition 3.4. *The dimension of the family of the sheaves constructed as in* (4) *is given by*

$$\begin{aligned}\dim \mathcal{F}(r) = &\dim \operatorname{Hom}(\mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d), \Omega^1_{\mathbb{P}^4}|_{X_d}) \\ &- \dim \operatorname{Aut}(\mathcal{O}_{X_d}(-2-r) \oplus \mathcal{O}_{X_d}(-d)).\end{aligned}$$

The next proposition gives us a tool to compute the dimension of the tangent space at a point $T^{\vee}_{\mathscr{D}}$ of the Gieseker–Maruyama moduli space of stable rank 2 reflexive sheaves on X_d.

Proposition 3.5. *Let $X_d \hookrightarrow \mathbb{P}^4$ be a smooth hypersurface of degree $d \in \{2,3,4,5\}$. If a sheaf $T_{\mathscr{D}}^{\vee}$ satisfies the exact sequence* (3)*, then:*

$$\dim \operatorname{Ext}^1(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) - \dim \operatorname{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \frac{d}{6}(5-d)(11-13d+8d^2+9r^2+6r(1+d))+1.$$

Proof. Indeed, applying the functor $\operatorname{Hom}(\cdot, T_{\mathscr{D}}^{\vee})$ to the exact sequence (3), we obtain the equality

$$\sum_{j=0}^{3}(-1)^j \dim \operatorname{Ext}^j(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \chi(TX_d \otimes T_{\mathscr{D}}^{\vee}) - \chi(T_{\mathscr{D}}^{\vee}(2+r)), \tag{13}$$

since $\operatorname{Ext}^i(\Omega^1_{X_d}, T_{\mathscr{D}}^{\vee}) \simeq H^i(TX_d \otimes T_{\mathscr{D}}^{\vee}))$ and $\operatorname{Ext}^i(\mathcal{O}_{X_d}(-2-r), T_{\mathscr{D}}^{\vee}) \simeq H^i(T_{\mathscr{D}}^{\vee}(2+r))$, for $0 \leq i \leq 3$.

Now, we twist the Euler exact sequence in $\mathbb{P}^4$ restricted to X_d

$$0 \to \mathcal{O}_{X_d} \to \mathcal{O}_{X_d}(1)^{\oplus 5} \to T\mathbb{P}^4|_{X_d} \to 0$$

and the exact sequence

$$0 \to TX_d \to T\mathbb{P}^4|_{X_d} \to \mathcal{O}_{X_d}(d) \to 0$$

by $\otimes T_{\mathscr{D}}^{\vee}$. Taking the Euler characteristic, we get

$$\chi(TX_d \otimes T_{\mathscr{D}}^{\vee}) = \chi(T_{\mathscr{D}}^{\vee}(2)) - \chi(T_{\mathscr{D}}^{\vee}(1)) - \chi(T_{\mathscr{D}}^{\vee}(d)). \tag{14}$$

The exact sequences in (13) and (14) imply that

$$\sum_{j=0}^{3}(-1)^j \dim \operatorname{Ext}^j(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \chi(T_{\mathscr{D}}^{\vee}(2)) - \chi(T_{\mathscr{D}}^{\vee}(1)) - \chi(T_{\mathscr{D}}^{\vee}(d)) - \chi(T_{\mathscr{D}}^{\vee}(2+r)). \tag{15}$$

Next, using the exact sequences (3), (6) and

$$0 \to \Omega^1_{\mathbb{P}^4}|_{X_d} \to \mathcal{O}_{X_d}(-1)^{\oplus 5} \to \mathcal{O}_{X_d} \to 0,$$

we have that

$$\chi(T_{\mathscr{D}}^{\vee}(t)) = 5 \cdot \chi(\mathcal{O}_{X_d}(t-1)) - \chi(\mathcal{O}_{X_d}(t)) - \chi(\mathcal{O}_{X_d}(t-d)) - \chi(\mathcal{O}_{X_d}(t-2-r)). \tag{16}$$

Moreover, the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^4}(-d) \xrightarrow{\cdot f} \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_{X_d} \to 0$$

implies that

$$\chi(\mathcal{O}_{X_d}(t)) = \chi(\mathcal{O}_{\mathbb{P}^4}(t)) - \chi(\mathcal{O}_{\mathbb{P}^4}(t-d)). \tag{17}$$

Using $\chi(\mathcal{O}_{\mathbb{P}^4}(k)) = \binom{k+4}{k}$ and from equations (15), (16) and (17) we get that

$$\sum_{j=0}^{3}(-1)^j \dim \operatorname{Ext}^j(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \frac{1}{6}d(d-5)(11-13d+8d^2+9r^2+6r(1+d)).$$

By Serre duality, we have that $\operatorname{Ext}^3(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) \simeq \operatorname{Hom}(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}(d-5))$, since $\omega_{X_d} \simeq \mathcal{O}_{X_d}(d-5)$. The stability of $T_{\mathscr{D}}^{\vee}$ implies that $\operatorname{Hom}(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}(d-5)) = 0$ for $d \in \{2,3,4\}$, since $\mu(T_{\mathscr{D}}^{\vee}(d-5)) < \mu(T_{\mathscr{D}}^{\vee})$. Thus, when $2 \leq d \leq 4$, we have that

$$\begin{aligned}\dim \operatorname{Ext}^1(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) - \dim \operatorname{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \\ \frac{1}{6}d(5-d)(11-13d+8d^2+9r^2+6r(1+d))+1.\end{aligned}$$

When $d=5$, we have that

$$\dim \operatorname{Ext}^1(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) - \dim \operatorname{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 0,$$

since $\operatorname{Ext}^3(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) \simeq \operatorname{Hom}(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee})$ and $\sum_{j=0}^{3}(-1)^j \dim \operatorname{Ext}^j(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 0$. □

Remark 3.6. When $d \geq 6$ it was not possible to calculate

$$\dim \operatorname{Ext}^1(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) - \dim \operatorname{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee})$$

using the arguments above, since it is not immediately clear that $\dim \operatorname{Ext}^3(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee})$ vanishes in this case.

4 Properties of the cotangent sheaf

Here we study the cohomology groups of the cotangent sheaf $T_{\mathscr{D}}^{\vee}$ of a generic codimension 1 distribution on X_d. We start with the following lemma.

Lemma 4.1. *If a sheaf $T_{\mathscr{D}}$ satisfies the exact sequence in* (3)*, then:*

1) $h^0(T_{\mathscr{D}}^\vee(t)) = 0$ *for* $t \leq 1$*;*

2) $h^1(T_{\mathscr{D}}^\vee) = 1$ *and* $h^1(T_{\mathscr{D}}^\vee(t)) = 0$ *for* $t \neq 0$;

3) $h^2(T_{\mathscr{D}}^\vee(t)) = h^0(\mathcal{O}_{X_d}(r - t + d - 3)$ *for* $t \geq 2d - 4$; *in particular,* $h^2(T_{\mathscr{D}}^\vee(t)) = 0$ *for* $t \geq r + d - 3$;

4) $h^3(T_{\mathscr{D}}^\vee(t)) = 0$ *for* $t \geq d - 3$.

Proof. For items 1) and 2), we consider the long exact sequence of cohomology obtained from the exact sequence in (3) twisted by $\mathcal{O}_{X_d}(t)$

$$\cdots \to H^0(\Omega^1_{X_d}(t)) \to H^0(T_{\mathscr{D}}^\vee(t)) \to 0 \to H^1(\Omega^1_{X_d}(t)) \to H^1(T_{\mathscr{D}}^\vee(t)) \to 0.$$

Being $h^0(\Omega^1_{X_d}(t)) = 0$ for $t \leq 1$ follows that $h^0(T_{\mathscr{D}}^\vee(t)) = 0$ for $t \leq 1$; now, as $h^1(\Omega^1_{X_d}(t)) = 0$ for $t \neq 0$ and $h^1(\Omega^1_{X_d}) = 1$, we get the item 2).

For items 3) and 4), we consider the long exact sequence of cohomology

$$\begin{aligned}\to H^2(\Omega^1_{X_d}(t)) \to H^2(T_{\mathscr{D}}^\vee(t)) \to H^3(\mathcal{O}_{X_d}(t - 2 - r)) \to \\ H^3(\Omega^1_{X_d}(t)) \to H^3(T_{\mathscr{D}}^\vee(t)) \to 0.\end{aligned}$$

We know that $h^2(\Omega^1_{X_d}(t)) = 0$ for $t \geq 2d - 4$ and $h^3(\Omega^1_{X_d}(t)) = 0$ for $t \geq d - 3$. So, $h^3(T_{\mathscr{D}}^\vee(t)) = 0$ for $t \geq d - 3$ and $h^2(T_{\mathscr{D}}^\vee(t)) = h^0(\mathcal{O}_{X_d}(r - t + d - 3))$ for $t \geq 2d - 4$, since, by Serre duality, $h^3(\mathcal{O}_{X_d}(t - 2 - r)) = h^0(\mathcal{O}_{X_d}(r - t + d - 3))$. □

Another important lemma is:

Lemma 4.2. *If $T_{\mathscr{D}}$ is a sheaf satisfying the exact sequence in* (3) *and TX_d is the tangent bundle on X_d, then*

$$H^3(TX_d \otimes T_{\mathscr{D}}^\vee) \simeq \mathrm{Ext}^3(\Omega^1_{X_d}, T_{\mathscr{D}}^\vee) = 0,$$

for $d = 2, \ldots, 5$.

Proof. Indeed, applying the functor $\mathrm{Hom}(\cdot, T_{\mathscr{D}}^\vee)$ to the exact sequence in (3), we get the long exact sequence in cohomology

$$\cdots \to \mathrm{Ext}^3(T_{\mathscr{D}}^\vee, T_{\mathscr{D}}^\vee) \to \mathrm{Ext}^3(\Omega^1_{X_d}, T_{\mathscr{D}}^\vee) \to H^3(T_{\mathscr{D}}^\vee(2 + r)) \to 0.$$

By Serre duality,

$$\mathrm{Ext}^3(T_{\mathscr{D}}^\vee, T_{\mathscr{D}}^\vee) \simeq \mathrm{Hom}(T_{\mathscr{D}}^\vee, T_{\mathscr{D}}^\vee(d - 5)).$$

Being $T_{\mathscr{D}}^{\vee}$ stable and $\mu(T_{\mathscr{D}}^{\vee}(d-5)) < \mu(T_{\mathscr{D}}^{\vee})$ (for $d = 2, 3, 4$) follows that $\mathrm{Ext}^3(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 0$.

By item 4) of Lemma 4.1, $H^3(T_{\mathscr{D}}^{\vee}(2+r)) = 0$ since $2 + r \geq d - 3$ and hence $\mathrm{Ext}^3(\Omega^1_{X_d}, T_{\mathscr{D}}^{\vee}) = 0$.

For the case $d = 5$, suppose that $\mathrm{Ext}^3(\Omega^1_{X_5}, T_{\mathscr{D}}^{\vee}) \neq 0$. By Serre duality,

$$\mathrm{Ext}^3(\Omega^1_{X_5}, T_{\mathscr{D}}^{\vee}) \simeq \mathrm{Hom}(T_{\mathscr{D}}^{\vee}, \Omega^1_{X_5}),$$

since $\omega_{X_5} \simeq \mathcal{O}_{X_5}$. So, there is a nonzero morphism $q : T_{\mathscr{D}}^{\vee} \to \Omega^1_{X_5}$. Note that $p \circ q \neq 0$, because if $p \circ q = 0$, we have the following commutative diagram

$$\begin{tikzcd}
 & T_{\mathscr{D}}^{\vee} \arrow[r, equal] \arrow[d, "\sigma"] & T_{\mathscr{D}}^{\vee} \arrow[d, "q"] \arrow[dr, "0 = p\circ q"] & & \\
0 \arrow[r] & \mathcal{O}_{X_d}(-2-r) \arrow[r] & \Omega^1_{X_d} \arrow[r, "p"] & T_{\mathscr{D}}^{\vee} \arrow[r] & 0,
\end{tikzcd}$$

i.e., $\sigma \in \mathrm{Hom}(T_{\mathscr{D}}^{\vee}, \mathcal{O}_{X_d}(-2-r)) \simeq H^0(T_{\mathscr{D}}(-2-r)) = 0$, since $T_{\mathscr{D}} \hookrightarrow TX_d$ and so $q = 0$.

Being $T_{\mathscr{D}}^{\vee}$ stable, and hence simple, it follows that $p \circ q = 1_{T_{\mathscr{D}}^{\vee}}$, and thus the exact sequence in (3) should split, i.e. $\Omega^1_{X_5} \simeq \mathcal{O}_{X_5}(-2-r) \oplus T_{\mathscr{D}}^{\vee}$ which contradicts the stability of $\Omega^1_{X_5}$. □

5 Generic distributions on quadric threefolds

Let X_2 denote a quadric threefold with ample line bundle $\mathcal{O}_{X_2}(1)$ whose first Chern class is denoted by H, i.e. $c_1(\mathcal{O}_{X_2}(1)) = H$. Recall that the cohomology ring $H^*(X_2, \mathbb{Z})$ of X_2 is generated by H, a line $L \in H^4(X_2, \mathbb{Z})$ and a point $P \in H^6(X_2, \mathbb{Z})$ with the relations: $H^2 = 2L$, $H.L = P$, $H^3 = 2P$. Remember that $c_1(\Omega^1_{X_2}) = -3H$, $c_2(\Omega^1_{X_2}) = 4H^2$ and $c_3(\Omega^1_{X_2}) = -2H^3$.

Our main goal here is to prove the first part of Theorem 1. We start this section by calculation the Chern classes of the tangent sheaf of a generic codimension 1 distribution on X_2.

Recall that given a generic distribution $\mathscr{D}$ on X_2, the integer $r := 1 - c_1(T_{\mathscr{D}})$ is called the *degree* of $\mathscr{D}$.

Lemma 5.1. *If a generic distribution $\mathscr{D}$ on X_2 has degree $r = 2k+1$, then the normalization of the sheaf $T_{\mathscr{D}}^{\vee}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{X_2}(-3-3k) \to \Omega^1_{X_2}(-k) \to T_{\mathscr{D}}^{\vee}(-k) \to 0, \tag{18}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (0, (3k^2 + 6k + 4)H^2, (8k^3 + 24k^2 + 26k + 10)H^3).$$

Proof. The exact sequence (18) implies that

$$\begin{aligned} c_1(T_{\mathscr{D}}^{\vee}(-k)) &= c_1(\Omega^1_{X_2}(-k)) - c_1(\mathcal{O}_{X_2}(-3-3k)) \\ &= c_1(\Omega^1_{X_2}) + 3c_1(\mathcal{O}_{X_2}(-k)) - c_1(\mathcal{O}_{X_2}(-3-3k)) \\ &= -3H + 3(-kH) - (-3-3k)H \\ &= 0, \end{aligned}$$

since $c_1(\Omega^1_{X_2}) = -3H$.

$$\begin{aligned} c_2(T_{\mathscr{D}}^{\vee}(-k)) &= c_2(\Omega^1_{X_2}(-k)) - c_1(T_{\mathscr{D}}^{\vee}(-k))c_1(\mathcal{O}_{X_2}(-3-3k)) \\ &= c_2(\Omega^1_{X_2}) + 2c_1(\Omega^1_{X_2})c_1(\mathcal{O}_{X_2}(-k)) + 3c_1(\mathcal{O}_{X_2}(-k))^2 \\ &= 4H^2 + 2(-3H)(-kH) + 3(-kH)^2 \\ &= (3k^2+6k+4)H^2, \end{aligned}$$

since $c_1(T_{\mathscr{D}}^{\vee}(-k)) = 0$ and $c_2(\Omega^1_{X_2}) = 4H^2$.

$$\begin{aligned} c_3(T_{\mathscr{D}}^{\vee}(-k)) =& c_3(\Omega^1_{X_2}(-k)) - c_1(\mathcal{O}_{X_2}(-3-3k)c_2(T_{\mathscr{D}}^{\vee}(-k)) \\ =& -2H^3 + (-k)H(4H^2) + k^2H^2(-3H) - k^3H^3 \\ &+ (3+3k)H(3k^2+6k+4)H^2 \\ =& (8k^3+24k^2+26k+10)H^3, \end{aligned}$$

since $c_3(\Omega^1_{X_2}) = -2H^3$ and $c_i(\mathcal{O}_{X_2}(k)) = 0$ for all $k \in \mathbb{Z}$ and $i \geq 2$. □

When $\mathscr{D}$ has degree $r = 2k$, we have that:

Lemma 5.2. *If a generic distribution $\mathscr{D}$ on X_2 has degree $r = 2k$, then the normalization of the sheaf $T_{\mathscr{D}}^{\vee}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{X_2}(-2-3k) \to \Omega^1_{X_2}(-k) \to T_{\mathscr{D}}^{\vee}(-k) \to 0, \tag{19}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (-H, (3k^2+3k+2)H^2, (8k^3+12k^2+8k+2)H^3).$$

Proof. The exact sequence (19) implies that

$$\begin{aligned} c_1(T_{\mathscr{D}}^{\vee}(-k)) &= c_1(\Omega^1_{X_2}(-k)) - c_1(\mathcal{O}_{X_2}(-2-3k)) \\ &= c_1(\Omega^1_{X_2}) + 3c_1(\mathcal{O}_{X_2}(-k)) - c_1(\mathcal{O}_{X_2}(-2-3k)) \\ &= -3H + 3(-kH) - (-2-3k)H \\ &= -H, \end{aligned}$$

since $c_1(\Omega^1_{X_2}) = -3H$.

$$\begin{aligned} c_2(T_{\mathscr{D}}^{\vee}(-k)) =& c_2(\Omega^1_{X_2}(-k)) - c_1(T_{\mathscr{D}}^{\vee}(-k))c_1(\mathcal{O}_{X_2}(-2-3k)) \\ =& c_2(\Omega^1_{X_2}) + 2c_1(\Omega^1_{X_2})c_1(\mathcal{O}_{X_2}(-k)) + 3c_1(\mathcal{O}_{X_2}(-k))^2 \\ & - (-H)(-2-3k)H \\ =& 4H^2 + 2(-3H)(-kH) + 3(-kH)^2 - 2H^2 - 3k^2 \\ =& (3k^2+3k+2)H^2, \end{aligned}$$

since $c_1(T_{\mathscr{D}}^{\vee}(-k)) = -H$ and $c_2(\Omega^1_{X_2}) = 4H^2$.

$$\begin{aligned} c_3(T_{\mathscr{D}}^{\vee}(-k)) =& c_3(\Omega^1_{X_2}(-k)) - c_1(\mathcal{O}_{X_2}(-2-3k)c_2(T_{\mathscr{D}}^{\vee}(-k)) \\ =& -2H^3 + (-k)H(4H^2) + k^2H^2(-3H) - k^3H^3 \\ & + (2+3k)H(3k^2+3k+2)H^2 \\ =& (8k^3+12k^2+8k+2)H^3, \end{aligned}$$

since $c_3(\Omega^1_{X_2}) = -2H^3$ and $c_i(\mathcal{O}_{X_2}(k)) = 0$ for all $k \in \mathbb{Z}$ and $i \geq 2$.

□

Note that the family $\mathcal{D}(2k+1)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension 1 distribution of degree $2k+1$ on X_2 has dimension

$$\begin{aligned} \dim \mathcal{D}(2k+1) &= \dim \operatorname{Hom}(\mathcal{O}_{X_2}(-3-3k), \Omega^1_{X_2}(-k)) - 1 \\ &= 8k^3 + 42k^2 + 69k + 34. \end{aligned}$$

Lemma 5.3. $h^2(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes \Omega^1_{\mathbb{P}^4}|_{X_2}) = h^3(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes \Omega^1_{\mathbb{P}^4}|_{X_2}) = 0.$

Proof. We twist the Euler exact sequence in $\mathbb{P}^4$ restricted to X_2 by $\otimes\Omega^1_{\mathbb{P}^4}|_{X_2}$; passing to cohomology, we get

$$\begin{gathered} 0 \to H^2(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes \Omega^1_{\mathbb{P}^4}|_{X_2}) \to H^3(\Omega^1_{\mathbb{P}^4}|_{X_2}) \to 5H^3(\Omega^1_{\mathbb{P}^4}|_{X_2}(1)) \to \\ H^3(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes \Omega^1_{\mathbb{P}^4}|_{X_2}) \to 0, \end{gathered}$$

since $H^2(\Omega^1_{\mathbb{P}^4}|_{X_2}(1)) = 0$.

As $H^3(\Omega^1_{\mathbb{P}^4}|_{X_2}) = H^3(\Omega^1_{\mathbb{P}^4}|_{X_2}(1)) = 0$, we have that $h^2(\mathrm{T}\mathbb{P}^4|_{X_2}\otimes\Omega^1_{\mathbb{P}^4}|_{X_2}) = h^3(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes \Omega^1_{\mathbb{P}^4}|_{X_2}) = 0$.

□

We prove the main result of this section.

Theorem 5.4. *For each $k \geq 1$, the moduli space of stable rank 2 reflexive sheaves on X_2 with Chern classes*

$$(c_1, c_2, c_3) = (0, (3k^2 + 6k + 4)H^2, (8k^3 + 24k^2 + 26k + 10)H^3)$$

contains an irreducible component of dimension $8k^3 + 42k^2 + 69k + 44$ containing the family of the tangent sheaves of a generic codimension 1 distribution of degree $2k + 1$ on X_2.

Proof. Initially note that, by the commutative diagram (7), each tangent sheaf $T^\vee_{\mathscr{D}}$ of a generic codimension 1 distribution $\mathscr{D}$ of degree $2k + 1$ can be given as the cokernel of the monomorphism

$$\sigma\,:\,\mathcal{O}_{X_2}(-3-3k) \oplus \mathcal{O}_{X_2}(-2-k) \to \Omega^1_{\mathbb{P}^4}|_{X_2}(-k).$$

By Proposition 3.4,

$$\begin{aligned}\dim \mathcal{F}(2k+1) =& \dim \operatorname{Hom}(\mathcal{O}_{X_2}(-3-3k) \oplus \mathcal{O}_{X_2}(-2-k), \Omega^1_{\mathbb{P}^4}|_{X_2}(-k)) \\ &- \dim \operatorname{Aut}(\mathcal{O}_{X_2}(-3-3k) \oplus \mathcal{O}_{X_2}(-2-k)) \\ =& 8k^3 + 42k^2 + 69k + 44,\end{aligned}$$

for $k \geq 0$. Thus, it is enough to argue that

$$\dim \operatorname{Ext}^1(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = \dim \mathcal{F}(2k+1) = 8k^3 + 42k^2 + 69k + 44,$$

for $k \geq 1$, and hence, by semicontinuity, we can conclude that

$$\dim \operatorname{Ext}^1(F, F) = \dim \mathcal{F}(2k+1) = 8k^3 + 42k^2 + 69k + 44,$$

for a generic sheaf $F \in \mathcal{F}(2k + 1)$. Or equivalent, we must show that

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = \dim \mathcal{F}(2k+1) - 36k^2 - 72k - 45 = 8k^3 + 6k^2 - 3k - 1,$$

since, by Proposition 3.5,

$$\dim \operatorname{Ext}^1(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) - \dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = 36k^2 + 72k + 45, \text{ for } k \geq 0.$$

Indeed, applying the functor $\mathrm{Hom}(\cdot, T_{\mathscr{D}}^{\vee}(-k))$ to the exact sequence (18), we get

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \dim \mathrm{Ext}^2(\Omega^1_{X_2}(-k), T_{\mathscr{D}}^{\vee}(-k)) = h^2(TX_2 \otimes T_{\mathscr{D}}^{\vee}), \quad (20)$$

since $H^1(T_{\mathscr{D}}^{\vee}(2k+3)) = H^2(T_{\mathscr{D}}^{\vee}(2k+3)) = 0$ by Lemma 4.1.

Now, we twist the standard normal bundle sequence

$$0 \to TX_2 \to T\mathbb{P}^4|_{X_2} \to \mathcal{O}_{X_2}(2) \to 0$$

by $\otimes T_{\mathscr{D}}^{\vee}$ and pass to cohomology. We get the exact sequence in cohomology

$$0 \to H^2(TX_2 \otimes T_{\mathscr{D}}^{\vee}) \to H^2(T\mathbb{P}^4|_{X_2} \otimes T_{\mathscr{D}}^{\vee}) \to H^2(T_{\mathscr{D}}^{\vee}(2)) \to 0,$$

since $H^1(T_{\mathscr{D}}^{\vee}(2)) = 0$ by item 2) of Lemma 4.1 and $H^3(TX_2 \otimes T_{\mathscr{D}}^{\vee}) = 0$ by Lemma 4.2. Thus, if $k \geq 1$, we have that

$$h^2(TX_2 \otimes T_{\mathscr{D}}^{\vee}) = h^2(T\mathbb{P}^4|_{X_2} \otimes T_{\mathscr{D}}^{\vee}) - h^0(\mathcal{O}_{X_2}(2k-2)), \quad (21)$$

since, by item 3) of Lemma 4.1, $h^2(T_{\mathscr{D}}^{\vee}(2)) = h^0(\mathcal{O}_{X_2}(2k-2))$.

In order to compute $h^2(T\mathbb{P}^4|_{X_2} \otimes T_{\mathscr{D}}^{\vee})$, we twist the exact sequences

$$0 \to \mathcal{O}_{X_2}(-2-k) \to T_{\sigma_1}(-k) \to T_{\mathscr{D}}^{\vee}(-k) \to 0$$

and

$$0 \to \mathcal{O}_{X_2}(-3-3k) \to \Omega^1_{\mathbb{P}^4}|_{X_2}(-k) \to T_{\sigma_1}(-k) \to 0$$

in the commutative diagram (8) by $\otimes \mathrm{T}\mathbb{P}^4|_{X_2}(k)$ and pass to cohomology, we get

$$\begin{aligned} 0 = H^2(\mathrm{T}\mathbb{P}^4|_{X_2}(-2)) \to & H^2(T\mathbb{P}^4|_{X_2} \otimes T_{\sigma_1}) \to \\ & H^2(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes T_{\mathscr{D}}^{\vee}) \to H^3(\mathrm{T}\mathbb{P}^4|_{X_2}(-2)) = 0, \end{aligned}$$

and

$$\begin{aligned} 0 = H^2(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes \Omega^1_{\mathbb{P}^4}|_{X_2}) \to & H^2(T\mathbb{P}^4|_{X_2} \otimes T_{\sigma_1}) \to \\ & H^3(\mathrm{T}\mathbb{P}^4|_{X_2}(-3-2k)) \to H^3(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes \Omega^1_{\mathbb{P}^4}|_{X_2}) = 0. \end{aligned}$$

note that $H^2(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes \Omega^1_{\mathbb{P}^4}|_{X_2}) = H^3(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes \Omega^1_{\mathbb{P}^4}|_{X_2}) = 0$ by Lemma 5.3.

It follows that

$$h^2(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes T_{\mathscr{D}}^{\vee}) = h^3(\mathrm{T}\mathbb{P}^4|_{X_2}(-3-2k)),$$

for $k \geq 1$. The Euler exact sequence in $\mathbb{P}^4$ restricted to X_2 implies that

$$h^3(\mathrm{T}\mathbb{P}^4|_{X_2}(-3-2k)) = 5h^0(\mathcal{O}_{X_2}(2k-1)) - h^0(\mathcal{O}_{X_2}(2k)),$$

for $k \geq 1$, since, by Serre duality, $h^3(\mathcal{O}_{X_2}(-t)) = h^0(\mathcal{O}_{X_2}(t-3))$ for all $t \in \mathbb{Z}$.

Therefore

$$\begin{aligned} h^2(\mathrm{T}\mathbb{P}^4|_{X_2} \otimes T_{\mathscr{D}}^{\vee}) &= 5h^0(\mathcal{O}_{X_2}(2k-1)) - h^0(\mathcal{O}_{X_2}(2k)) \\ &= \frac{1}{3}(2k-1)(2k+1)(8k+3), \end{aligned} \tag{22}$$

for $k \geq 1$.

The equations (20), (21) and (22) give us

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 8k^3 + 6k^2 - 3k - 1,$$

for $k \geq 1$, as we desired.

□

Remark 5.5. When $k = 0$, we can still conclude that the stable rank 2 reflexive sheaves F on X_2, given by the short exact sequence

$$0 \to \mathcal{O}_{X_2}(-3) \oplus \mathcal{O}_{X_2}(-2) \to \Omega^1_{\mathbb{P}^4}|_{X_2} \to F \to 0,$$

are smooth points of the moduli space of stable rank 2 reflexive sheaves with Chern classes $(c_1, c_2, c_3) = (0, 4H^2, 6H^3)$ within an irreducible component of dimension 45, since $\mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 0$ (see equations (20), (21) and (22)). However, these sheaves only form a family of dimension 44 within this irreducible component.

Similarly, the family $\mathcal{D}(2k)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension 1 distribution of degree $2k$ on X_2 has dimension

$$\begin{aligned} \dim \mathcal{D}(2k) &= \dim \mathrm{Hom}(\mathcal{O}_{X_2}(-2-3k), \Omega^1_{X_2}(-k)) - 1 \\ &= 8k^3 + 30k^2 + 33k + 9. \end{aligned}$$

By Proposition 3.4, the family of the stable rank 2 reflexive sheaves F on X_2 given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_{X_2}(-2-3k) \oplus \mathcal{O}_{X_2}(-2-k) \to \Omega^1_{\mathbb{P}^4}|_{X_2}(-k)$$

has dimension

$$\begin{aligned}\dim \mathcal{F}(2k) = &\dim \operatorname{Hom}(\mathcal{O}_{X_2}(-2-3k) \oplus \mathcal{O}_{X_2}(-2-k), \Omega^1_{\mathbb{P}^4}|_{X_2}(-k)) \\ &- \dim \operatorname{Aut}(\mathcal{O}_{X_2}(-2-3k) \oplus \mathcal{O}_{X_2}(-2-k)) \\ =& \, 8k^3 + 30k^2 + 33k + 19,\end{aligned}$$

for $k \geq 1$ and $\dim \mathcal{F}(0) = 18$.

By Proposition 3.5,

$$\dim \operatorname{Ext}^1(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) - \dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = 36k^2 + 36k + 18,$$

for $k \geq 0$.

Moreover, doing an analogue calculation as in the proof of Theorem 5.4 we get that

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = 8k^3 - 6k^2 - 3k + 1,$$

for $k \geq 1$ and $\dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = 0$, if $k = 0$.

As in the case $c_1 = 0$, we obtain the following theorem.

Theorem 5.6. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X_2 with Chern classes*

$$(c_1, c_2, c_3) = (-H, (3k^2 + 3k + 2)H^2, (16k^3 + 12k^2 + 8k - 2)H^3)$$

contains an irreducible component of dimension 18 and $8k^3 + 30k^2 + 33k + 19$, for $k = 0$ and $k \geq 1$, respectively, containing the family of the tangent sheaves of a generic codimension 1 distribution of degree $2k$ on X_2. Moreover, this component is nonsingular in the case $k = 0$.

In the next section we will do an analogue study on a smooth cubic threefold.

6 Generic distributions on cubic threefolds

Let $X_3 \hookrightarrow \mathbb{P}^4$ denote a smooth cubic threefold with ample line bundle $\mathcal{O}_{X_3}(1)$ whose first Chern class is denoted by H, i.e. $c_1(\mathcal{O}_{X_3}(1)) = H$. The cohomology ring $H^*(X_3, \mathbb{Z})$ of X_3 is generated by H, a line $L \in H^4(X_3, \mathbb{Z})$ and a point

$P \in H^6(X_3, \mathbb{Z})$ with the relations: $H^2 = 3L,\ H.L = P,\ H^3 = 3P$. We know that $c_1(\Omega^1_{X_3}) = -2H, c_2(\Omega^1_{X_3}) = 4H^2$ and $c_3(\Omega^1_{X_3}) = 2H^3$.

Recall that given a generic distribution $\mathscr{D}$ on X_3, the integer $r := -c_1(T_{\mathscr{D}})$ is called the *degree* of $\mathscr{D}$.

The next lemma gives us the Chern classes of the tangent sheaf of a generic distribution on X_3.

Lemma 6.1. *If a generic distribution $\mathscr{D}$ on X_3 has degree $r = 2k$, then the normalization of the sheaf $T^\vee_{\mathscr{D}}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{X_3}(-2-3k) \to \Omega^1_{X_3}(-k) \to T^\vee_{\mathscr{D}}(-k) \to 0. \tag{23}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (0, (3k^2+4k+4)H^2, (8k^3+16k^2+16k+10)H^3).$$

Proof. The exact sequence (23) implies that

$$\begin{aligned} c_1(T^\vee_{\mathscr{D}}(-k)) &= c_1(\Omega^1_{X_3}(-k)) - c_1(\mathcal{O}_{X_3}(-2-3k)) \\ &= c_1(\Omega^1_{X_3}) + 3c_1(\mathcal{O}_{X_3}(-k)) - c_1(\mathcal{O}_{X_3}(-2-3k)) \\ &= -2H + 3(-kH) - (-2-3k)H \\ &= 0, \end{aligned}$$

since $c_1(\Omega^1_{X_3}) = -2H$.

$$\begin{aligned} c_2(T^\vee_{\mathscr{D}}(-k)) &= c_2(\Omega^1_{X_3}(-k)) - c_1(T^\vee_{\mathscr{D}}(-k))c_1(\mathcal{O}_{X_3}(-2-3k)) \\ &= c_2(\Omega^1_{X_3}) + 2c_1(\Omega^1_{X_3})c_1(\mathcal{O}_{X_3}(-k)) + 3c_1(\mathcal{O}_{X_3}(-k))^2 \\ &= 4H^2 + 2(-2H)(-kH) + 3(-kH)^2 \\ &= (3k^2+4k+4)H^2, \end{aligned}$$

since $c_1(T^\vee_{\mathscr{D}}(-k)) = 0$ and $c_2(\Omega^1_{X_3}) = 4H^2$.

$$\begin{aligned} c_3(T^\vee_{\mathscr{D}}(-k)) &= c_3(\Omega^1_{X_3}(-k)) - c_1(\mathcal{O}_{X_3}(-2-3k)c_2(T^\vee_{\mathscr{D}}(-k)) \\ &= 2H^3 + (-k)H(4H^2) + k^2H^2(-2H) - k^3H^3 \\ &\quad + (2+3k)H(3k^2+4k+4)H^2 \\ &= (8k^3+16k^2+16k+10)H^3, \end{aligned}$$

since $c_3(\Omega^1_{X_3}) = 2H^3$ and $c_i(\mathcal{O}_{X_3}(k)) = 0$ for all $k \in \mathbb{Z}$ and $i \geq 2$.

□

When $\mathscr{D}$ has degree $r = 2k+1$, we have the following lemma.

Lemma 6.2. *If a generic distribution $\mathscr{D}$ on X_3 has degree $r = 2k+1$, then the normalization of the sheaf $T_{\mathscr{D}}^{\vee}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{X_3}(-4-3k) \to \Omega^1_{X_3}(-1-k) \to T_{\mathscr{D}}^{\vee}(-1-k) \to 0. \qquad (24)$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (-H, (3k^2+7k+7)H^2, (8k^3+28k^2+38k+23)H^3).$$

Proof. The exact sequence (24) implies that

$$\begin{aligned} c_1(T_{\mathscr{D}}^{\vee}(-1-k)) &= c_1(\Omega^1_{X_3}(-1-k)) - c_1(\mathcal{O}_{X_3}(-4-3k)) \\ &= c_1(\Omega^1_{X_3}) + 3c_1(\mathcal{O}_{X_3}(-1-k)) - c_1(\mathcal{O}_{X_3}(-4-3k)) \\ &= -2H + 3(-1-k)H - (-4-3k)H \\ &= -H, \end{aligned}$$

since $c_1(\Omega^1_{X_3}) = -2H$.

$$\begin{aligned} c_2(T_{\mathscr{D}}^{\vee}(-1-k)) =& \, c_2(\Omega^1_{X_3}(-1-k)) \\ & - c_1(T_{\mathscr{D}}^{\vee}(-1-k))c_1(\mathcal{O}_{X_3}(-4-3k)) \\ =& \, c_2(\Omega^1_{X_3}) + 2c_1(\Omega^1_{X_3})c_1(\mathcal{O}_{X_3}(-1-k)) \\ & + 3c_1(\mathcal{O}_{X_3}(-1-k))^2 + H(-4-3k)H \\ =& \, 4H^2 + 2(-2H)(-1-k)H + 3(-1-k)^2H^2 \\ & - (4+3k)H^2 \\ =& \, (3k^2+7k+7)H^2, \end{aligned}$$

since $c_1(T_{\mathscr{D}}^{\vee}(-1-k)) = -H$ and $c_2(\Omega^1_{X_3}) = 4H^2$.

$$\begin{aligned} c_3(T_{\mathscr{D}}^{\vee}(-1-k)) =& \, c_3(\Omega^1_{X_3}(-1-k)) \\ & - c_1(\mathcal{O}_{X_3}(-4-3k)c_2(T_{\mathscr{D}}^{\vee}(-1-k)) \\ =& \, 2H^3 + (-1-k)H(4H^2) + (-1-k)^2H^2(-2H) \\ & + (-1-k)^3H^3 + (4+3k)H(3k^2+7k+7)H^2 \\ =& \, (8k^3+28k^2+38k+23)H^3, \end{aligned}$$

since $c_3(\Omega^1_{X_3}) = 2H^3$ and $c_i(\mathcal{O}_{X_3}(k)) = 0$ for all $k \in \mathbb{Z}$ and $i \geq 2$.

$\square$

Note that the family $\mathcal{D}(2k)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension 1 distribution of degree $2k$ on X_3 has dimension

$$\begin{aligned}\dim \mathcal{D}(2k) &= \dim \operatorname{Hom}(\mathcal{O}_{X_3}(-2-3k), \Omega^1_{X_3}(-k)) - 1 \\ &= 12k^3 + 42k^2 + 36k + 9.\end{aligned}$$

Lemma 6.3. $h^2(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes \Omega^1_{\mathbb{P}^4}|_{X_3}) = h^3(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes \Omega^1_{\mathbb{P}^4}|_{X_3}) = 0.$

Proof. We twist the Euler exact sequence in $\mathbb{P}^4$ restricted to X_3 by $\otimes\Omega^1_{\mathbb{P}^4}|_{X_3}$; passing to cohomology, we get

$$\begin{aligned}0 \to H^2(\mathrm{T}\mathbb{P}^4|_{X_3}\otimes\Omega^1_{\mathbb{P}^4}|_{X_3}) \to H^3(\Omega^1_{\mathbb{P}^4}|_{X_3}) \to 5H^3(\Omega^1_{\mathbb{P}^4}|_{X_3}(1)) \to \\ H^3(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes \Omega^1_{\mathbb{P}^4}|_{X_3}) \to 0,\end{aligned}$$

since $H^2(\Omega^1_{\mathbb{P}^4}|_{X_3}(1)) = 0$.

As $H^3(\Omega^1_{\mathbb{P}^4}|_{X_3}) = H^3(\Omega^1_{\mathbb{P}^4}|_{X_3}(1)) = 0$, we get $h^2(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes \Omega^1_{\mathbb{P}^4}|_{X_3}) = h^3(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes \Omega^1_{\mathbb{P}^4}|_{X_3}) = 0$. □

We prove the main result of this section.

Theorem 6.4. *For each $k \geq 1$, the moduli space of stable rank 2 reflexive sheaves on X_3 with Chern classes*

$$(c_1, c_2, c_3) = (0, (3k^2 + 4k + 4)H^2, (8k^3 + 16k^2 + 16k + 10)H^3)$$

contains an irreducible component of dimension $12k^3 + 42k^2 + 36k + 49$ containing the family of the tangent sheaves of a generic codimension 1 distribution of degree $2k$ on X_3.

Proof. Initially note that, by the commutative diagram (7), each tangent sheaf $T^\vee_{\mathscr{D}}$ of a generic codimension 1 distribution $\mathscr{D}$ of degree $2k$ can be given as the cokernel of the monomorphism

$$\sigma \,:\, \mathcal{O}_{X_3}(-2-3k) \oplus \mathcal{O}_{X_3}(-3-k) \to \Omega^1_{\mathbb{P}^4}|_{X_3}(-k).$$

By Proposition 3.4,

$$\begin{aligned}\dim \mathcal{F}(2k) = {}& \dim \operatorname{Hom}(\mathcal{O}_{X_3}(-2-3k) \oplus \mathcal{O}_{X_3}(-3-k), \Omega^1_{\mathbb{P}^4}|_{X_3}(-k)) \\ &- \dim \operatorname{Aut}(\mathcal{O}_{X_3}(-2-3k) \oplus \mathcal{O}_{X_3}(-3-k)) \\ ={}& 12k^3 + 42k^2 + 36k + 49,\end{aligned}$$

if $k \geq 1$ and $\dim \mathcal{F}(0) = 44$. Thus, it is enough to argue that

$$\dim \mathrm{Ext}^1(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \dim \mathcal{F}(2k) = 12k^3 + 42k^2 + 36k + 49,$$

for $k \geq 1$, and hence, by semicontinuity, we can conclude that

$$\dim \mathrm{Ext}^1(F, F) = \dim \mathcal{F}(2k) = 12k^3 + 42k^2 + 36k + 49,$$

for a generic sheaf $F \in \mathcal{F}(2k)$. Or equivalent, we must to show that

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \dim \mathcal{F}(2k) - 36k^2 - 48k - 45 = 12k^3 + 6k^2 - 12k + 4,$$

since, by Proposition 3.5,

$$\dim \mathrm{Ext}^1(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) - \dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 36k^2 + 48k + 45, \text{ for } k \geq 0.$$

Indeed, applying the functor $\mathrm{Hom}(\cdot, T_{\mathscr{D}}^{\vee}(-k))$ to the exact sequence (23), we get the equality

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \dim \mathrm{Ext}^2(\Omega^1_{X_3}(-k), T_{\mathscr{D}}^{\vee}(-k)) = h^2(TX_3 \otimes T_{\mathscr{D}}^{\vee}), \quad (25)$$

since $H^1(T_{\mathscr{D}}^{\vee}(2+2k)) = H^2(T_{\mathscr{D}}^{\vee}(2+2k)) = 0$ by Lemma 4.1.

Now, we twist the exact sequence

$$0 \to TX_3 \to T\mathbb{P}^4|_{X_3} \to \mathcal{O}_{X_3}(3) \to 0$$

by $\otimes T_{\mathscr{D}}^{\vee}$ and then pass to cohomology, we get the exact sequence in cohomology

$$0 \to H^2(TX_3 \otimes T_{\mathscr{D}}^{\vee}) \to H^2(T\mathbb{P}^4|_{X_3} \otimes T_{\mathscr{D}}^{\vee}) \to H^2(T_{\mathscr{D}}^{\vee}(3)) \to 0,$$

since $H^1(T_{\mathscr{D}}^{\vee}(3)) = 0$ by item 2) of Lemma 4.1 and $H^3(TX_3 \otimes T_{\mathscr{D}}^{\vee}) = 0$ by Lemma 4.2. Thus,

$$h^2(TX_3 \otimes T_{\mathscr{D}}^{\vee}) = h^2(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes T_{\mathscr{D}}^{\vee}) - h^0(\mathcal{O}_{X_3}(2k-3)), \quad (26)$$

since, by item 3) of Lemma 4.1, $h^2(T_{\mathscr{D}}^{\vee}(3)) = h^0(\mathcal{O}_{X_3}(2k-3))$.

In order to compute $h^2(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes T_{\mathscr{D}}^{\vee})$, we twist the exact sequences

$$0 \to \mathcal{O}_{X_3}(-3-k) \to T_{\sigma_1}(-k) \to T_{\mathscr{D}}^{\vee}(-k) \to 0$$

and

$$0 \to \mathcal{O}_{X_3}(-2-3k) \to \Omega^1_{\mathbb{P}^4}|_{X_3}(-k) \to T_{\sigma_1}(-k) \to 0$$

in the commutative diagram (8) by $\otimes \mathrm{T}\mathbb{P}^4|_{X_3}(k)$ and then pass to cohomology, we get, for each $k \geq 1$,

$$0 = H^2(\mathrm{T}\mathbb{P}^4|_{X_3}(-3)) \to H^2(T\mathbb{P}^4|_{X_3} \otimes T_{\sigma_1}) \to \\ H^2(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes T_{\mathscr{D}}^{\vee}) \to H^3(\mathrm{T}\mathbb{P}^4|_{X_3}(-3)) = 0$$

and

$$0 = H^2(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes \Omega^1_{\mathbb{P}^4}|_{X_3}) \to H^2(T\mathbb{P}^4|_{X_3} \otimes T_{\sigma_1}) \to \\ H^3(\mathrm{T}\mathbb{P}^4|_{X_3}(-2-2k)) \to H^3(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes \Omega^1_{\mathbb{P}^4}|_{X_3}) = 0.$$

note that $H^2(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes \Omega^1_{\mathbb{P}^4}|_{X_3}) = H^3(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes \Omega^1_{\mathbb{P}^4}|_{X_3}) = 0$ by Lemma 6.3.

It follows that

$$h^2(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes T_{\mathscr{D}}^{\vee}) = h^3(\mathrm{T}\mathbb{P}^4|_{X_3}(-2-2k)),$$

for $k \geq 1$. The Euler exact sequence in $\mathbb{P}^4$ restricted to X_3 implies that

$$h^3(\mathrm{T}\mathbb{P}^4|_{X_3}(-2-2k)) = 5h^0(\mathcal{O}_{X_3}(2k-1)) - h^0(\mathcal{O}_{X_3}(2k)),$$

for $k \geq 1$, since, by Serre duality, $h^3(\mathcal{O}_{X_3}(-t)) = h^0(\mathcal{O}_{X_3}(t-2))$ for all $t \in \mathbb{Z}$.

Therefore,

$$h^2(\mathrm{T}\mathbb{P}^4|_{X_3} \otimes T_{\mathscr{D}}^{\vee}) = 5h^0(\mathcal{O}_{X_3}(2k-1)) - h^0(\mathcal{O}_{X_3}(2k)) = 16k^3 - 6k^2 + k - 1, \tag{27}$$

for $k \geq 1$.

Joining the equations (25), (26) and (27), we have that

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 12k^3 + 6k^2 - 12k + 4, \text{ for } k \geq 1.$$

□

Remark 6.5. When $k = 0$, we can still conclude that the stable rank 2 reflexive sheaves F on X_3 given by short exact sequence

$$0 \to \mathcal{O}_X(-2) \oplus \mathcal{O}_X(-3) \to \Omega^1_{\mathbb{P}^4}|_X \to F \to 0$$

are smooth points of the moduli space of stable rank 2 reflexive sheaves with Chern classes $(c_1, c_2, c_3) = (0, 4H^2, 10H^3)$ within an irreducible component of dimension 45, since $\mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 0$ (see equations (25) and (26)). However, these sheaves only form a family of dimension 44 within this irreducible component.

Similarly, the family $\mathcal{D}(2k+1)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension 1 distribution of degree $2k+1$ on X_3 has dimension

$$\begin{aligned}\dim \mathcal{D}(2k+1) &= \dim \operatorname{Hom}(\mathcal{O}_{X_3}(-4-3k), \Omega^1_{X_3}(-1-k)) - 1\\ &= 12k^3 + 60k^2 + 87k + 39.\end{aligned}$$

By Proposition 3.4, the family of the stable rank 2 reflexive sheaves F on X_3 given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_{X_3}(-4-3k) \oplus \mathcal{O}_{X_3}(-4-k) \to \Omega^1_{\mathbb{P}^4}|_{X_3}(-1-k)$$

has dimension

$$\begin{aligned}&\dim \mathcal{F}(2k+1)\\ &\quad= \dim \operatorname{Hom}(\mathcal{O}_{X_3}(-4-3k) \oplus \mathcal{O}_{X_3}(-4-k), \Omega^1_{\mathbb{P}^4}|_{X_3}(-1-k))\\ &\qquad - \dim \operatorname{Aut}(\mathcal{O}_{X_3}(-4-3k) \oplus \mathcal{O}_{X_3}(-4-k))\\ &\quad= 12k^3 + 60k^2 + 87k + 79,\end{aligned}$$

for $k \geq 1$ and $\dim \mathcal{F}(1) = 78$.

By Proposition 3.5,

$$\dim \operatorname{Ext}^1(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) - \dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = 36k^2 + 84k + 78,$$

for $k \geq 0$.

Following the proof of Theorem 6.4, it is easy to show that

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = 12k^3 + 24k^2 + 3k + 1,$$

if $k \geq 1$ and $\dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = 0$, for $k = 0$.

For the case $c_1 = -1$, we establish the following theorem:

Theorem 6.6. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X_3 with Chern classes*

$$(c_1, c_2, c_3) = (-H, (3k^2 + 7k + 7)H^2, (8k^3 + 28k^2 + 38k + 23)H^3)$$

contains an irreducible component of dimension 78, if $k = 0$, and $12k^3 + 60k^2 + 87k + 79$ for $k \geq 1$, containing the family of the tangent sheaves of a generic codimension 1 distribution of degree $2k+1$ on X_3. Moreover, this component is nonsingular in the case $k = 0$.

7 Generic distributions on quartic threefolds

Throughout this section $X_4 \hookrightarrow \mathbb{P}^4$ denotes a smooth quartic threefold with ample line bundle $\mathcal{O}_{X_4}(1)$ whose first Chern class is denoted by H, i.e. $c_1(\mathcal{O}_{X_4}(1)) = H$. The cohomology ring $H^*(X_4, \mathbb{Z})$ of X_4 is generated by H, a line $L \in H^4(X_4, \mathbb{Z})$ and a point $P \in H^6(X_4, \mathbb{Z})$ with the relations: $H^2 = 4L,\ H.L = P,\ H^3 = 4P$.
We know that $c_1(\Omega^1_{X_4}) = -H, c_2(\Omega^1_{X_4}) = 6H^2$ and $c_3(\Omega^1_{X_4}) = 14H^3$. Recall that given a generic distribution $\mathscr{D}$ on X_4, the integer $r := -1 - c_1(T_{\mathscr{D}})$ is called the *degree* of $\mathscr{D}$.

Our first step is to calculate the Chern classes of a generic distribution $\mathscr{D}$ on X_4 of odd degree, say $r = 2k + 1$.

Lemma 7.1. *If a generic distribution $\mathscr{D}$ on X_4 has degree $r = 2k + 1$, then the normalization of the sheaf $T^\vee_{\mathscr{D}}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{X_4}(-4 - 3k) \to \Omega^1_{X_4}(-1 - k) \to T^\vee_{\mathscr{D}}(-1 - k) \to 0. \tag{28}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (0, (3k^2 + 8k + 11)H^2, (8k^3 + 32k^2 + 54k + 50)H^3)$$

Proof. The exact sequence (28) implies that

$$\begin{aligned} c_1(T^\vee_{\mathscr{D}}(-1 - k)) &= c_1(\Omega^1_{X_4}(-1 - k)) - c_1(\mathcal{O}_{X_4}(-4 - 3k)) \\ &= c_1(\Omega^1_{X_4}) + 3c_1(\mathcal{O}_{X_4}(-1 - k)) - c_1(\mathcal{O}_{X_4}(-4 - 3k)) \\ &= -H + 3(-1 - k)H - (-4 - 3k)H \\ &= 0, \end{aligned}$$

since $c_1(\Omega^1_{X_4}) = -H$.

$$\begin{aligned} c_2(T^\vee_{\mathscr{D}}(-1 - k)) &= c_2(\Omega^1_{X_4}(-1 - k)) - c_1(T^\vee_{\mathscr{D}}(-1 - k))c_1(\mathcal{O}_{X_4}(-4 - 3k)) \\ &= c_2(\Omega^1_{X_4}) + 2c_1(\Omega^1_{X_4})c_1(\mathcal{O}_{X_4}(-1 - k)) \\ &\quad + 3c_1(\mathcal{O}_{X_4}(-1 - k))^2 \\ &= 6H^2 + 2(-H)(-1 - k)H + 3(-1 - k)^2H^2 \\ &= (3k^2 + 8k + 11)H^2, \end{aligned}$$

since $c_1(T^\vee_{\mathscr{D}}(-1 - k)) = 0$ and $c_2(\Omega^1_{X_4}) = 6H^2$.

$$\begin{aligned} c_3(T_{\mathscr{D}}^{\vee}(-1-k)) &= c_3(\Omega_{X_4}^1(-1-k)) - c_1(\mathcal{O}_{X_4}(-4-3k))c_2(T_{\mathscr{D}}^{\vee}(-1-k)) \\ &= 14H^3 + (-1-k)H(6H^2) + (-1-k)^2H^2(-H) \\ &\quad + (-1-k)^3H^3 + (4+3k)H(3k^2+8k+11)H^2 \\ &= (8k^3+32k^2+54k+50)H^3, \end{aligned}$$

since $c_3(\Omega_{X_4}^1) = 14H^3$ and $c_i(\mathcal{O}_{X_4}(k)) = 0$ for all $k \in \mathbb{Z}$ and $i \geq 2$.

□

When $\mathscr{D}$ has degree $r = 2k$, we have the following lemma.

Lemma 7.2. *If a generic distribution $\mathscr{D}$ on X_4 has degree $r = 2k$, then the normalization of the sheaf $T_{\mathscr{D}}^{\vee}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{X_4}(-3-3k) \to \Omega_{X_4}^1(-1-k) \to T_{\mathscr{D}}^{\vee}(-1-k) \to 0. \tag{29}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (-H, (3k^2+5k+8)H^2, (8k^3+20k^2+28k+30)H^3).$$

Proof. The exact sequence (29) implies that

$$\begin{aligned} c_1(T_{\mathscr{D}}^{\vee}(-1-k)) &= c_1(\Omega_{X_4}^1(-1-k)) - c_1(\mathcal{O}_{X_4}(-3-3k)) \\ &= c_1(\Omega_{X_4}^1) + 3c_1(\mathcal{O}_{X_4}(-1-k)) - c_1(\mathcal{O}_{X_4}(-3-3k)) \\ &= -H + 3(-1-k)H - (-3-3k)H \\ &= -H, \end{aligned}$$

since $c_1(\Omega_{X_4}^1) = -H$.

$$\begin{aligned} c_2(T_{\mathscr{D}}^{\vee}(-1-k)) &= c_2(\Omega_{X_4}^1(-1-k)) - c_1(T_{\mathscr{D}}^{\vee}(-1-k))c_1(\mathcal{O}_{X_4}(-3-3k)) \\ &= c_2(\Omega_{X_4}^1) + 2c_1(\Omega_{X_4}^1)c_1(\mathcal{O}_{X_4}(-1-k)) \\ &\quad + 3c_1(\mathcal{O}_{X_4}(-1-k))^2 - (-H)(-3-3k)H \\ &= 6H^2 + 2(-H)(-1-k)H + 3(-1-k)^2H^2 \\ &\quad + (-3-3k)H^2 \\ &= (3k^2+5k+8)H^2, \end{aligned}$$

since $c_1(T_{\mathscr{D}}^{\vee}(-1-k)) = -H$ and $c_2(\Omega_{X_4}^1) = 6H^2$.

$$\begin{aligned}
c_3(T_{\mathscr{D}}^{\vee}(-1-k)) &= c_3(\Omega^1_{X_4}(-1-k)) - c_1(\mathcal{O}_{X_4}(-3-3k)c_2(T_{\mathscr{D}}^{\vee}(-1-k))) \\
&= 14H^3 + (-1-k)H(6H^2) + (-1-k)^2H^2(-H) \\
&\quad + (-1-k)^3H^3 + (3+3k)H(3k^2+5k+8)H^2 \\
&= (8k^3+20k^2+28k+30)H^3,
\end{aligned}$$

since $c_3(\Omega^1_{X_4}) = 14H^3$ and $c_i(\mathcal{O}_{X_4}(k)) = 0$ for all $k \in \mathbb{Z}$ and $i \geq 2$.

□

Note that the family $\mathcal{D}(2k+1)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension 1 distribution of degree $2k+1$ on X_4 has dimension

$$\begin{aligned}
\dim \mathcal{D}(2k+1) &= \dim \operatorname{Hom}(\mathcal{O}_{X_4}(-4-3k), \Omega^1_{X_4}(-1-k)) - 1 \\
&= 16k^3 + 76k^2 + 86k + 40,
\end{aligned}$$

for $k \geq 1$ and $\dim \mathcal{D}(1) = 39$.

Lemma 7.3. $h^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes \Omega^1_{\mathbb{P}^4}|_{X_4}) = 5$ *and* $h^3(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes \Omega^1_{\mathbb{P}^4}|_{X_4}) = 0$.

Proof. We twist the Euler exact sequence in $\mathbb{P}^4$ restricted to X_4 by $\otimes\Omega^1_{\mathbb{P}^4}|_{X_4}$; passing to cohomology, we get

$$\begin{aligned}
0 \to H^2(\mathrm{T}\mathbb{P}^4|_{X_4}\otimes\Omega^1_{\mathbb{P}^4}|_{X_4}) \to H^3(\Omega^1_{\mathbb{P}^4}|_{X_4}) \to 5H^3(\Omega^1_{\mathbb{P}^4}|_{X_4}(1)) \to \\
H^3(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes \Omega^1_{\mathbb{P}^4}|_{X_4}) \to 0.
\end{aligned}$$

It follows that, $h^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes \Omega^1_{\mathbb{P}^4}|_{X_4}) = 5$ and $h^3(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes \Omega^1_{\mathbb{P}^4}|_{X_4}) = 0$, since $h^3(\Omega^1_{\mathbb{P}^4}|_{X_4}) = 5$ and $h^3(\Omega^1_{\mathbb{P}^4}|_{X_4}(1)) = 0$.

□

We prove the main result of this section.

Theorem 7.4. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X_4 with Chern classes,*

$$(c_1, c_2, c_3) = (0, (3k^2+8k+11)H^2, (8k^3+32k^2+54k+50)H^3),$$

contains an irreducible component of dimension (139 *if* $k=0$*)* $16k^3 + 76k^2 + 86k + 145$ *for* $k \geq 1$ *containing the family of the tangent sheaves of a generic codimension 1 distribution of degree* $2k+1$ *on* X_4.

Proof. Initially note that, by the commutative diagram (7), each tangent sheaf $T_{\mathscr{D}}^{\vee}$ of a generic codimension 1 distribution $\mathscr{D}$ of degree $2k+1$ can be given as the cokernel of the monomorphism

$$\sigma : \mathcal{O}_{X_4}(-4-3k) \oplus \mathcal{O}_{X_4}(-5-k) \to \Omega^1_{\mathbb{P}^4}|_{X_4}(-1-k).$$

By Proposition 3.4,

$$\begin{aligned}\dim \mathcal{F}(2k+1) = &\dim \operatorname{Hom}(\mathcal{O}_{X_4}(-4-3k) \oplus \mathcal{O}_{X_4}(-5-k), \Omega^1_{\mathbb{P}^4}|_{X_4}(-1-k))\\ &- \dim \operatorname{Aut}(\mathcal{O}_{X_4}(-4-3k) \oplus \mathcal{O}_{X_4}(-5-k))\\ =&16k^3 + 76k^2 + 86k + 145,\end{aligned}$$

if $k \geq 1$ and $\dim \mathcal{F}(1) = 139$. Thus, it is enough to argue that

$$\dim \operatorname{Ext}^1(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \dim \mathcal{F}(2k+1) = 16k^3 + 76k^2 + 86k + 145, \ \ (k \geq 1),$$

and

$$\dim \operatorname{Ext}^1(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \dim \mathcal{F}(1) = 139,$$

if $k = 0$, and hence, by semicontinuity, we can conclude that

$$\dim \operatorname{Ext}^1(F, F) = \dim \mathcal{F}(2k+1) = 16k^3 + 76k^2 + 86k + 145, \ \ (k \geq 1),$$

for a generic sheaf $F \in \mathcal{F}(2k+1)$ and

$$\dim \operatorname{Ext}^1(F, F) = \dim \mathcal{F}(1) = 139,$$

for a generic sheaf $F \in \mathcal{F}(1)$. Or equivalent, we must to show that

$$\dim \operatorname{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \dim \mathcal{F}(2k+1) - 24k^2 - 64k - 85 = 16k^3 + 52k^2 + 22k + 60,$$

if $k \geq 1$ and

$$\dim \operatorname{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \dim \mathcal{F}(1) - 85 = 54,$$

since, by Proposition 3.5,

$$\dim \operatorname{Ext}^1(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) - \dim \operatorname{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 24k^2 + 64k + 85,$$

for $k \geq 0$.

Indeed, applying the functor $\operatorname{Hom}(\cdot, T_{\mathscr{D}}^{\vee}(-1-k))$ to the exact sequence (28), we get

$$0 \to \operatorname{Ext}^2(\Omega^1_{X_4}, T_{\mathscr{D}}^{\vee}) \to \operatorname{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) \to \operatorname{Ext}^2(\mathcal{O}_{X_4}, T_{\mathscr{D}}^{\vee}(3+2k)) \to 0,$$

since, by Lemma 4.1, $\mathrm{Ext}^1(\mathcal{O}_{X_4}, T_{\mathscr{D}}^\vee(3+2k)) \simeq H^1(T_{\mathscr{D}}^\vee(3+2k)) = 0$ and, by stability of $T_{\mathscr{D}}^\vee$, $\mathrm{Ext}^3(T_{\mathscr{D}}^\vee, T_{\mathscr{D}}^\vee) \simeq \mathrm{Hom}(T_{\mathscr{D}}^\vee, T_{\mathscr{D}}^\vee(-1)) = 0$.

When $k \geq 1$, we have that

$$\begin{aligned}\dim \mathrm{Ext}^2(T_{\mathscr{D}}^\vee, T_{\mathscr{D}}^\vee) &= \dim \mathrm{Ext}^2(\Omega^1_{X_4}, T_{\mathscr{D}}^\vee) - \dim \mathrm{Ext}^2(\mathcal{O}_{X_4}, T_{\mathscr{D}}^\vee(3+2k)) \\ &= h^2(TX_4 \otimes T_{\mathscr{D}}^\vee) - h^2(T_{\mathscr{D}}^\vee(3+2k)),\end{aligned} \tag{30}$$

and for $k=0$, we have that

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^\vee, T_{\mathscr{D}}^\vee) = h^2(TX_4 \otimes T_{\mathscr{D}}^\vee) - 1, \tag{31}$$

since, by Lemma 4.1, $h^2(T_{\mathscr{D}}^\vee(3)) = 1$.

Now, we twist the exact sequence

$$0 \to TX_4 \to T\mathbb{P}^4|_{X_4} \to \mathcal{O}_{X_4}(4) \to 0$$

by $\otimes T_{\mathscr{D}}^\vee$; and passing to cohomology, we have that

$$0 \to H^2(TX_4 \otimes T_{\mathscr{D}}^\vee) \to H^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes T_{\mathscr{D}}^\vee) \to H^2(T_{\mathscr{D}}^\vee(4)) \to 0,$$

since, by Lemma 4.1, $h^1(T_{\mathscr{D}}^\vee(4)) = 0$ and, by Lemma 4.2, $h^3(TX_4 \otimes T_{\mathscr{D}}^\vee) = 0$.

It follows that

$$h^2(TX_4 \otimes T_{\mathscr{D}}^\vee) = h^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes T_{\mathscr{D}}^\vee) - h^0(\mathcal{O}_{X_4}(2k-2)), \tag{32}$$

since, by item 3) of Lemma 4.1, $h^2(T_{\mathscr{D}}^\vee(4)) = h^0(\mathcal{O}_{X_4}(2k-2))$.

In order to compute $h^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes T_{\mathscr{D}}^\vee)$, we twist the exact sequences

$$0 \to \mathcal{O}_{X_4}(-5-k) \to T_{\sigma_1}(-1-k) \to T_{\mathscr{D}}^\vee(-1-k) \to 0$$

and

$$0 \to \mathcal{O}_{X_4}(-4-3k) \to \Omega^1_{\mathbb{P}^4}|_{X_4}(-1-k) \to T_{\sigma_1}(-1-k) \to 0$$

in the commutative diagram (8) by $\otimes \mathrm{T}\mathbb{P}^4|_{X_4}(1+k)$; and passing to cohomology, we get, for each $k \geq 0$,

$$0 \to H^2(T\mathbb{P}^4|_{X_4} \otimes T_{\sigma_1}) \to H^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes T_{\mathscr{D}}^\vee) \to H^3(\mathrm{T}\mathbb{P}^4|_{X_4}(-4)) \to 0,$$

since $H^2(\mathrm{T}\mathbb{P}^4|_{X_4}(-4)) = H^3(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes T_{\sigma_1}) = 0$ and

$$\begin{aligned}0 \to H^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes \Omega^1_{\mathbb{P}^4}|_{X_4}) \to H^2(T\mathbb{P}^4|_{X_4} \otimes T_{\sigma_1}) \to \\ H^3(\mathrm{T}\mathbb{P}^4|_{X_4}(-3-2k)) \to 0,\end{aligned}$$

since $H^2(\mathrm{T}\mathbb{P}^4|_{X_4}(-3-2k)) = 0$ for all $k \in \mathbb{Z}$ and $H^3(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes \Omega^1_{\mathbb{P}^4}|_{X_4}) = 0$ by Lemma 7.3. So,

$$h^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes T_{\mathscr{D}}^{\vee}) = h^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes T_{\sigma_1}) + 40, \tag{33}$$

since $h^3(\mathrm{T}\mathbb{P}^4|_{X_4}(-4)) = 40$. Also

$$h^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes T_{\sigma_1}) = h^3(\mathrm{T}\mathbb{P}^4|_{X_4}(-3-2k)) + 5, \tag{34}$$

since, by Lemma 7.3, $h^2(\mathrm{T}\mathbb{P}^4|_{X_4} \otimes \Omega^1_{\mathbb{P}^4}|_{X_4}) = 5$.

The Euler exact sequence in $\mathbb{P}^4$ restricted to X_4 implies that

$$h^3(\mathrm{T}\mathbb{P}^4|_{X_4}(-3-2k)) = 5h^0(\mathcal{O}_{X_4}(2k+1)) - h^0(\mathcal{O}_{X_4}(2k+2)),$$

for $k \geq 1$, since, by Serre duality, $h^3(\mathcal{O}_{X_4}(-t)) = h^0(\mathcal{O}_{X_4}(t-1))$ for all $t \in \mathbb{Z}$.

Therefore,

$$h^3(\mathrm{T}\mathbb{P}^4|_{X_4}(-3-2k)) = \frac{2}{3}(2k+1)(16k^2+22k+15), \tag{35}$$

for $k \geq 0$.

When $k = 0$, the equations (31), (32), (33), (34) and (35) give us

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 54.$$

When $k \geq 1$, the equations (30), (32), (33), (34) and (35) give us

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 16k^3 + 52k^2 + 22k + 60.$$

□

Similarly, the family $\mathcal{D}(2k)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension 1 distribution of degree $2k$ on X_4 has dimension

$$\begin{aligned} \dim \mathcal{D}(2k) &= \dim \mathrm{Hom}(\mathcal{O}_{X_4}(-3-3k), \Omega^1_{X_4}(-1-k)) - 1 \\ &= 16k^3 + 52k^2 + 22k + 14, \end{aligned}$$

if $k \geq 1$ and $\dim \mathcal{D}(0) = 9$. By Proposition 3.4, the family of the stable rank 2 reflexive sheaves F on X_4 is given as the cokernel of the monomorphism

$$\sigma\,:\,\mathcal{O}_{X_4}(-3-3k)\oplus\mathcal{O}_{X_4}(-5-k)\to\Omega^1_{\mathbb{P}^4}|_{X_4}(-1-k)$$

has dimension

$$\dim\mathcal{F}(2k)=\begin{cases}99, & k=0,\\ 208, & k=1,\end{cases}$$

and, for each $k\geq 2$,

$$\begin{aligned}\dim\mathcal{F}(2k)=&\dim\operatorname{Hom}(\mathcal{O}_{X_4}(-3-3k)\oplus\mathcal{O}_{X_4}(-5-k),\Omega^1_{\mathbb{P}^4}|_{X_4}(-1-k))\\ &-\dim\operatorname{Aut}(\mathcal{O}_{X_4}(-3-3k)\oplus\mathcal{O}_{X_4}(-5-k))\\ =&16k^3+52k^2+22k+119.\end{aligned}$$

By Proposition 3.5,

$$\dim\operatorname{Ext}^1(T_{\mathscr{D}}^\vee,T_{\mathscr{D}}^\vee)-\dim\operatorname{Ext}^2(T_{\mathscr{D}}^\vee,T_{\mathscr{D}}^\vee)=24k^2+40k+59,\ \text{for } k\geq 0.$$

Following the proof of Theorem 7.4, it is easy to show that

$$\dim\operatorname{Ext}^2(T_{\mathscr{D}}^\vee,T_{\mathscr{D}}^\vee)=\begin{cases}40, & k=0,\\ 85, & k=1,\end{cases}$$

and, for each $k\geq 2$,

$$\dim\operatorname{Ext}^2(T_{\mathscr{D}}^\vee,T_{\mathscr{D}}^\vee)=16k^3+28k^2-18k+60.$$

For the case $c_1=-1$, we establish the following theorem:

Theorem 7.5. *For each $k\geq 0$, the moduli space of stable rank 2 reflexive sheaves on X_4 with Chern classes*

$$(c_1,c_2,c_3)=(-H,(3k^2+5k+8)H^2,(8k^3+20k^2+28k+30)H^3)$$

contains an irreducible component of dimension (99, if $k=0$; 208, if $k=1$) $16k^3+52k^2+22k+119$ containing the family of the tangent sheaves of a generic codimension 1 distribution of degree $2k$ on X_4.

In the next section we will do an analogue study on a smooth quintic threefold.

8 Generic distributions on quintic threefolds

Throughout this section $X_5 \hookrightarrow \mathbb{P}^4$ denotes a smooth quintic threefold with ample line bundle $\mathcal{O}_{X_5}(1)$ whose first Chern class is denoted by H, i.e. $c_1(\mathcal{O}_{X_5}(1)) = H$. The cohomology ring $H^*(X_5,\mathbb{Z})$ of X_5 is generated by H, a line $L \in H^4(X_5,\mathbb{Z})$ and a point $P \in H^6(X_5,\mathbb{Z})$ with the relations: $H^2 = 5L$, $H.L = P$, $H^3 = 5P$. We know that $c_1(\Omega^1_{X_5}) = 0$, $c_2(\Omega^1_{X_5}) = 10H^2$ and $c_3(\Omega^1_{X_5}) = 40H^3$.

Recall that given a generic distribution $\mathscr{D}$ on X, the integer $r := -2-c_1(T_{\mathscr{D}})$ is called the *degree* of $\mathscr{D}$.

We start this section by calculating the Chern classes of a generic distribution $\mathscr{D}$ on X_5 of even degree, say $r = 2k$.

Lemma 8.1. *If a generic distribution $\mathscr{D}$ on X_5 has degree $r = 2k$, then the normalization of the sheaf $T^\vee_{\mathscr{D}}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{X_5}(-3-3k) \to \Omega^1_{X_5}(-1-k) \to T^\vee_{\mathscr{D}}(-1-k) \to 0, \tag{36}$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (0, (3k^2+6k+13)H^2, (8k^3+24k^2+44k+68)H^3).$$

Proof. The exact sequence (36) implies that

$$\begin{aligned} c_1(T^\vee_{\mathscr{D}}(-1-k)) &= c_1(\Omega^1_{X_5}(-1-k)) - c_1(\mathcal{O}_{X_5}(-3-3k)) \\ &= c_1(\Omega^1_{X_5}) + 3c_1(\mathcal{O}_{X_5}(-1-k)) - c_1(\mathcal{O}_{X_5}(-3-3k)) \\ &= 3(-1-k)H - (-3-3k)H \\ &= 0, \end{aligned}$$

since $c_1(\Omega^1_{X_5}) = 0$.

$$\begin{aligned} c_2(T^\vee_{\mathscr{D}}(-1-k)) &= c_2(\Omega^1_{X_5}(-1-k)) - c_1(T^\vee_{\mathscr{D}}(-1-k))c_1(\mathcal{O}_{X_5}(-3-3k)) \\ &= c_2(\Omega^1_{X_5}) + 2c_1(\Omega^1_{X_5})c_1(\mathcal{O}_{X_5}(-1-k)) \\ &\quad + 3c_1(\mathcal{O}_{X_5}(-1-k))^2 \\ &= 10H^2 + 3(-1-k)^2H^2 \\ &= (3k^2+6k+13)H^2, \end{aligned}$$

since $c_1(T^\vee_{\mathscr{D}}(-1-k)) = 0$ and $c_2(\Omega^1_{X_5}) = 10H^2$.

$$\begin{aligned} c_3(T^\vee_{\mathscr{D}}(-1-k)) &= c_3(\Omega^1_{X_5}(-1-k)) - c_1(\mathcal{O}_{X_5}(-3-3k))c_2(T^\vee_{\mathscr{D}}(-1-k)) \\ &= 40H^3 + (-1-k)H(10H^2) + (-1-k)^3H^3 \\ &\quad + (3+3k)H(3k^2+6k+13)H^2 \\ &= (8k^3+24k^2+44k+68)H^3, \end{aligned}$$

since $c_3(\Omega^1_{X_5}) = 40H^3$ and $c_i(\mathcal{O}_{X_5}(k)) = 0$ for all $k \in \mathbb{Z}$ and $i \geq 2$.

□

When $\mathscr{D}$ has degree $r = 2k+1$, we have the following lemma.

Lemma 8.2. *If a generic distribution $\mathscr{D}$ on X_5 has degree $r = 2k+1$, then the normalization of the sheaf $T^\vee_{\mathscr{D}}$ fits into the short exact sequence*

$$0 \to \mathcal{O}_{X_5}(-5-3k) \to \Omega^1_{X_5}(-2-k) \to T^\vee_{\mathscr{D}}(-2-k) \to 0, \qquad (37)$$

for $k \geq 0$ and its Chern classes are

$$(c_1, c_2, c_3) = (-H, (3k^2+9k+17)H^2, (8k^3+36k^2+74k+97)H^3).$$

Proof. The exact sequence (37) implies that

$$\begin{aligned} c_1(T^\vee_{\mathscr{D}}(-2-k)) &= c_1(\Omega^1_{X_5}(-2-k)) - c_1(\mathcal{O}_{X_5}(-5-3k)) \\ &= c_1(\Omega^1_{X_5}) + 3c_1(\mathcal{O}_{X_5}(-2-k)) - c_1(\mathcal{O}_{X_5}(-5-3k)) \\ &= 3(-2-k)H - (-5-3k)H \\ &= -H, \end{aligned}$$

since $c_1(\Omega^1_{X_5}) = 0$.

$$\begin{aligned} c_2(T^\vee_{\mathscr{D}}(-2-k)) &= c_2(\Omega^1_{X_5}(-2-k)) - c_1(T^\vee_{\mathscr{D}}(-2-k))c_1(\mathcal{O}_{X_5}(-5-3k)) \\ &= c_2(\Omega^1_{X_5}) + 2c_1(\Omega^1_{X_5})c_1(\mathcal{O}_{X_5}(-2-k)) \\ &\quad + 3c_1(\mathcal{O}_{X_5}(-2-k))^2 + H(-5-3k)H \\ &= 10H^2 + 3(-2-k)^2H^2 - (5+3k)H^2 \\ &= (3k^2+9k+17)H^2, \end{aligned}$$

since $c_1(T^\vee_{\mathscr{D}}(-2-k)) = -H$ and $c_2(\Omega^1_{X_5}) = 10H^2$.

$$\begin{aligned}c_3(T_{\mathscr{D}}^{\vee}(-2-k)) &= c_3(\Omega^1_{X_5}(-2-k)) - c_1(\mathcal{O}_{X_5}(-5-3k))c_2(T_{\mathscr{D}}^{\vee}(-2-k))\\ &= 40H^3 + (-2-k)H(10H^2) + (-2-k)^3H^3\\ &\quad + (5+3k)H(3k^2+9k+17)H^2\\ &= (8k^3+36k^2+74k+97)H^3,\end{aligned}$$

since $c_3(\Omega^1_{X_5}) = 40H^3$ and $c_i(\mathcal{O}_{X_5}(k)) = 0$ for all $k \in \mathbb{Z}$ and $i \geq 2$.

□

Note that the family $\mathcal{D}(2k)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension 1 distribution of degree $2k$ on X_5 has dimension

$$\begin{aligned}\dim \mathcal{D}(2k) &= \dim \operatorname{Hom}(\mathcal{O}_{X_5}(-3-3k), \Omega^1_{X_5}(-1-k)) - 1\\ &= 20k^3 + 60k^2 - 15k + 44,\end{aligned}$$

for $k \geq 0$.

Lemma 8.3. $h^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) - h^3(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) = -1.$

Proof. We twist the Euler exact sequence in $\mathbb{P}^4$ restricted to X_5 by $\otimes\Omega^1_{\mathbb{P}^4}|_{X_5}$; passing to cohomology, we get

$$\begin{aligned}0 \to H^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) \to H^3(\Omega^1_{\mathbb{P}^4}|_{X_5}) \to 5H^3(\Omega^1_{\mathbb{P}^4}|_{X_5}(1))\\ \to H^3(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) \to 0.\end{aligned}$$

It follows that,

$$\begin{aligned}h^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) &- h^3(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5})\\ &= h^3(\Omega^1_{\mathbb{P}^4}|_{X_5}) - 5h^3(\Omega^1_{\mathbb{P}^4}|_{X_5}(1))\\ &= -1,\end{aligned}$$

since $h^3(\Omega^1_{\mathbb{P}^4}|_{X_5}) = 24$ and $h^3(\Omega^1_{\mathbb{P}^4}|_{X_5}(1)) = 5$.

□

We prove the main result of this section.

Theorem 8.4. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X_5 with Chern classes*

$$(c_1, c_2, c_3) = (0, (3k^2+6k+13)H^2, (8k^3+24k^2+44k+68)H^3)$$

contains an irreducible component of dimension, (198 if $k = 0$, 323 if $k = 1$) $20k^3+60k^2-15k+268$ for $k \geq 2$, containing the family of the tangent sheaves of a generic codimension 1 distribution of degree $2k$ on X_5.

Proof. Initially note that, by the commutative diagram (7), each tangent sheaf $T_{\mathscr{D}}^{\vee}$ of a generic codimension 1 distribution $\mathscr{D}$ of degree $2k$ can be given as the cokernel of the monomorphism

$$\sigma\,:\,\mathcal{O}_{X_5}(-3-3k)\oplus\mathcal{O}_{X_5}(-6-k)\to\Omega^1_{\mathbb{P}^4}|_{X_5}(-1-k).$$

By Proposition 3.4,

$$\begin{aligned}\dim\mathcal{F}(2k)=&\dim\mathrm{Hom}(\mathcal{O}_{X_5}(-3-3k)\oplus\mathcal{O}_{X_5}(-6-k),\Omega^1_{\mathbb{P}^4}|_{X_5}(-1-k))\\&-\dim\mathrm{Aut}(\mathcal{O}_{X_5}(-3-3k)\oplus\mathcal{O}_{X_5}(-6-k))\\=&\,20k^3+60k^2-15k+268,\end{aligned}$$

if $k\geq 2$, and

$$\dim\mathcal{F}(2k)=\begin{cases}198, & \text{if } k=0,\\ 323, & \text{if } k=1.\end{cases}$$

Thus, it is enough to argue that

$$\dim\mathrm{Ext}^1(T_{\mathscr{D}}^{\vee},T_{\mathscr{D}}^{\vee})=\dim\mathcal{F}(2k)=20k^3+60k^2-15k+268,\ \ (k\geq 2)$$

and

$$\dim\mathrm{Ext}^1(T_{\mathscr{D}}^{\vee},T_{\mathscr{D}}^{\vee})=\dim\mathcal{F}(2k)=\begin{cases}198, & \text{if } k=0,\\ 323, & \text{if } k=1,\end{cases}$$

and hence, by semicontinuity, we can conclude that

$$\dim\mathrm{Ext}^1(F,F)=\dim\mathcal{F}(2k)=20k^3+60k^2-15k+268,\ \ (k\geq 2)$$

and

$$\dim\mathrm{Ext}^1(F,F)=\dim\mathcal{F}(2k)=\begin{cases}198, & \text{if } k=0,\\ 323, & \text{if } k=1,\end{cases}$$

for a generic sheaf $F\in\mathcal{F}(2k)$. Or equivalent, we must to show that

$$\dim\mathrm{Ext}^2(T_{\mathscr{D}}^{\vee},T_{\mathscr{D}}^{\vee})=\dim\mathcal{F}(2k)=20k^3+60k^2-15k+268,$$

if $k\geq 2$, and

$$\dim\mathrm{Ext}^2(T_{\mathscr{D}}^{\vee},T_{\mathscr{D}}^{\vee})=\dim\mathcal{F}(2k)=\begin{cases}198, & \text{if } k=0,\\ 323, & \text{if } k=1,\end{cases}$$

since, by Proposition 3.5,

$$\dim\mathrm{Ext}^1(T_{\mathscr{D}}^{\vee},T_{\mathscr{D}}^{\vee})=\dim\mathrm{Ext}^2(T_{\mathscr{D}}^{\vee},T_{\mathscr{D}}^{\vee}),$$

for $k\geq 0$.

Applying the functor $\operatorname{Hom}(\cdot, T_{\mathscr{D}}^{\vee}(-1-k))$ to the exact sequence (36), we get

$$\dim \operatorname{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = h^2(TX_5 \otimes T_{\mathscr{D}}^{\vee}) - h^2(T_{\mathscr{D}}^{\vee}(2+2k)) + 1 \tag{38}$$

if $k \geq 0$, since, by Lemma 4.1 $h^1(T_{\mathscr{D}}^{\vee}(2+2k)) = 0$; by Lemma 4.2 $h^3(TX_5 \otimes T_{\mathscr{D}}^{\vee}) = 0$ and $\dim \operatorname{Ext}^3(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 1$, because $\operatorname{Ext}^3(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) \simeq \operatorname{Hom}(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee})$.

Now, we twist the exact sequence

$$0 \to TX_5 \to T\mathbb{P}^4|_{X_5} \to \mathcal{O}_{X_5}(5) \to 0$$

by $\otimes T_{\mathscr{D}}^{\vee}$; and passing to cohomology, we have that

$$0 \to H^2(TX_5 \otimes T_{\mathscr{D}}^{\vee}) \to H^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes T_{\mathscr{D}}^{\vee}) \to H^2(T_{\mathscr{D}}^{\vee}(5)) \to 0,$$

since, by Lemma 4.1, $h^1(T_{\mathscr{D}}^{\vee}(5)) = 0$ and, by Lemma 4.2, $h^3(TX_5 \otimes T_{\mathscr{D}}^{\vee}) = 0$. Hence,

$$h^2(TX_5 \otimes T_{\mathscr{D}}^{\vee}) = h^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes T_{\mathscr{D}}^{\vee}) - h^2(T_{\mathscr{D}}^{\vee}(5)). \tag{39}$$

Using the exact sequence (36), it is easy to see that

$$h^2(T_{\mathscr{D}}^{\vee}(5)) = h^0(\mathcal{O}_{X_5}(2k-3)) + 1, \tag{40}$$

since $h^2(\Omega^1_{X_5}(5)) = 1$, $h^2(\Omega^1_{X_5}(5)) = 0$ and $h^3(\mathcal{O}_{X_5}(3-2k)) = h^0(\mathcal{O}_{X_5}(2k-3))$. In order to compute $h^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes T_{\mathscr{D}}^{\vee})$, we twist the exact sequences

$$0 \to \mathcal{O}_{X_5}(-6-k) \to T_{\sigma_1}(-1-k) \to T_{\mathscr{D}}^{\vee}(-1-k) \to 0$$

and

$$0 \to \mathcal{O}_{X_5}(-3-3k) \to \Omega^1_{\mathbb{P}^4}|_{X_5}(-1-k) \to T_{\sigma_1}(-1-k) \to 0$$

in the commutative diagram (8) by $\otimes \mathrm{T}\mathbb{P}^4|_{X_5}(1+k)$ and then pass to cohomology, we get, for each $k \geq 0$,

$$0 \to H^2(T\mathbb{P}^4|_{X_5} \otimes T_{\sigma_1}) \to H^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes T_{\mathscr{D}}^{\vee}) \to H^3(\mathrm{T}\mathbb{P}^4|_{X_5}(-5)) \to 0,$$

since $H^2(\mathrm{T}\mathbb{P}^4|_{X_5}(-5)) = H^3(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes T_{\sigma_1}) = 0$ and

$$\begin{aligned} 0 \to H^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) \to H^2(T\mathbb{P}^4|_{X_5} \otimes T_{\sigma_1}) \to H^3(\mathrm{T}\mathbb{P}^4|_{X_5}(-2-2k)) \\ \to H^3(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) \to 0, \end{aligned}$$

since $h^2(\mathrm{T}\mathbb{P}^4|_{X_5}(t)) = 0$, for all $t \neq 0$ and $h^3(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes T_{X_5}) = 0$. So,

$$\begin{aligned} h^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes T_{\mathscr{D}}^{\vee}) &= h^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes T_{\sigma_1}) + h^3(\mathrm{T}\mathbb{P}^4|_{X_5}(-5)) \\ &= h^3(\mathrm{T}\mathbb{P}^4|_{X_5}(-2-2k)) + h^3(\mathrm{T}\mathbb{P}^4|_{X_5}(-5)) \\ &\quad + h^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) - h^3(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) \\ &= h^3(\mathrm{T}\mathbb{P}^4|_{X_5}(-2-2k)) + 224, \end{aligned} \tag{41}$$

Since, by Lemma 8.3, $h^2(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) - h^3(\mathrm{T}\mathbb{P}^4|_{X_5} \otimes \Omega^1_{\mathbb{P}^4}|_{X_5}) = -1$ and $h^3(\mathrm{T}\mathbb{P}^4|_{X_5}(-5)) = 225$.

Now, we twist the exact sequence

$$0 \to \mathrm{T}\mathbb{P}^4(-5) \to \mathrm{T}\mathbb{P}^4 \to \mathrm{T}\mathbb{P}^4|_{X_5} \to 0$$

by $\otimes \mathcal{O}_{X_5}(-2-2k)$; passing to cohomology, we get

$$h^3(\mathrm{T}\mathbb{P}^4|_{X_5}(-2-2k)) = \frac{5}{6}(32k^3 + 36k^2 + 46k + 12), \tag{42}$$

for $k \geq 0$. Joining the equations (38), (39), (40), (41) and (42), we get

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = 20k^3 + 60k^2 - 15k + 268,$$

if $k \geq 2$, and

$$\dim \mathrm{Ext}^2(T_{\mathscr{D}}^{\vee}, T_{\mathscr{D}}^{\vee}) = \begin{cases} 198, & k = 0, \\ 323, & k = 1, \end{cases}$$

as desired. $\square$

Similarly, the family $\mathcal{D}(2k+1)$ of the stable rank 2 sheaves obtained as tangent sheaves of a generic codimension 1 distribution of degree $2k+1$ on X has dimension

$$\begin{aligned} \dim \mathcal{D}(2k+1) &= \dim \mathrm{Hom}(\mathcal{O}_{X_5}(-5-3k), \Omega^1_{X_5}(-2-k)) - 1 \\ &= 20k^3 + 90k^2 + 60k + 54, \end{aligned}$$

if $k \geq 1$ and $\dim \mathcal{D}(1) = 39$. By Proposition 3.4, the family of the stable rank 2 reflexive sheaves F on X_5 is given as the cokernel of the monomorphism

$$\sigma \,:\, \mathcal{O}_{X_5}(-5-3k) \oplus \mathcal{O}_{X_5}(-7-k) \to \Omega^1_{\mathbb{P}^4}|_{X_5}(-2-k)$$

has dimension

$$\dim \mathcal{F}(2k+1) = \begin{cases} 248, & \text{if } k=0, \\ 446, & \text{if } k=1, \end{cases}$$

and, for each $k \geq 2$,

$$\begin{aligned}\dim \mathcal{F}(2k+1) = &\dim \operatorname{Hom}(\mathcal{O}_{X_5}(-5-3k) \oplus \mathcal{O}_{X_5}(-7-k), \Omega^1_{\mathbb{P}^4}|_{X_5}(-2-k)) \\ &- \dim \operatorname{Aut}(\mathcal{O}_{X_5}(-5-3k) \oplus \mathcal{O}_{X_5}(-7-k)) \\ = &20k^3 + 90k^2 + 60k + 278.\end{aligned}$$

By Proposition 3.5,

$$\dim \operatorname{Ext}^1(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = \dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}), \text{for } k \geq 0.$$

Following the proof of Theorem 8.4, it is easy to show that

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = \begin{cases} 248, & \text{if } k=0, \\ 446, & \text{if } k=1, \end{cases}$$

and, for each $k \geq 2$,

$$\dim \operatorname{Ext}^2(T^\vee_{\mathscr{D}}, T^\vee_{\mathscr{D}}) = 20k^3 + 90k^2 + 60k + 278.$$

For the case $c_1 = -1$, we establish the following theorem.

Theorem 8.5. *For each $k \geq 0$, the moduli space of stable rank 2 reflexive sheaves on X_5 with Chern classes*

$$(c_1, c_2, c_3) = (-H, (3k^2 + 9k + 17)H^2, (8k^3 + 36k^2 + 74k + 97)H^3),$$

contains an irreducible component of dimension (248 if $k = 0$, 446 if $k = 1$) $20k^3 + 90k^2 + 60k + 278$ for $k \geq 2$, containing the family of the tangent sheaves of a generic codimension 1 distribution of degree $2k+1$ on X_5.

9 Local complete intersection distributions on threefold hypersurfaces

The simplest way to construct codimension 1 distributions of local complete intersection type on X_d is to consider two non trivial twisted vector fields $\nu_i \in H^0(TX_d(k_i))$, $i = 1, 2$, such that the cokernel of the induced monomorphism $\nu : \mathcal{O}_{X_d}(-k_1) \oplus \mathcal{O}_{X_d}(-k_2) \to TX_d$ is a torsion free sheaf. Then we obtain a codimension 1 distribution

$$0 \to \mathcal{O}_{X_d}(-k_1) \oplus \mathcal{O}_{X_d}(-k_2) \xrightarrow{\nu} TX_d \to \mathscr{I}_Z(k_1 + k_2 + 5 - d) \to 0,$$

whose singular locus Z is a curve see [1, Lemma 2.1]. Some properties of these *split distributions* have been studied in [3].

In this section, we show how to construct examples of codimension 1 distributions on X_d such that its tangent sheaf is non split locally free sheaf. We set

$$\gamma_d := \min\{t \in \mathbb{Z} \mid TX_d(t) \text{ is globally generated}\}.$$

Let E be a stable rank 2 vector bundle on X_d with Chern classes $c_1(E) = 0$ (so that $E^* \simeq E$) and $c_2(E) = dL$ such that $E(1)$ globally generated. It follows that $E \otimes TX_d(\gamma_d + t)$ is also globally generated for all $t \geq 1$. By Ottaviani's Bertini-type theorem [9, Teorema 2.8], there is a monomorphism $\phi : E(-2-t) \to \Omega^1_{X_d}$ such that $\operatorname{coker}\phi$ is a torsion free sheaf of rank 1, for each $t \geq 1$, i.e, there are non generic codimension 1 distributions on X_d

$$\mathscr{D} \ : \ 0 \to E(-\gamma_{X_d} - t) \xrightarrow{\phi} TX_d \to \mathscr{I}_Z(r+2) \to 0$$

of degree $r = c_1(TX_d) - c_1(E(-\gamma_d - t)) - 2 = 3 - d + 2t + 2\gamma_d$, where d is the degree of X_d.

Let us now consider an explicit family of codimension 1 distributions of local complete intersection type, constructed via the technique described above. To be precise, consider a monad of the following form

$$\mathcal{M}_\bullet : \mathcal{O}_{X_d}(-1) \xrightarrow{\alpha} \mathcal{O}_{X_d}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{X_d}(1), \tag{43}$$

i.e., $\mathcal{M}_\bullet$ is a complex of sheaves on X_d with α being a bundle monomorphism and β being an epimorphism; it follows that $E := \ker(\beta)/\operatorname{im}(\alpha)$ is a rank 2 locally free sheaf with $c_1(E) = 0$ and $c_2(E) = dL$. One can also check that E is stable, see [6, Proposition 4].

Next we will establish the existence of monads as in (43).

Example 9.1. Consider the monad

$$\mathcal{M}_1 : \mathcal{O}_{X_d}(-1) \xrightarrow{\alpha} \mathcal{O}_{X_d}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{X_d}(1),$$

where

$$\alpha^T := (X_1 \ \ -X_0 \ \ X_4 \ \ -X_3) \quad \text{and} \quad \beta := (X_0 \ \ X_1 \ \ X_2 \ \ X_3).$$

We have that $\mathcal{M}_1$ is well defined for each $p = (x_0 : x_1 : x_2 : x_3 : x_4) \in X_d$, the matrices α and β have maximum rank 1 for every point in X_d and $\beta.\alpha = 0$. Moreover, its cohomology sheaf E is a rank 2 locally free sheaf on X_d.

Lemma 9.2. *If E is a rank 2 locally free sheaf given as the cohomology sheaf of a monad as displayed in* (43)*, then $E(1)$ is globally generated.*

Proof. We consider $K := \ker \beta$; to show that $E(1)$ is globally generated it is enough to argue that $K(1)$ is globally generated since we have the epimorphism $K(1) \twoheadrightarrow E(1)$.

The formula

$$\wedge^q K^* \simeq \wedge^r K^* \otimes \wedge^{r-q} K, \qquad r = \operatorname{rk} K,$$

gives us

$$K(1) \simeq K \otimes \det K^* \simeq K \otimes \wedge^3 K^* \simeq \wedge^2 K^*.$$

The exact sequence

$$0 \to K \to \mathcal{O}_{X_d}^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_{X_d}(1) \to 0$$

implies that K^* globally generated as an image of $\mathcal{O}_{X_d}^{\oplus 4}$. Therefore, $\wedge^2 K^*$ is globally generated and hence $K(1)$ is also. □

Therefore, we have the following theorem.

Theorem 9.3. *If E is a rank 2 locally free sheaf with Chern classes $c_1(E) = 0$ and $c_2 = 2L$ such that $E(1)$ globally generated, then, for every $t \geq 1$, there is a codimension 1 distribution on X_d of degree $r = 3 - d + 2t + 2\gamma_X$ such that its tangent sheaf is a non split locally free sheaf given by*

$$\mathscr{D} \ : \ 0 \to E(-\gamma_d - t) \xrightarrow{\phi} TX_d \to \mathscr{I}_Z(r+2) \to 0.$$

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