

THE VAN DER POL EQUATION: QUALITATIVE
AND NUMERICAL STUDY

ECUACIÓN DE VAN DER POL: ESTUDIO
CUALITATIVO Y NUMÉRICO

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Abstract

This expository paper aims at the study of nonlinear equations, focused on the van der Pol equation, including deduction, qualitative analysis, and numerical examples. The van der Pol equation is deduced using an electrical circuit as a physical model. The qualitative analysis is divided into two parts: the theoretical enunciation and its application. The main theorems used in this study are Poincaré-Bendixson's and Lyapunov's. The construction of a Lyapunov function is also performed. Finally, a series of numerical examples are graphically presented using computational tools such as Python and Octave. The phase portraits and temporal behavior of the van der Pol equation are exhibited, along with the basin of attraction obtained experimentally, compared with the basin of attraction yielded by the Lyapunov function. Therefore, the numerical study provides a visual representation of the results stated in the qualitative analysis.

Keywords: van der Pol equation; Lyapunov function; qualitative analysis; numerical integration.

Resumen

Este artículo expositivo tiene como objetivo el estudio de ecuaciones no lineales, enfocado en la ecuación de van der Pol, incluyendo deducción, análisis cualitativo y ejemplos numéricos. La ecuación de van der Pol es deducida utilizando un circuito eléctrico como modelo físico. El análisis cualitativo está dividido en dos partes: enunciación teórica y sus aplicaciones. Los teoremas principales usados en este estudio son los de Poincaré-Bendixson y de Lyapunov. Se hace también la construcción de una función de Lyapunov. Finalmente, una serie de ejemplos numéricos son ilustrados gráficamente utilizando herramientas computacionales como Python y Octave. Son exhibidos retratos de fase y comportamientos temporales, así como la cuenca de atracción obtenida experimentalmente, en comparación con la obtenida por la función de Lyapunov. Por consiguiente, el estudio numérico proporciona una representación visual de los resultados determinados en el análisis cualitativo.

Palabras clave: ecuación de van der Pol; función de Lyapunov; análisis cualitativo; integración numérica.

Mathematics Subject Classification: Primary 34C05, 34D20; Secondary 34C23, 34C26

1 Introduction

Balthasar van der Pol (1889-1959) was a Dutch physicist pioneer in radio communication who contributed substantially for international telecommunication, establishing, for example, the first radio-telephonic communication between the Netherlands and the East Indies [2]. This accomplishment made him a Knight of the Order of Orange Nassau, but his achievements were not restricted to practical engineering. While working with triodes, his mathematical approach formed the

basis of much of the modern theory on nonlinear oscillations. In 1926, "On relaxation oscillations" ([7]) was published on the Philosophical Magazine, presenting what is now known as the van der Pol equation.

Van der Pol, along with van der Mark, studied the heartbeats [8], realizing that they are governed by relaxation oscillations, and proposed an electrical model of the heart. Later, G. Austin et al. [1] studied the Parkinson tremors modeled by van der Pol equation, stating that, at least theoretically, it is possible to diminish facilitating paths and increase the inhibitors of this pathology. Moreover, the van der Pol equation models many other kinds of electrical, mechanical, chemical and biological oscillations.

His experiments and works on nonlinear oscillations influenced the history of dynamics and motivated other researchers such as Cambridge mathematicians Cartwright and Littlewood. They studied extensively the forced van der Pol equation, being the first ones to show the existence of singular solutions. They also concluded that this forced equation has bistable parameter regime and that there does not exist a smooth boundary between the basins of attraction of the stable periodic orbits. Their work on the van der Pol system led to the discovery of what is now called stable periodic orbits. Other historical details can be found in references [2] and [6] – [9]. A new line of research on the van der Pol equation is the study using fractional calculus, as in [3].

This paper studies the van der Pol equation, from deduction to numerical examples. In Section 2, the van der Pol equation is deduced using a simplified electrical circuit. Section 3 presents the theory used to analyse the equations qualitatively. Section 4 develops the van der Pol equation using first an a priori analysis and then applying the theorems. Section 5 approximates the van der Pol equation for small and great nonlinearities. The numerical examples used to illustrate the study are found in Section 6. The main codes are presented in appendices A and B.

2 Deduction

In this section, the van der Pol equation will be deduced by utilizing a similar circuit as the one used by the Dutch physicist in his experiments. The original triode valve can be modeled by a current source controlled by voltage with a nonlinear relationship of the form $i(v) = -av + bv^3$. The circuit to be analyzed is shown in Fig. 1.

Kirchhoff's Current Law (KCL) is an application of the conservation of charge in circuits and states that the total current entering a junction must be equal to the total current leaving the same junction.

Denoting i_F , i_C and i_L the currents entering the upper node through the current source, the capacitor and the inductor, respectively, and applying KCL,

$$i_F + i_C + i_L = 0. \quad (2.1)$$

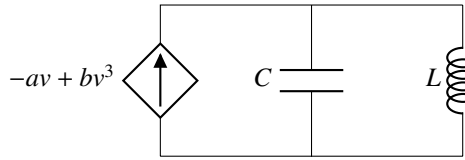


Figure 1: Oscillator circuit.

Since the elements are in parallel, they are at the same potential, i.e., $v_C = v_L = v$, where v is a function of time ($v = v(t')$). Moreover, the current source is time dependent and driven by $i_F(t) = -av(t') + bv^3(t')$.

The current through a capacitor of capacitance C charged to a potential difference $v(t')$ is given by $i_C = C \frac{d}{dt'}(v(t'))$. And the current through an inductor of inductance L under the same potential $v(t')$ is given by $i_L = \frac{1}{L} \int_{t'_0}^{t'} v(x') dx'$.

By substituting the current values in (2.1),

$$-av(t') + bv^3(t') + C \frac{dv(t')}{dt'} + \frac{1}{L} \int_{t'_0}^{t'} v(x) dx = 0.$$

Deriving with respect to t' and setting the zero potential at t'_0 ,

$$-a \frac{d}{dt'}(v(t')) + 3bv^2(t') \frac{d}{dt'}(v(t')) + C \frac{d^2}{dt'^2}(v(t')) + \frac{1}{L} v(t') = 0.$$

Rearranging,

$$\frac{d^2}{dt'^2}(v(t')) + \frac{1}{C}(3bv^2(t') - a) \frac{d}{dt'}(v(t')) + \frac{1}{LC} v(t') = 0.$$

The resonance frequency in an LC circuit is given by $\omega_0 = \frac{1}{\sqrt{LC}}$. By the following change of variables,

$$\begin{aligned} \omega_0 t' &= t, \\ v &= \sqrt{\frac{a}{3b}} x, \end{aligned}$$

and utilizing the fluxional notation for time derivatives,

$$\ddot{x} + \frac{a}{\omega_0}(x^2 - 1)\dot{x} + x = 0$$

Writing $\epsilon = \frac{a}{\omega_0}$,

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0, \quad (2.2)$$

Which is the normalized van der Pol equation.

3 Qualitative theory for nonlinear systems

Due to the difficulty of obtaining analytical solutions for nonlinear differential equations, a qualitative analysis is necessary. In this sense, Poincaré-Bendixson's and Lyapunov's theorems provide the tools needed to study the behavior of a system without in fact calculating it.

In the current section, methods to determine the stability of the equilibrium points will be presented, as well as the concept of a limit cycle and how to determine whether or not a system has a periodic solution.

From now on, it is convenient to approach the van der Pol equation as a system of two first-order differential equations. There are two possibilities, the usual system (3.1) and the Liénard system (3.2). Both approaches lead to the same result, but the latter will be used in this study.

The usual system is

$$\begin{cases} \dot{x} = \hat{y}, \\ \dot{\hat{y}} = -\epsilon(x^2 - 1)\hat{y} - x, \end{cases} \quad (3.1)$$

and the Liénard system is

$$\begin{cases} \dot{x} = y - \epsilon(\frac{x^3}{3} - x), \\ \dot{y} = -x. \end{cases} \quad (3.2)$$

Before getting into the subject, it is important to define some concepts.

For $f : D \rightarrow R^n$, consider the following nonlinear system

$$\dot{x} = f(x, t),$$

where D is an open and connected subset of R^n and x_0 an equilibrium point, i.e., $f(x_0, t) = 0$ for all $t \geq t_0$.

Definition 3.1. An equilibrium point $x = x_0$ is said to be (locally) stable, in the sense of Lyapunov, if, for each $\epsilon > 0$ there is a $\delta > 0$, such that if $\|x(t_0) - x_0\| \leq \delta$, then

$$\|x(t) - x_0\| \leq \epsilon, \quad \forall t \in [t_0, +\infty[.$$

Definition 3.2. An equilibrium point $x = x_0$ is said to be asymptotically stable if it is stable and there is a δ such that when $\|x(t_0) - x_0\| \leq \delta$, then

$$\|x(t) - x_0\| \rightarrow 0, \quad t \rightarrow +\infty.$$

Definition 3.3. Let ϕ^t be a flow in D and $d(P_1, P_2)$ the distance between points P_1 and P_2 , a point p is an ω -limit point of x to ϕ^t if there exists a sequence t_k going

to infinity as k goes to infinity such that $\lim_{k \rightarrow \infty} d(\phi^k(x), p) = 0$. The set of all of the ω -limit points is called the ω -limit set and may be denoted $\omega(p)$. The α -limit set of x is defined the same way, but with t_k going to minus infinity.

3.1 Lyapunov's indirect method

Lyapunov's indirect method considers a linearization of the system to study its local stability. In this sense, if a linearization exists, its stability implies local stability for the original nonlinear system.

An equation can be linearized by the Jacobian matrix with respect to the control variables evaluated at the equilibrium point. If the real part of all eigenvalues is negative, the equilibrium point is locally stable. If no eigenvalue is zero, and the real part of one eigenvalue is positive, the equilibrium point is locally unstable. On the other hand, if there is an eigenvalue equal to zero, the linear approximation is not sufficient to determine the equilibrium state of a system.

3.2 Limit cycle and Poincaré-Bendixson's theorem

Studying a nonlinear system, it is important to check if the ω -limit set has periodic solutions, the limit cycles. With respect to the stability, they can be classified as:

1. Asymptotically stable, if all the neighboring trajectories approach the limit cycle as time approaches infinity.
2. Unstable, if all the neighboring trajectories approach the limit cycle as time approaches minus infinity.
3. Semi-stable, if one trajectory approaches the limit cycle as time approaches infinity and the other as time approaches minus infinity.

On this matter, the following theorem may be used to determine, from the ω -limit set, if there exists a limit cycle.

Theorem 3.4. (*Poincaré-Bendixson*) *Let $D \subseteq \mathbb{R}^2$ be an open set and $f : D \rightarrow \mathbb{R}^2$ a vector field of class C^1 with a finite number of equilibrium points. If $p \in D$ is such that the positive trajectory $\phi^t(p)$ for $t \geq 0$ is contained in a compact set $K \subset D$, then:*

1. *If $\omega(p)$ contains only equilibrium points, then $\omega(p)$ consists of a single equilibrium point.*
2. *If $\omega(p)$ contains only regular points, then $\omega(p)$ consists of a single closed trajectory.*
3. *If $\omega(p)$ consists of equilibrium points and regular trajectories, then for all regular trajectory $\gamma \subset \omega(p)$ there exists equilibrium points $z_i, z_j \in \omega(p)$ such that $\alpha(\gamma) = z_i$ and $\omega(\gamma) = z_j$.*

The proof of the previous theorem can be found in [4].

3.3 Lyapunov's direct method

Lyapunov's direct method utilizes the generalization of an energy function, the Lyapunov function. Its initial approach consists of finding a candidate to this function and then analyzing its behavior.

In a physical interpretation, if the energy of a system is positive and nonincreasing, it eventually settles at a minimum energy state. Therefore the system will be stable for that domain. This is Lyapunov's stability theorem, formally stated after the definition of a Lyapunov function.

Definition 3.5. (*Lyapunov function*) Let $f : D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ be vector field in the open set $D \subset \mathbb{R}^n$, $A \subset D$ and $x_0 \in A$ an equilibrium point of f . Let $V : A \rightarrow \mathbb{R}$, V is a Lyapunov function of f at x_0 if:

1. $V(x_0) = 0, V(x) > 0$, for each $x \in A \setminus \{x_0\}$.
2. $V(\phi(t_1, x)) \geq V(\phi(t_2, x))$, for each $x \in D, t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$ and $\phi(t_1, x), \phi(t_2, x) \in A$.

Theorem 3.6. (*Lyapunov stability*) Let $t_1 < t_2, t_1, t_2 \in \mathbb{R}$ and $\phi(t_1, x), \phi(t_2, x) \in A$:

1. If $V(\phi(t_1, x)) \geq V(\phi(t_2, x)), \forall x \in D$, the equilibrium point is stable.
2. If $V(\phi(t_1, x)) > V(\phi(t_2, x)), \forall x \in D$, the equilibrium point is asymptotically stable.

The proof of this theorem can be found in [5].

4 Qualitative study

4.1 General analysis

At first glance, the van der Pol equation (2.2) may be divided into two cases, $\epsilon < 0$ and $\epsilon > 0$.

Consider $|\epsilon|$ sufficiently great, e.g., $|\epsilon| \geq 0$, and let x be a small initial point. The quadratic part of (2.2) tends to zero and the equation could be written as (4.1), the equation of a linear harmonic oscillator.

$$\ddot{x} - \epsilon \dot{x} + x = 0. \quad (4.1)$$

1. If $\epsilon < 0$, the damping term is positive. Thus the solutions of x have a decreasing behavior towards the origin. Also:

- (a) For $0 > \epsilon > -2$, $x(t) = e^{\epsilon t}(A_1 \cos(\sqrt{4 - \epsilon^2}t) + A_2 \sin(\sqrt{4 - \epsilon^2}t))$.
- (b) For $\epsilon = -2$, $x(t) = e^{\epsilon t}(A_1 + A_2t)$.
- (c) For $-2 > \epsilon$, $x(t) = e^{\epsilon t}(A_1 e^{(\sqrt{\epsilon^2 - 4})t} + A_2 e^{-(\sqrt{\epsilon^2 - 4})t})$.

2. If $\epsilon > 0$, the damping term is negative. Thus the solutions of x could, a priori, increase indefinitely.

- (a) For $0 < \epsilon < 2$, $x(t) = e^{\epsilon t}(A_1 \cos(\sqrt{4 - \epsilon^2}t) + A_2 \sin(\sqrt{4 - \epsilon^2}t))$.
- (b) For $\epsilon = 2$, $x(t) = e^{\epsilon t}(A_1 + A_2t)$.
- (c) For $\epsilon > 2$, $x(t) = e^{\epsilon t}(A_1 e^{(\sqrt{\epsilon^2 - 4})t} + A_2 e^{-(\sqrt{\epsilon^2 - 4})t})$.

Where A_1 and A_2 are constants determined by the boundary conditions.

On the other hand, x does not increase indefinitely. When $x \gg 1$, the constant term of the van der Pol equation (2.2) tends to zero, leading to equation (4.2).

$$\ddot{x} + \epsilon x^2 \dot{x} + x = 0. \quad (4.2)$$

Therefore, the damping term becomes positive. This limits the growth of x . This indicates that for $\epsilon > 0$ there exists a limit cycle asymptotically stable between both above-mentioned oscillations ($0 < x \ll 1$ and $x \gg 1$).

However, Equation (4.2) for $\epsilon < 0$ with initial conditions such that $x \gg 1$ becomes a nonlinear ODE with a negative damping term. But once there is no limiting term, the solutions of x do grow indefinitely.

Finally, this prior analysis indicates that for $\epsilon < 0$ there should be an attracting factor to the origin (a stable equilibrium point), and a repelling one, depending on the initial conditions; whilst for $\epsilon > 0$ there should exist a repulsive factor around the origin and an asymptotically stable limit cycle.

In the following sections, these statements will be validated using the theorems formulated in section 3.

4.2 Application of Lyapunov's indirect theorem

As stated in section 3.1, it is possible to use Lyapunov's indirect method to determine the local stability of an equilibrium point by the linear approximation of a nonlinear system.

Applying this idea to the van der Pol system (3.2), since the only equilibrium point is the origin, then the Jacobian matrix of (3.2) is

$$\begin{pmatrix} \frac{\partial \dot{x}}{\partial x}(0,0) & \frac{\partial \dot{x}}{\partial y}(0,0) \\ \frac{\partial \dot{y}}{\partial x}(0,0) & \frac{\partial \dot{y}}{\partial y}(0,0) \end{pmatrix} = \begin{pmatrix} \epsilon & 1 \\ -1 & 0 \end{pmatrix}.$$

The eigenvalues can be calculated by $\det(Jac - \lambda I) = 0$. So

$$\det \begin{pmatrix} \epsilon - \lambda & 1 \\ -1 & \epsilon - \lambda \end{pmatrix} = 0,$$

$$\lambda^2 - \epsilon\lambda + 1 = 0,$$

so that,

$$\lambda = \frac{\epsilon \pm \sqrt{\epsilon^2 - 4}}{2}.$$

In general, the eigenvalues are either imaginary, positive or negative numbers. For $\epsilon < 0$, the real part of all eigenvalues is negative, and thus the origin is locally stable. As for $\epsilon > 0$, the real part of all eigenvalues is positive, which yields that the origin is locally unstable. On the other hand, when $\epsilon = 0$, the van der Pol equation (2.2) reduces to a linear equation with solutions of the form $x(t) = a \cos(t + \phi)$.

4.3 Application of Poincaré-Bendixson's theorem

In order to apply the Poincaré-Bendixson's theorem, knowing the behavior of the equation and derivatives - both in polar and cartesian coordinates - is of great use.

Let $x = a \cos \theta$ and $y = a \sin \theta$, where $a = a(t)$ and $\theta = \theta(t)$. System (3.2) becomes

$$\begin{cases} \frac{d}{dt}(a \cos \theta) = a \sin \theta - \epsilon \left(\frac{a^2 \cos^2 \theta}{3} - 1 \right), \\ \frac{d}{dt}(a \sin \theta) = -a \cos \theta. \end{cases} \quad (4.3)$$

Firstly, from (4.3), treat θ constant and derive a . After derivation, multiply the first equation by $\cos \theta$, the second one by $\sin \theta$ and take the sum, leading to Equation (4.4).

$$\dot{a} = -\epsilon a \cos^2 \theta \left(\frac{a^2 \cos^2 \theta}{3} - 1 \right). \quad (4.4)$$

Now, from (4.3), let a be a constant and derive θ . After derivation, multiply the first equation by $-\sin \theta$, the second one by $\cos \theta$ and take the sum, leading to Equation (4.5).

$$\dot{\theta} = -1 + \epsilon \sin \theta \cos \theta \left(\frac{a^2 \cos^2 \theta}{3} - 1 \right) \quad (4.5)$$

Writing (4.4) and (4.5) as a system, (3.2) becomes (4.6) in polar coordinates.

$$\begin{cases} \dot{a} = -\epsilon a \cos^2 \theta \left(\frac{a^2 \cos^2 \theta}{3} - 1 \right), \\ \dot{\theta} = -1 + \epsilon \sin \theta \cos \theta \left(\frac{a^2 \cos^2 \theta}{3} - 1 \right). \end{cases} \quad (4.6)$$

Equation (4.4) may be studied for $\epsilon < 0$ and for $\epsilon > 0$. Since $a > 0$, $0 \leq \cos^2 \theta \leq 1$ and $x^2 = a^2 \cos^2 \theta$, the sign of \dot{a} is determined only by the sign of ϵ and $1 - \frac{x^2}{3}$. Therefore,

1. For $\epsilon < 0$,
 - (a) $-\sqrt{3} < x < 0$ or $0 < x < \sqrt{3} \rightarrow \dot{a} < 0$,
 - (b) $x = \pm\sqrt{3}$ or $x = 0 \rightarrow \dot{a} = 0$,
 - (c) $|x| > \sqrt{3} \rightarrow \dot{a} > 0$.

This behavior is illustrated in Fig. 2.

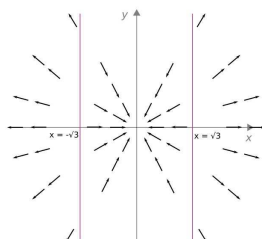


Figure 2: Radius behavior for $\epsilon < 0$.

All trajectories in $(-\sqrt{3}, \sqrt{3}) \times \mathbb{R}$ have a decreasing radius a , except at the origin, where it is constant. Outside this domain, the radius increases indefinitely for all trajectories. At both limits, the radius is constant.

Inside this domain, the trajectories accumulate on the origin, which makes the ω -limit set to have an equilibrium point.

2. For $\epsilon > 0$,
 - (a) $|x| > \sqrt{3} \rightarrow \dot{a} < 0$,
 - (b) $x = \pm\sqrt{3}$ or $x = 0 \rightarrow \dot{a} = 0$,
 - (c) $-\sqrt{3} < x < 0$ or $0 < x < \sqrt{3} \rightarrow \dot{a} > 0$.

This behavior is shown in Fig. 3.

All trajectories in $(-\sqrt{3}, \sqrt{3}) \times \mathbb{R}$ have an increasing radius a , except at the origin, where it is constant. Outside this domain, the radius decreases. At both limits, the radius is also constant.

In this sense, no trajectory accumulates on the origin and consequently the ω -limit set does not have equilibrium points.

So as to set a limit on y , the derivatives in cartesian coordinates might be analyzed. From (3.2), let $F(x) = \epsilon(\frac{x^3}{3} - x)$. Then,

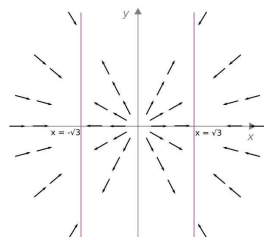


Figure 3: Radius behavior for $\epsilon > 0$.

1. $\dot{x} < 0 \iff y < F(x)$,
 $\dot{x} = 0 \iff y = F(x)$,
 $\dot{x} > 0 \iff y > F(x)$.
2. $\dot{y} < 0 \iff x > 0$,
 $\dot{y} = 0 \iff x = 0$,
 $\dot{y} > 0 \iff x < 0$.

At $x = 0$, y is constant. Also x is constant at $y = F(x)$. At $x > 0$ (respectively, $x < 0$), y decreases (respectively, increases). At $y > F(x)$ (respectively, $y < F(x)$), x increases (respectively, decreases). The greater the distance $|x - 0|$ (respectively, $|y - F(x)|$), the greater the module $|\dot{y}|$ (respectively, $|\dot{x}|$). These relations are captured in Fig. 4.

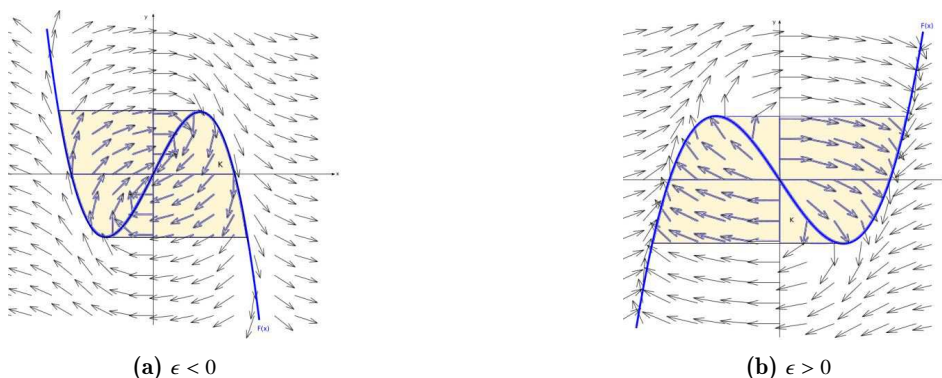


Figure 4: Cartesian derivatives behavior (normalized).

Let K the compact set bounded by $y = \pm \frac{2}{3}\epsilon$ and $F(x)$. Then, for $\epsilon < 0$, all trajectories with initial value in K tend to the origin, i.e., the equilibrium point is an ω -limit point. This implies, using Poincaré-Bendixson's theorem, that the ω -limit set consists of a single equilibrium point.

On the other hand, for $\epsilon > 0$, all trajectories with an initial value in $K \setminus \{(0, 0)\}$ go outwards the origin, which means that the ω -limit set contains only regular points. Therefore, $\omega(p)$ consists of a single closed trajectory, the limit cycle.

The following lemmas are consequences of the derivative analysis and prove the Poincaré-Bendixson's theorem for the van der Pol equation for $\epsilon > 0$.

Lemma 4.1. *All trajectories $\gamma = (x, y)$ of F starting at a point $A = (0, a)$ where $a > 0$ (respectively, $a < 0$) of sufficiently large module intersect back the vertical axis at a point $D = (0, d)$ where $d < 0$ (respectively, $d > 0$).*

Lemma 4.2. *Point D satisfies $d > -a$.*

So, for every trajectory with initial points in a compact K , the van der Pol system is limited and ω -limit set is a closed trajectory. Fig. 5 illustrates this behavior.

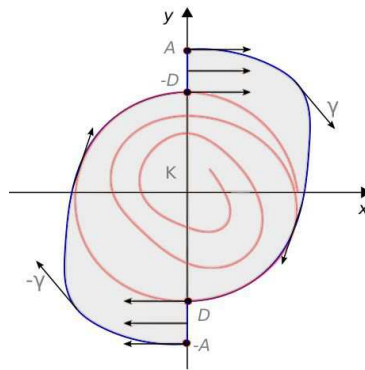


Figure 5: Trajectories of the van der Pol System

4.4 Application of Lyapunov's direct method

The method of the variable gradient [5] will be used in the current section as a method of finding a Lyapunov function.

Assume $\nabla V(x, y) = g(x, y) = (g_1, g_2) = (c_1x + c_2y, c_3x + c_4y)$ and consider the symmetry $\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}$, i.e., $\frac{\partial g_1}{\partial y} = \frac{\partial g_2}{\partial x}$.

Take c_1, c_2, c_3 and c_4 as constants in x and in y . Thus $\frac{\partial g_1}{\partial y} = c_2; \frac{\partial g_2}{\partial x} = c_3$, and then $c_1 = c_3 = k$ and $g(x, y) = (c_1x + ky, kx + c_4y)$.

Now, considering $k = 0$, we have that $g(x, y) = (c_1x, c_4y)$.

As $\dot{V}(x, y) = \nabla V(x, y) \cdot f(x, y) = g(x, y) \cdot f(x, y)$, where, for the van der Pol equation written as a Liénard system (3.2), $f(x, y) = (y - \epsilon(\frac{x^3}{3} - x), -x)$.

Thus,

$$\dot{V}(x, y) = c_1xy - \epsilon c_1\left(\frac{x^4}{3} - x^2\right) - c_4xy. \tag{4.7}$$

Integrating (4.7), $V(x, y) = \int_0^x g_1(s_1, 0) ds_1 + \int_0^y g_1(x, s_2) ds_2$.

The Lyapunov function candidate is

$$V(x, y) = c_1 \frac{x^2}{2} + c_4 \frac{y^2}{2}. \tag{4.8}$$

Condition 1 of the definition of a Lyapunov function is easily satisfied letting $c_1, c_4 > 0$, since $V(0, 0) = 0$ and $V(x, y) > 0; \forall (x, y) \in \mathbb{R}^2 \setminus (0, 0)$.

Letting $c_1 = c_4$, we have that $\dot{V}(x, y) = -\epsilon c_1 x^2 \left(\frac{x^2}{3} - 1\right)$.

For $\epsilon < 0$, $\dot{V} < 0$ when $-\sqrt{3} < x < 0$ or $0 < x < \sqrt{3}$. Furthermore, as discussed in section 4.3, System (3.2) may be bounded by $y = \pm \frac{2}{3}\epsilon$ in the y direction. Since the origin is an equilibrium point and the Lyapunov function decreases, every trajectory in a domain D bounded by $-\sqrt{3} < x < \sqrt{3}$ and $-\frac{2}{3}\epsilon < y < \frac{2}{3}\epsilon$, except $x = 0$, tend to the origin. Thus, the origin is an asymptotically stable point.

As for $\epsilon > 0$, $\dot{V} < 0$ when $|x| > \sqrt{3}$. Once more, y may be bounded by $y = \pm \frac{2}{3}\epsilon$. Now, for a set bounded by the domain D defined above, the Lyapunov function increases. On the other hand, it decreases outside this domain. Since there is a limit cycle, this behavior shows that the regular trajectories are asymptotically stable.

5 Approximations to the van der Pol equation

5.1 Weak nonlinear oscillations

For small nonlinearities ($|\epsilon| \ll 1$), the solutions for the van der Pol system (3.2) is approximately $x = a \cos(t + \phi)$, $y = -a \sin(t + \phi)$.

In polar coordinates, the representative point $(a(t), \theta(t))$ moves around the origin with an angular speed of $\dot{\theta} \approx -1$ and period $T \approx 2\pi$.

From (4.6), the van der Pol equation (2.2) may be written as

$$\frac{da}{d\theta} = \frac{\dot{a}}{\dot{\theta}} = \frac{\epsilon a \cos^2 \theta \left(\frac{a^2 \cos^2 \theta}{3} - 1\right)}{\epsilon \sin \theta \cos \theta \left(\frac{a^2 \cos^2 \theta}{3} - 1\right) - 1}. \tag{5.1}$$

Multiplying both numerator and denominator on equation (5.1) by $\epsilon \sin \theta \cos \theta \left(\frac{a^2 \cos^2 \theta}{3} - 1\right) + 1$,

$$\frac{da}{d\theta} = \epsilon a \cos^2 \theta \left(\frac{a^2 \cos^2 \theta}{3} - 1\right) + O(\epsilon^2). \tag{5.2}$$

The right-hand side of Equation (5.2) could be expressed as a Fourier series as

$$\frac{da}{d\theta} = p_0 + \sum_{n=1}^{\infty} (p_n \cos n\theta + q_n \sin n\theta) + O(\epsilon^2). \quad (5.3)$$

Integrating the right-hand side of Equation (5.3) over the range $\theta_0 \leq \theta \leq \theta_0 + 2\pi$, the terms under the summation sign is zero and (5.3) is written, with an error of order ϵ^2 , as

$$\frac{da}{d\theta} = p_0 = \frac{1}{2\pi} \int_0^{2\pi} \epsilon a \cos^2 \theta \left(\frac{a^2 \cos^2 \theta}{3} - 1 \right) d\theta. \quad (5.4)$$

Since $\dot{\theta} \approx -1$ and integrating the right-hand side of Equation (5.4),

$$\frac{da}{dt} \approx -\frac{\epsilon a}{2} \left(\frac{a^2}{4} - 1 \right). \quad (5.5)$$

Integrating Equation (5.5) with respect to a and t and writing a in terms of t ,

$$a(t) = \frac{2}{\sqrt{1 - e^{-\epsilon(t+C)}}}. \quad (5.6)$$

Where C is the constant of integration.

Substituting Equation (5.6) on the general solution for small ϵ ($x = a \cos t + \phi$),

$$x(t) = \frac{2 \cos(t + \phi)}{\sqrt{1 - e^{-\epsilon(t+C)}}}. \quad (5.7)$$

Where ϕ is a phase constant to set the initial point value. Equation (5.7) is, therefore, the approximate solution for $|\epsilon| \ll 1$.

5.2 Relaxation oscillations

On the other hand, for $|\epsilon| \gg 1$, Equation (5.7) does not hold, but it is still possible to study the system's behavior over time.

Let $y_1 = \frac{y}{\epsilon}$ and $G(x) = \frac{x^3}{3} - x$, system (3.2) then becomes

$$\begin{cases} \dot{x} = \epsilon(y_1 - G(x)), \\ \dot{y}_1 = -\frac{x}{\epsilon}. \end{cases}$$

Let $y_1 - G(x) \approx O(1)$. So $\dot{x} \approx O(\epsilon)$ and $y_1 \approx O(\epsilon^{-1})$. When $\epsilon \rightarrow \infty$, $\dot{x} \rightarrow \infty$ and $y_1 \rightarrow 0$. At $y_1 = G(x)$, x does not increase over time, but decreases at $y_1 < G(x)$ and increases at $y_1 > G(x)$.

What happens to the trajectory with an initial point P_0 sufficiently away from $G(x)$ is a rapid and almost horizontal travel until it reaches $G(x)$. At that moment, it moves slowly in both x and y_1 directions. Subsequently, there is another rapid horizontal movement followed by a further slow travel in both directions.

The slow trajectories are displaced from $G(x)$ by an error of order ϵ^{-1} , whereas the rapid trajectories are displaced from $y_1 = G(x_{max})$ by an error also of order ϵ^{-1} .

Fig. 6 expresses graphically this behavior.

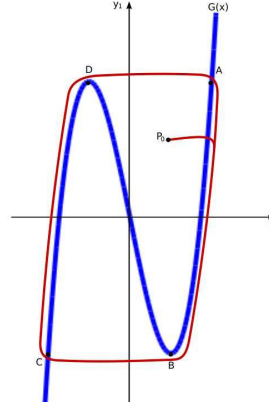


Figure 6: Trajectory for the van der Pol system when $\epsilon \gg 1$ and with initial point P_0 .

In $x - y_1$ coordinates, points A , B , C and D are easily calculated, resulting in $A = (2, \frac{2}{3})$, $B = (1, -\frac{2}{3})$, $C = (-2, -\frac{2}{3})$ and $D = (-1, \frac{2}{3})$. A to B and C to D are slow trajectories, whilst D to A and B to C are fast trajectories. The slow buildup followed by the rapid change in system’s state characterizes these types of oscillations as relaxation oscillations.

It is possible, using the idea of a relaxation oscillation, to estimate the period T for the van der Pol oscillator for $\epsilon \gg 1$.

$$\Delta t_{DA} = \Delta t_{BC} \approx O(\epsilon^{-1}),$$

$$\Delta t_{AB} = \Delta t_{CD} \approx \frac{T}{2} = \int_{t_A}^{t_B} dt.$$

As $y_1 \approx G(x)$ for the slow trajectories,

$$\frac{dy_1}{dt} \approx \frac{dG}{dx} \cdot \frac{dx}{dt} = (x^2 - 1) \frac{dx}{dt}.$$

Moreover, $\frac{dy_1}{dt} = \frac{-x}{\epsilon}$ and writing dt as function of dx ,

$$dt = \epsilon(x - 1/x)dx.$$

Therefore the period for the oscillations may be calculated by

$$\frac{T}{2} = - \int_{x_A}^{x_B} \epsilon(x - 1/x) dx.$$

Since $x_A = 2$ and $x_B = 1$, with an error of order ϵ^{-1} ,

$$T = \epsilon(3 - 2 \ln(2)).$$

Fig. 7 sketches the $x \times t$ behavior for the van der Pol oscillator for $\epsilon \gg 1$.

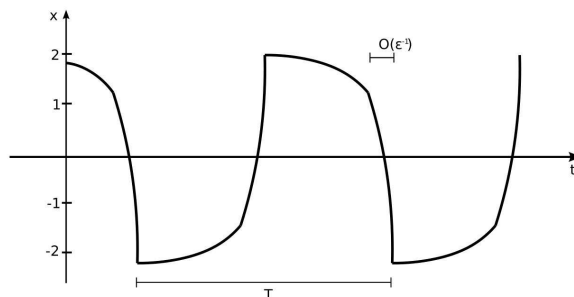


Figure 7: $x \times t$ sketch for the van der Pol oscillator for $\epsilon \gg 1$.

6 Numerical study

In order to illustrate graphically the analyses made thus far, the van der Pol system is studied using numerical integration obtained using GNU Octave and Python. The main codes are presented in appendices A and B.

6.1 Phase portrait

Figs. 8 to 14 shows the normalized field vector and the trajectories of the van der Pol equation for different values of ϵ .

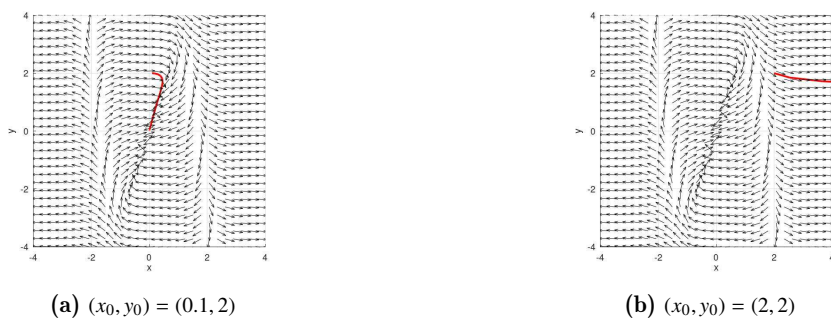


Figure 8: Vector field and the trajectories of the van der Pol equation for $\epsilon = -4$ for different initial points.

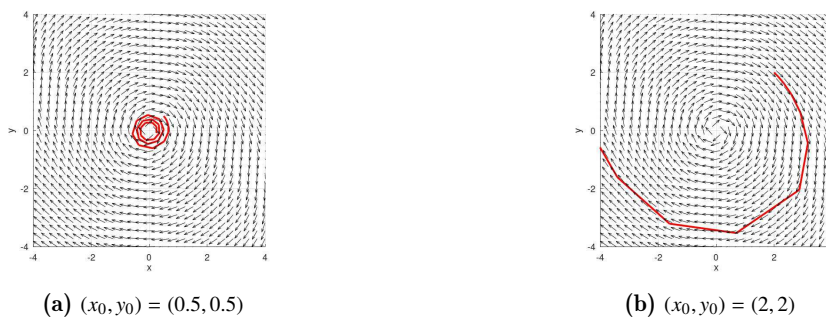


Figure 9: Vector field and the trajectories of the van der Pol equation for $\epsilon = -0.1$ for different initial points.

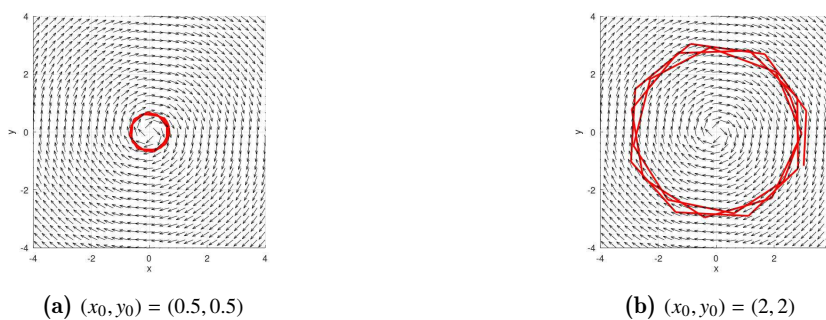


Figure 10: Vector field and the trajectories of the van der Pol equation for $\epsilon = -0.01$ for different initial points.

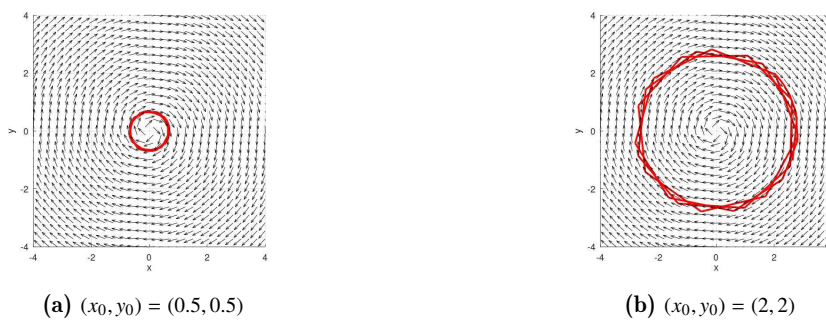


Figure 11: Vector field and the trajectories of the van der Pol equation for $\epsilon = 0$ for different initial points.

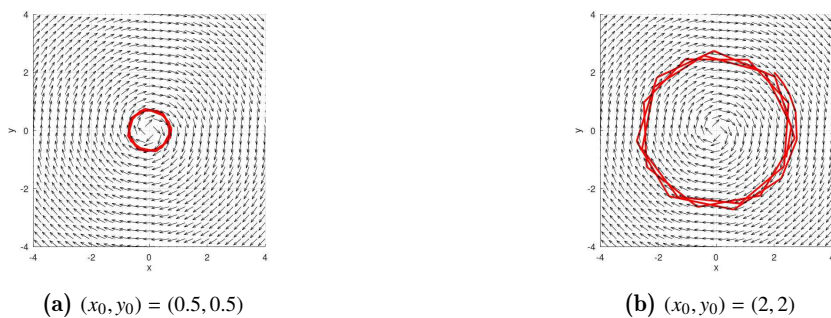


Figure 12: Vector field and the trajectories of the van der Pol equation for $\epsilon = 0.01$ for different initial points.

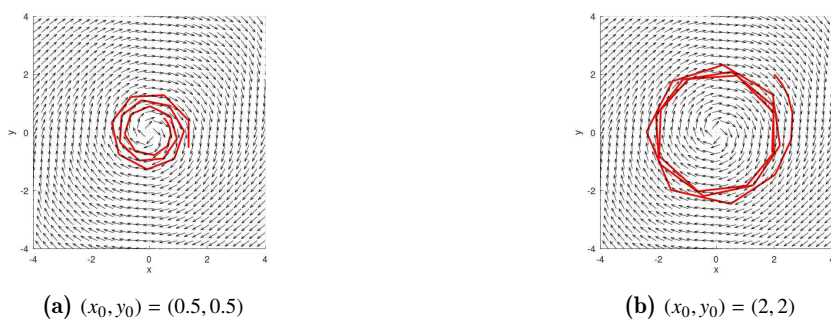


Figure 13: Vector field and the trajectories of the van der Pol equation for $\epsilon = 0.1$ for different initial points.

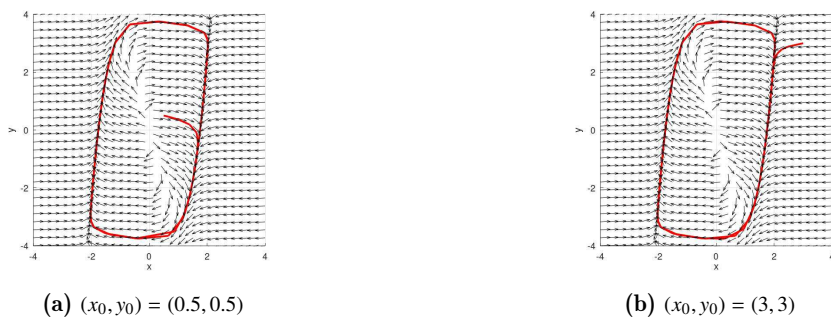


Figure 14: Vector field and the trajectories of the van der Pol equation for $\epsilon = 4$ for different initial points.

The relationship of ϵ and the system's linearity is easily noticed. Figs. 11a and 11b are the trajectories of a linear system, since $\epsilon = 0$. These solutions are sinusoidal, which are characterized by a circular path. As stated in section 5.1, for small values of ϵ , i.e., $|\epsilon| \ll 1$, the oscillations are weakly non-linear and the trajectories tend to a circular path. This behavior is captured in Figs. 9a, 13a and 13b.

However, Fig. 8b shows that, for initial conditions sufficiently away from the origin, the trajectories do not converge to any point. For $\epsilon \ll -1$, the non-linear term is significant, and, for a given domain, all trajectories tend to the origin, i.e., the origin is an asymptotically stable point (Fig. 9a), whilst outside this domain, the trajectories do not accumulate on any point (Fig. 8b). This behavior illustrates the results yielded by applying Poincaré-Bendixson's theorem (section 4.3) and also analyzing the basin of attraction using a Lyapunov function (section 4.4).

Finally, Fig.14 shows the phase portrait for $\epsilon \gg 1$. For initial points close to the origin, the trajectories move away from this point. Also, all trajectories tend to the limit cycle. These behaviors illustrate that the origin is an unstable equilibrium point and that the limit cycle is asymptotically stable, as stated in section 3.

6.2 Temporal behavior

Fig. 15 shows the temporal behavior of the van der Pol equation for different positive values of ϵ .

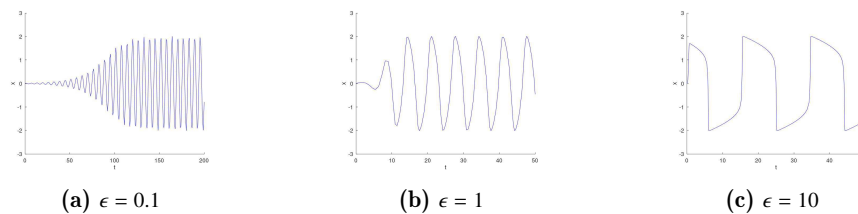


Figure 15: Temporal behavior of the van der Pol equation with initial value (0.01, 0.01).

As discussed in Section 5.1, for small values of $|\epsilon|$ (Fig. 15a), the sinusoidal function is modulated in amplitude by $a(t)$ in Equation (5.6). For considerable values of ϵ , i.e., $\epsilon \gg 1$, it is noticeable the relaxation behavior of the oscillation, as analyzed in Section 5.2. The greater the value of ϵ , the fastest the oscillation from a positive to a negative peak. For smaller ϵ , but not small enough to have a sinusoidal behavior (e.g., Fig. 15b), the time between a positive and a negative peak is greater.

6.3 Basin of attraction

After finding a Lyapunov function for the van der Pol system, Equation (4.8), a basin of attraction may be found. The case studied here will be for $\epsilon < 0$, but it is equally applicable for positive values of ϵ . As in Section 3.3, the basin of attraction for $\epsilon < 0$ is $-\sqrt{3} < x < \sqrt{3}$ and $\frac{2}{3}\epsilon < y < \frac{2}{3}\epsilon$.

An algorithm may be implemented in order to find an empirical basin of attraction and to compare with the one obtained using the Lyapunov function. Initially, a region is defined, creating a mesh. Each point of the mesh is then used as an initial point to integrate numerically the van der Pol equation. As stated in Poincaré-Bendixson's theorem, if the ω -limit set has an equilibrium point, it consists of only one equilibrium point. In this matter, the ω -limit set for each initial point is compared to the origin (equilibrium point). If they are the same point, then the initial point is in the basin of attraction. After each point in the mesh is tested, the points in the basin of attraction may be plotted and compared to the basin of attraction from the Lyapunov function, as Fig. 16 shows.

The higher the mesh, the more accurate is the empirical basin of attraction, requiring also more computational process. The mesh used in this work consists of 2500 elements.

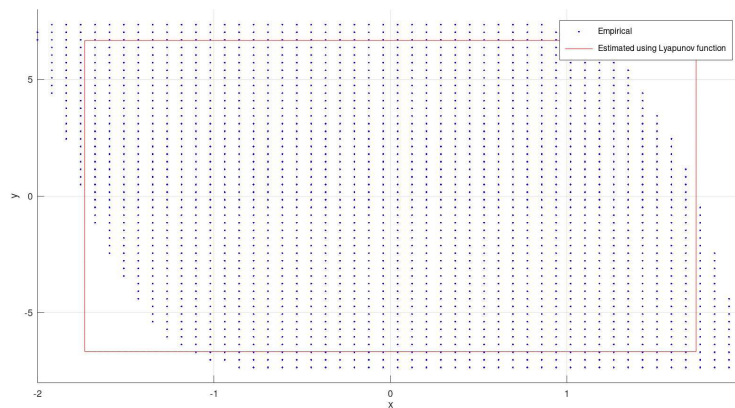


Figure 16: Basin of attraction for the van der Pol equation for $\epsilon = -10$.

7 Conclusion

Throughout this paper, the van der Pol equation has been studied using qualitative theory and also numerical integration. The equation was deduced by analyzing an electrical circuit as a physical motivator. Then, Lyapunov's and Poincaré-Bendixson's theorems were stated, followed by a general investigation of the van

der Pol equation and application of the theorems. Also, approximations to the van der Pol system have been made in order to identify the temporal behavior. Finally, computation has been applied to a more graphical examination of the proprieties and characteristics yielded in this paper.

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Appendix

A Octave code

```
e = 0;
xInicial = 0;
yInicial = 0;

##funcao para integrar a eq van der pol
function [t, y] = EVDP(e, xInicial, yInicial)

    ##atribui a eq de van der pol
    ##evdp = @(t,y) [y(2); (e*(1-y(1)^2)*y(2))-y(1)];
    evdp = @(t,y) [y(2)-e*((y(1)^3)/3-y(1)); -y(1)];
    ##y(1) - x
    ##y(2) - y

    ##resolve a edo
    tinicial = 0;
    tfinal = 100;
    [t,y] = ode45 (evdp, [tinicial, tfinal],
    [xInicial, yInicial]);
    ##t - matriz coluna com os valores do tempo
    ##y - matriz com duas colunas. A primeira sao
    ##os valores de x e a segunda, de y

endfunction
```

This is the core code used to integrate numerically the van der Pol equation using GNU Octave. With this function, all graphics used in this work may be plotted.

B Python code

```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.animation as animation
import math
from scipy.integrate import odeint
from scipy.integrate import solve_ivp

# gera a linha do tempo a ser analisada
tInicial = 0
tFinal = 200
nT = 200
t = np.linspace(tInicial, tFinal, nT)

def EVDP (e, xInicial, yInicial):
    # definicao da constante epsilon e do ponto inicial
    e = e
    xInicial = xInicial
    yInicial = yInicial

    # definicao do sistema da eq vdp
    def vdP_EDO(t, z, e):
        x, y = z
        return [y-e*(x**3/3-x), -x]

    # solucoes da equacao de van der pol (x,y) =
    (sol.y[0], sol.y[1])
    # e t = sol.t
    sol = solve_ivp(vdP_EDO, [tInicial, tFinal], \
    [xInicial, yInicial], args=(e,), t_eval = t)

    return sol
```

This is the core code used to integrate numerically the van der Pol equation using Python. With this function, all graphics used in this work may be plotted.

References

- [1] G. Austin, W. Hayward, C. Tsai, A. Kuykendall, *Parkinsonian tremor: some aspects of an experimental model and its solution*. *Confinia Neurologica* **26**(1965), no. 3-5, 389–403. DOI: [10.1159/000104056](https://doi.org/10.1159/000104056)

- [2] M. L. Cartwright, *Balthazar van der Pol*. J. London Math. Soc. **35**(1960), 367–376. DOI: [10.1112/jlms/s1-35.3.367](https://doi.org/10.1112/jlms/s1-35.3.367)
- [3] A. Fleitas, J. A. Méndez-Bermúdez, J. E. Nápoles Valdés, J. M. Sigarreta Almira, *On fractional Liénard-type systems*. Rev. Mexicana Fís. **65**(2019), no. 6, 618–625. DOI: [10.31349/revmexfis.65.618](https://doi.org/10.31349/revmexfis.65.618)
- [4] M. W. Hirsch, S. Smale, R. L. Devaney, *Differential equations, dynamical systems, and an introduction to chaos*. Third. Elsevier/Academic Press, Amsterdam, 2013, xiv+418. DOI: [10.1016/B978-0-12-382010-5.00001-4](https://doi.org/10.1016/B978-0-12-382010-5.00001-4)
- [5] H. J. Marquez, *Nonlinear control systems: analysis and design*. Vol. 161. John Wiley, NJ, 2003.
- [6] J. E. Nápoles Valdes, *A century of qualitative theory of ordinary differential equations*. Lect. Mat. **25**(2004), no. 1, 59–111.
- [7] B. van der Pol, *LXXXVIII. On relaxation-oscillations*. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **2**(1926), no. 11, 978–992. DOI: [10.1080/14786442608564127](https://doi.org/10.1080/14786442608564127)
- [8] B. van der Pol, J. van der Mark, *LXXII. The heartbeat considered as a relaxation oscillation, and an electrical model of the heart*. The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science **6**(1928), no. 38, 763–775. DOI: [10.1080/14786441108564652](https://doi.org/10.1080/14786441108564652)
- [9] J. N. Valdés, *Differential equations and contemporaneity*. Revista Brasileira de História da Matemática **7**(2007), no. 14, 213–232.