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A ROBUST STABILITY CRITERION IN THE HEAT EQUATION WITH A CONFORMABLE FRACTIONAL DERIVATIVE DEFINED ON A RADIALLY SYMMETRIC SPHERE

UN CRITERIO DE ESTABILIDAD ROBUSTA EN LA ECUACIÓN DE CALOR CON UNA DERIVADA FRACCIONARIA CONFORMABLE GENERAL DEFINIDA SOBRE UNA ESFERA RADIALMENTE SIMÉTRICA

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Abstract

In this paper, we present a robust stability criterion for a heat equation with axial symmetry and with a general time-conformable fractional derivative defined on a sphere. The heat equation is assumed to have a heat source that is represented as a Fourier series with coefficients described by bounded, piecewise continuous functions. The robust stability criterion establishes conditions to guarantee that the solution of the heat equation, along with its partial derivative with respect to the radial axis and its general time-conformable fractional derivative, remains bounded by a predetermined value. The robust stability criterion is obtained by extending the concept of stability under constant-acting perturbations applied to systems of ordinary differential equations. The results are illustrated numerically.

Keywords: general conformable fractional derivative; heat equation; Fourier series; robust stability.

Resumen

En este artículo, presentamos un criterio de estabilidad robusta para una ecuación de calor con simetría axial y con derivada fraccionaria general conformable en el tiempo definida en una esfera. Se supone que la ecuación de calor admite una fuente de calor externa que se representa como una serie de Fourier con coeficientes descritos por funciones continuas a trozos y acotadas. El criterio de estabilidad robusta establece condiciones para garantizar que la solución de la ecuación de calor, así como su derivada parcial con respecto al eje radial y su derivada fraccionaria conformable general en el tiempo, son funciones acotadas por un valor constante prefijado. El criterio de estabilidad robusta se obtiene por una extensión del concepto de estabilidad bajo perturbaciones de acción constante que se aplica a sistemas de ecuaciones diferenciales ordinarias. Los resultados se ilustran numéricamente.

Palabras clave: derivada fraccionaria conformable general; ecuación de calor; series de Fourier; estabilidad robusta.

Mathematics Subject Classification: Primary: 93D09, 6A33. Secondary: 93B03, 34A26, 42A16.

1. Introduction

The term fractional derivative has originated in 1695 when L'Hôpital asked Leibniz in a letter about the meaning of the derivative $\frac{d^n}{dx^n}$ when $n = \frac{1}{2}$; see, for example, [14]. This generated the development of fractional calculus, leading to the proposal of various definitions for fractional derivatives and integrals. A review of some of these definitions can be found in [15, 19].

In recent years, the study of fractional calculus has been the focus of attention of different researchers due to the introduction of a new concept of local fractional derivative due to Khalil, Al Horani, Yousef and Sababheh [12]: *the conformable* *fractional derivative*. The conformable fractional derivative satisfies some properties from elementary calculus [1]. These properties have made possible the study of problems related to the existence, uniqueness, boundedness and stability of solutions to some conformable fractional differential equations [7, 17, 24].

Furthermore, research on the existence of solutions to some conformable fractional partial differential equations has been considered. In [11], the authors examine the existence of solutions of the conformable fractional heat equation using Fourier series, while in [4] the same method is used to find the solution of the conformable fractional heat equation defined in a cylinder. Regarding the conformable fractional heat equation with axial symmetry defined on a plate, the existence of its solutions is studied in [5]. A different method for finding solutions of the conformable fractional heat equation is discussed in [8] using the Fourier transform.

The concept of the conformable fractional derivative introduced by Khalil and his collaborators has generated different extensions in recent years, some particular cases are addressed in [2, 3, 10, 16, 21, 22]. In this paper, we adopt the concept of general conformable fractional derivative introduced in [22]. Using this concept, we propose a method to establish a robust stability criterion for the heat equation with axial symmetry and a general time-conformable fractional derivative defined on a sphere assuming heat sources belonging to a prefixed set of functions. The importance of robust stability lies in its applications to control theory; see, for example, [6] The robust stability criterion is determined by extending the concept of stability under constant-acting perturbations introduced by Duboshin and Malkin for systems of ordinary differential equations; see, for example, [9]. This criterion ensures that the solution of the heat equation, its general time-conformable fractional derivative and its first partial derivative with respect to the radial axis do not exceed a predetermined value.

We organize the paper as follows. Section 2 presents the main properties of the general conformable fractional derivative. Section 3 outlines the statement of the problem under study. The justification for the existence of solutions that can be expressed as Fourier series is formulated in Section 4. The robust stability criterion is presented in Section 5. Finally, the conclusions of the paper are given in Section 7.

2. Preliminaries

This section summarizes the properties of the general conformable fractional derivative introduced in [22].

Definition 1. The constant function $\varphi(t,\alpha) \equiv 1$ and the continuous functions $\varphi\colon [0,\infty)\times(0,1] \to \mathbb{R}$ that satisfy

- $\varphi(t,1) = 1$ for all $t > 0$,
- $\varphi(t,\alpha) \neq 0$ for all $(t,\alpha) \in [0,\infty)\times(0,1]$,
- $\varphi(\,\cdot\,,\alpha) \neq \varphi(\,\cdot\,,\beta)$, where $\alpha \neq \beta$ and $\alpha,\beta \in (0,1]$,

are called *conformable fractional functions*.

Definition 2. Let $p\colon [0,\infty) \to \mathbb{R}$ and let φ be a conformable fractional function. The *general conformable fractional derivative* of p of order $\alpha \in (0,1]$ at $t > 0$ is defined as

$$D^{\alpha,\varphi}p(t) = \lim_{\epsilon\to 0}\frac{p(t+\epsilon\varphi(t,\alpha)) - p(t)}{\epsilon},$$

provided that the limit exists. If the limit $\lim_{t\to 0^+} D^{\alpha,\varphi}p(t)$ exists, we denote its value by $D^{\alpha,\varphi}p(0)$. That is,

$$D^{\alpha,\varphi}p(0) = \lim_{t\to 0^+} D^{\alpha,\varphi}p(t).$$

In what follows, we use the expression "the function p is α-differentiable" when the function p has a general conformable fractional derivative of order $\alpha \in (0,1]$. It is known that there are functions that are α-differentiable at a point t_0 but are not differentiable at that point; see for example [1], where the author shows some examples that correspond to the choice $\varphi(t,\alpha) = t^{1-\alpha}$ as a particular case.

The particular choice of certain functions φ leads to some particular cases of conformable fractional derivatives known in the literature; see [22]. Some particular cases are the following:

- If $\varphi(t,\alpha) \equiv 1$, then $D^{\alpha,\varphi}$ coincides with the classical derivative.
- If $\varphi(t,\alpha) = t^{1-\alpha}$, then $D^{\alpha,\varphi}$ reduces to the conformable fractional derivative introduced in [12].
- If $\varphi(t,\alpha) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha+1)}t^{1-\alpha}$ with $\beta > -1$, then $D^{\alpha,\varphi}$ reduces to the generalized fractional derivative considered in [2].

The following properties of the general conformable fractional derivative are satisfied:

Lemma 1 ([22]). If $p\colon [0,\infty) \to \mathbb{R}$ is α-differentiable at $t > 0$, $\alpha \in (0,1]$, then p is continuous in t.

Lemma 2 ([22])**.** Let $\alpha \in (0,1]$ and let p and q be functions α-differentiable at $t > 0$. Then

- $D^{\alpha,\varphi}(k_1 p + k_2 q)(t) = k_1 D^{\alpha,\varphi} p(t) + k_2 D^{\alpha,\varphi} q(t)$ for all $k_1, k_2 \in \mathbb{R}$,
- $D^{\alpha,\varphi}(p \cdot q)(t) = p(t) D^{\alpha,\varphi} q(t) + q(t) D^{\alpha,\varphi} p(t)$,
- $D^{\alpha,\varphi}\left(\dfrac{p}{q}\right)(t) = \dfrac{q(t) D^{\alpha,\varphi} p(t) - p(t) D^{\alpha,\varphi} q(t)}{q(t)^2}$.
- If p is differentiable at t, then
$$D^{\alpha,\varphi} p(t) = \varphi(t,\alpha)\frac{\mathrm{d}p}{\mathrm{d}t}(t).$$

In particular, if $l(t) = c$ with $c \in \mathbb{R}$, $p(t) = t^m$ with $m \in \mathbb{R}$ and $q(t) = e^{\lambda t}$ with $\lambda \in \mathbb{R}$, then [22]

$$D^{\alpha,\varphi} l(t) = 0, \quad D^{\alpha,\varphi} p(t) = m t^{m-1} \varphi(t,\alpha), \quad D^{\alpha,\varphi} q(t) = \lambda e^{\lambda t} \varphi(t,\alpha).$$

Definition 3 ([22])**.** Let $p\colon [0,\infty) \to \mathbb{R}$. The *general conformable fractional integral* is defined as

$$I^{\alpha,\varphi} p(t) = \int_0^t \frac{p(s)}{\varphi(s,\alpha)}\,\mathrm{d}s,$$

provided that integral exists for all $t > 0$.

Once again, in what follows, we use the expression "the function p is α-integrable" when the function p has a general conformable fractional integral of order $\alpha \in (0,1]$.

The following relations between the operators $D^{\alpha,\varphi}$ and $I^{\alpha,\varphi}$ hold.

Lemma 3 ([22])**.** Let $\alpha \in (0,1]$. If $p\colon [0,\infty) \to \mathbb{R}$ is continuous and $I^{\alpha,\varphi} p(t)$ exists for $t > 0$, then

$$D^{\alpha,\varphi} I^{\alpha,\varphi} p(t) = p(t).$$

Lemma 4 ([22])**.** Let $\alpha \in (0,1]$. If $p\colon [0,\infty) \to \mathbb{R}$ is differentiable, then

$$I^{\alpha,\varphi} D^{\alpha,\varphi} p(t) = p(t) - p(0).$$

We recall that a function $p\colon [0,\infty) \to \mathbb{R}$ is strictly increasing if, for any two numbers a and b such that $a < b$, we have $p(a) < p(b)$. The function p is strictly decreasing if, for any two numbers a and b such that $a < b$, we have $p(a) > p(b)$. With the help of the mean value theorem for the general conformable fractional derivative presented in [22], the following result is obtained.

Lemma 5. Suppose that

$$h(t) = \int_0^t \frac{\mathrm{d}s}{\varphi(s,\alpha)},$$

is strictly increasing on $(0,\infty)$. Let $p\colon [0,\infty) \to \mathbb{R}$ be continuous, such that $D^{\alpha,\varphi}p(t)$ exists over $(0,\infty)$. If $D^{\alpha,\varphi}p(t) > 0$ (respectively $D^{\alpha,\varphi}p(t) < 0$) for all $t > 0$, then p is strictly increasing (respectively strictly decreasing).

Proof. We first observe that $D^{\alpha,\varphi}h(t) = 1$. Let $t \in (a,b)$ and define

$$q(t) = p(t) - p(a) - \frac{p(t) - p(a)}{h(b) - h(a)}(h(t) - h(a)).$$

According to Rolle's theorem, there exists $c \in (a,b)$ such that $D^{\alpha,\varphi}(q(c)) = 0$, so $D^{\alpha,\varphi}p(c) = \frac{p(b)-p(a)}{h(b)-h(a)}$; see [22]. The conclusion follows from this expression. □

The concept of general time-conformable fractional derivative is introduced according to Definition 2 as follows:

Definition 4. Let $A \subset \mathbb{R}^m$ be a non-empty set and let $p\colon [0,\infty) \times A \to \mathbb{R}$ be a function of $m+1$ variables: $(t,\boldsymbol{x}) = (t,x_1,\dots,x_m)$. The *general time-conformable fractional derivative* of order $\alpha \in (0,1]$ of p at $t > 0$ is defined as

$$D_t^{\alpha,\varphi}p(t,\boldsymbol{x}) = \lim_{\epsilon\to 0}\frac{p(t+\epsilon\varphi(t,\alpha),\boldsymbol{x}) - p(t,\boldsymbol{x})}{\epsilon},$$

provided that the limit exists. If the limit $\lim_{t\to 0^+} D_t^{\alpha,\varphi}p(t,\boldsymbol{x})$ exists, we denote its value by $D_t^{\alpha,\varphi}p(0,\boldsymbol{x})$. That is,

$$D_t^{\alpha,\varphi}p(0,\boldsymbol{x}) = \lim_{t\to 0^+} D_t^{\alpha,\varphi}p(t,\boldsymbol{x}).$$

If $\varphi(t,\alpha) \equiv 1$, then the general time-conformable fractional derivative is reduced to the usual first-order partial derivative with respect to t. Also, if p has a first-order partial derivative with respect to t, then

$$D_t^{\alpha,\varphi}p(t,\boldsymbol{x}) = \varphi(t,\alpha)\frac{\partial p(t,\boldsymbol{x})}{\partial t}.$$

3. Statement of the main result

Before stating our main result, we provide the following notation. We consider the orthogonal set $\{j_0(\mu_n r)\}_{n\in\mathbb{N}}$ of the spherical Bessel functions of order zero, where

$$j_\nu(s) = \sqrt{\frac{\pi}{2s}}J_{\nu+\frac{1}{2}}(s) = \sqrt{\frac{\pi}{2}}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!\Gamma\left(n+\nu+\frac{3}{2}\right)}\left(\frac{s}{2}\right)^{2n+\nu},$$

with $\nu \geq 0$ and where the sequence $\{\mu_n\}_{n\in\mathbb{N}}$ is defined by $\mu_n = \frac{\pi}{R}n$ with $R \in (0,\pi)$. It is well known that $|j_0(s)| \leq 1$ for all $s \in (0,\infty)$; see e.g., [20]. We set $\delta > 0$ and denote by $\mathcal{D}_\delta$ the family of sequences $\{\delta_n\}_{n\in\mathbb{N}}$ of nonnegative real numbers satisfying $\sum_{n=1}^{\infty} \delta_n = \delta$, and for each $\{\delta_n\}_{n\in\mathbb{N}} \in \mathcal{D}_\delta$, we consider the sequence of sets $\{\mathcal{U}_{\delta_n}\}_{n\in\mathbb{N}}$ defined by

$$\mathcal{U}_{\delta_n} = \{u \in \mathcal{PC}(\mathbb{R}) : |u(t)| \leq \delta_n\},$$

where $\mathcal{PC}(\mathbb{R})$ represents the set of piecewise continuous functions defined on $\mathbb{R}$. Finally, we consider the set

$$\mathcal{U}_\Omega^\delta = \left\{\sum_{n=1}^{\infty} u_n(t) j_0(\mu_n r) : u_n \in \mathcal{U}_{\delta_n}, \{\delta_n\}_{n\in\mathbb{N}} \in \mathcal{D}_\delta\right\}.$$

The following norm is considered in the set $\mathcal{U}_\Omega^\delta$:

$$\|u\|_{L^\infty(\Omega)} = \sup_{(t,r)\in\Omega} |u(t,r)|.$$

From the representation of the heat sources u in $\mathcal{U}_\Omega^\delta$ as a Fourier series, we observe that the inequality $|u(t,r)| \leq \delta$ is satisfied for all $(t,r) \in \Omega$, since

$$|u(t,r)| \leq \sum_{n=1}^{\infty} |u_n(t)| \leq \sum_{n=1}^{\infty} \delta_n = \delta.$$

Therefore, if $u \in \mathcal{U}_\Omega^\delta$, then $\|u\|_{L^\infty(\Omega)} \leq \delta$. Similarly, on the set $\mathcal{U}_{\delta_n}$ the following norm is considered

$$\|u_n\|_{L^\infty(0,T)} = \sup_{t\in[0,T]} |u_n(t)|.$$

It is clear that if $u_n \in \mathcal{U}_{\delta_n}$, then $\|u_n\|_{L^\infty(0,T)} \leq \delta_n$.

In this paper, we consider the perturbed heat equation with axial symmetry and with a general time-conformable fractional derivative defined on a sphere of radius R, and whose perturbation is described by a heat source $u \in \mathcal{U}_\Omega^\delta$. That is, the partial differential equation with a general time-conformable fractional derivative described by:

$$D_t^{\alpha,\varphi} y = \kappa_\alpha \left(\frac{\partial^2 y}{\partial r^2} + \frac{1}{r}\frac{\partial y}{\partial r}\right) + u(t,r), \tag{3.1}$$

where $(t,r) \in \Omega = [0,T] \times [0,R]$, $T > 0$ is finite and κ_α is the generalized thermal diffusivity constant. The general time-conformable fractional derivative used in (3.1) is defined under the choice of a positive conformable fractional function $\varphi\colon [0,T] \times (0,1] \to \mathbb{R}$ with the property that the function

$$h(t) = \int_0^t \frac{\mathrm{d}s}{\varphi(s,\alpha)},$$

is well defined on $[0, T]$. It follows that h is a strictly increasing function. We assume that $h(0^+) = \lim_{t\to 0^+} h(t)$ is finite. The equation (3.1) is complemented with the following initial and boundary conditions:

$$y(0, r) = 0, \qquad r \in [0, R], \tag{3.2}$$

$$y(t, R) = 0, \quad |y(t, 0)| < \infty, \quad t \in [0, T]. \tag{3.3}$$

The initial-boundary value problem (3.1)–(3.3) is a particular case of the diffusion equation in polar coordinates analyzed in [13], where the existence of solutions and the continuous dependence on source function and initial-boundary conditions of the solution is shown.

We consider that the thermal diffusivity constant κ_α and the maximum value of the polar radius R satisfy the inequality

$$\frac{R}{2\pi} \leq \kappa_\alpha. \tag{3.4}$$

There are other cases that are of interest and that depend on the values of R and κ_α. These cases are presented later in Remark 3.

In what follows, we consider that the solutions of the initial-boundary value problem (3.1)–(3.3) with a heat source $u \in \mathcal{U}_\Omega^\delta$ can be expressed as

$$y(t, r) = \sum_{n=1}^{\infty} y_n(t) j_0(\mu_n r), \quad (t, r) \in \Omega, \tag{3.5}$$

where $y_n : [0, T] \to \mathbb{R}$ are functions (generalized Fourier coefficients) that need to be determined. These functions satisfy the condition $y_n(0) = 0$, $n \in \mathbb{N}$, which is obtained from the initial condition $y(0, r) = 0$ for $r \in [0, R]$.

The set of solutions y of the initial-boundary value problem (3.1)–(3.3) associated with a heat source $u \in \mathcal{U}_\Omega^\delta$ that are expressed in the form (3.5) is denoted by $\mathcal{Y}_\Omega$. The set $\mathcal{Y}_\Omega$ is non-empty, because if (3.1) does not admit heat sources, that is, if we choose $\bar{u} \equiv 0$, then the solution of the initial-boundary value problem (3.1)–(3.3) coincides with the trivial solution $\bar{y} \equiv 0$, that is, $\bar{y} \in \mathcal{Y}_\Omega$. Such a solution is called *unperturbed*. We note that $\bar{y}$ is the unique unperturbed solution of (3.1) with initial-boundary conditions (3.2)–(3.3). In the set $\mathcal{Y}_\Omega$, the following norm is considered:

$$\|y\|_{\mathcal{Y}_\Omega} = \max\left\{ \sup_{(t,r)\in\Omega} |y(t, r)|, \sup_{(t,r)\in\Omega} \left|D_t^{\alpha,\varphi} y(t, r)\right|, \sup_{(t,r)\in\Omega} \left|\frac{\partial y}{\partial r}(t, r)\right| \right\}.$$

The introduction of this norm allows us to compare solutions y associated with non-trivial heat sources $u \in \mathcal{U}_\Omega^\delta$ with the unperturbed solution $\bar{y}$, that is, we can analyze the real number $\|y - \bar{y}\|_{\mathcal{Y}_\Omega} = \|y\|_{\mathcal{Y}_\Omega}$. This allows us to introduce a concept of robust stability for the initial-boundary value problem (3.1)–(3.3) with heat sources $u \in \mathcal{U}_\Omega^\delta$, in the sense of the following:

Definition 5. The unperturbed solution $\bar{y} \equiv 0$ of the heat equation with general time-conformable fractional derivative (3.1) is called robustly stable with respect to heat sources $u \in \mathcal{U}^{\delta}_{\Omega}$, if for all $\epsilon > 0$ there exists $\eta(\epsilon) > 0$ such that under the condition: $\|u\|_{L^\infty(\Omega)} \leq \delta < \eta(\epsilon)$, any other solution $y \in \mathcal{Y}_\Omega$ of the initial-boundary value problem (3.1)–(3.3) satisfies the inequality $\|y\|_{\mathcal{Y}_\Omega} < \epsilon$.

Definition 5 is a natural extension of the concept of stability under constant-acting perturbations introduced by Duboshin and Malkin, which is applied to systems of ordinary differential equations; see, e.g., [9]. Concepts analogous to those of Definition 5 have been considered in [18, 23]. In this paper, we establish the conditions that guarantee the robust stability of the unperturbed solution $\bar{y} \equiv 0$ of the initial-boundary value problem (3.1)–(3.4) in the sense of Definition 5.

4. Application of the Fourier method and properties of generalized Fourier coefficients

In this section, we present the justification for applying the Fourier method of separation of variables to determine the solutions of the initial-boundary value problem (3.1)–(3.3) when we consider heat sources $u \in \mathcal{U}^{\delta}_{\Omega}$.

It is well known that justifying the application of the Fourier method to determine solutions of the initial-boundary value problem (3.1)–(3.3) when heat sources $u \in \mathcal{U}^{\delta}_{\Omega}$ are chosen, consists in proving the uniform convergence on Ω of the Fourier series (3.5), as well as three other series obtained by term-by-term differentiation of the following functions:

$$D_t^{\alpha,\varphi} y, \quad \frac{1}{r}\frac{\partial y}{\partial r}, \quad \frac{\partial^2 y}{\partial r^2}. \tag{4.1}$$

The condition of existence is obtained as follows:

Theorem 1. Suppose that $u(r,t) = \sum_{n=1}^{\infty} u_n(t) j_0(\mu_n r)$ is a heat source for the initial-boundary value problem (3.1)–(3.3) such that $u_n \in \mathcal{U}_{\delta_n}$ for each $n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} \delta_n$ is convergent, then the function y expressed in the form (3.5) is a unique solution (in the almost everywhere sense) of the initial-boundary value problem (3.1)–(3.3) associated with this heat source, that is, $y \in \mathcal{Y}_\Omega$.

The proof of this theorem is presented in Section 6.

From Theorem 1, it follows that for each choice of a heat source $u \in \mathcal{U}^{\delta}_{\Omega}$, the initial-boundary value problem (3.1)–(3.3) always admits a solution that can be expressed in the form (3.5).

We choose a sequence $\{\delta_n\}_{n\in\mathbb{N}} \in \mathcal{D}_\delta$ and consider a heat source represented by $u(t,r) = \sum_{n=1}^{\infty} u_n(t) j_0(\mu_n r)$, with $u_n \in \mathcal{U}_{\delta_n}$ for all $n \in \mathbb{N}$. Let y be the solution associated with this heat source that admits the representation (3.5).

After substituting the expressions for y and u into the heat equation with a general time-conformable fractional derivative (3.1), we find that the n-th Fourier coefficient y_n and the function $u_n \in \mathcal{U}_{\delta_n}$ must satisfy the following general conformable fractional differential equation with an external perturbation:

$$D^{\alpha,\varphi} y_n + \kappa_\alpha \mu_n^2 y_n = u_n(t), \quad y_n(0) = 0. \tag{4.2}$$

The initial condition $y_n(0) = 0$ in (4.2) is obtained from the initial condition (3.3).

We denote by $\mathcal{Y}_{\delta_n}$ the set of solutions y_n of (4.2) that are associated with an external perturbation $u_n \in \mathcal{U}_{\delta_n}$. If the general conformable fractional differential equation (4.2) does not admit an external perturbation, that is, if we choose $\bar{u}_n \equiv 0$, then the corresponding solution of (4.2) is the trivial solution $\bar{y}_n \equiv 0$, that is, it follows that $\bar{y}_n \in \mathcal{Y}_{\delta_n}$. The solution $\bar{y}_n$ is called *unperturbed*.

Using the method described in [22], we find that the solution of (4.2) is expressed by

$$y_n(t) = \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)} \, \mathrm{d}s. \tag{4.3}$$

These solutions have the following properties:

(a) From the definition of general conformable fractional derivative, it follows that

$$D^{\alpha,\varphi} y_n(t) = u_n(t) - \kappa_\alpha \mu_n^2 \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)} \, \mathrm{d}s. \tag{4.4}$$

(b) If the external perturbations $u_n^{\pm}(t) = \pm\delta_n$ are chosen in $\mathcal{U}_{\delta_n}$, then the following Fourier coefficients are obtained:

$$y_n^{\pm}(t) = \pm \frac{\delta_n}{\kappa_\alpha \mu_n^2} \left(1 - e^{-\kappa_\alpha \mu_n^2 (h(t)-h(0^+))}\right).$$

We note that y_n^+ is strictly increasing and y_n^- is strictly decreasing on $[0,T]$, because $D^{\alpha,\varphi} y_n^+(t) > 0$ and $D^{\alpha,\varphi} y_n^-(t) < 0$ for $t \in [0,T]$.

(c) For any other choice $u_n \in \mathcal{U}_{\delta_n}$, from the inequality $u_n^-(t) \leq u_n(t) \leq u_n^+(t)$ for $t \in [0,T]$, we observe that the solution y_n of (4.2) associated with this perturbation satisfies the inequality

$$y_n^-(t) \leq y_n(t) \leq y_n^+(t), \quad t \in [0,T].$$

If y_n is a solution of the general conformable fractional differential equation (4.2) associated with an external perturbation $u_n \in \mathcal{U}_{\delta_n}$, from ($b$) and ($c$), it follows that

$$|y_n(t)| \leq \frac{\delta_n}{\kappa_\alpha \mu_n^2} \left| 1 - e^{-\kappa_\alpha \mu_n^2 (h(t)-h(0^+))} \right| < \frac{\delta_n}{\kappa_\alpha \mu_n^2},$$

where the last inequality follows from the fact that h is strictly increasing. We deduce from this inequality that for each choice $u_n \in \mathcal{U}_{\delta_n}$, the corresponding solution y_n satisfies

$$\sup_{t\in[0,T]} |y_n(t)| < \frac{\delta_n}{\kappa_\alpha \mu_n^2}. \tag{4.5}$$

Using the same argument, it is shown that the function $D^{\alpha,\varphi} y_n$ satisfies

$$\sup_{t\in[0,T]} |D^{\alpha,\varphi} y_n(t)| < 2\delta_n, \tag{4.6}$$

which follows from (4.4) and (4.5).

Inequalities (4.5) and (4.6) allow us to introduce a concept of robust stability analogous to the concept of stability under constant-acting perturbations due to Duboshin and Malkin for systems of ordinary differential equations; see, e.g., [9]. For this purpose, we consider the following norm for the solution of (4.2):

$$\|y_n\|_{\mathcal{Y}_{\delta_n}} = \max\left\{ \sup_{t\in[0,T]} |y_n(t)|, \sup_{t\in[0,T]} |D^{\alpha,\varphi} y_n(t)| \right\}.$$

Definition 6. The unperturbed solution $\bar{y}_n \equiv 0$ of the problem (4.2) is robustly stable with respect to external perturbations $u_n \in \mathcal{U}_{\delta_n}$, if for all $\epsilon_n > 0$ there exists $\eta_n(\epsilon_n) > 0$ such that under the condition: $\|u_n\|_{L^\infty(0,T)} \leq \delta_n < \eta_n(\epsilon_n)$, any other solution of the general conformable fractional differential equation (4.2) satisfies the inequality $\|y_n\|_{\mathcal{Y}_{\delta_n}} < \epsilon_n$.

We establish the robust stability criterion for the unperturbed solution of (4.2). Let $\epsilon_n > 0$. From inequalities (4.5) and (4.6), it follows that

$$\|y_n\|_{\mathcal{Y}_{\delta_n}} \leq 2\delta_n \max\left\{ \frac{1}{2\kappa_\alpha \mu_n^2}, 1 \right\}.$$

Therefore, the inequality $\|y_n\|_{\mathcal{Y}_{\delta_n}} < \epsilon_n$ is valid if

$$\delta_n < \eta_n(\epsilon_n) = \frac{\epsilon_n}{2\max\left\{ \frac{1}{2\kappa_\alpha \mu_n^2}, 1 \right\}} = \frac{\epsilon_n}{2}, \tag{4.7}$$

where the last inequality is obtained from (3.4). Therefore, if the inequality $0 < \delta_n < \frac{1}{2}\epsilon_n$ holds, then $\|y_n\|_{\mathcal{Y}_{\delta_n}} < \epsilon_n$, that is, the unperturbed solution of (4.2) is robustly stable in the sense of Definition 6.

Remark 1. We note that the values on the right-hand side of the inequalities (4.5) and (4.6) do not depend on the order $\alpha \in (0, 1]$ of the general conformable fractional derivative in (4.2), or the fractional conformable function φ that is used to define the general conformable fractional derivative. As a consequence, the same property holds for the robust stability criterion (4.7) of the unperturbed solution $\bar{y}_n$ for each $n \in \mathbb{N}$.

5. Robust stability criterion

In this section, we obtain a robust stability criterion for the unperturbed solution $\bar{y}$ of the heat equation (3.1) with initial-boundary conditions (3.2)–(3.3) when we consider arbitrary heat sources $u \in \mathcal{U}_{\Omega}^{\delta}$. The robust stability criterion is obtained from the properties of the Fourier coefficients of the solution y, which is expressed in the form (3.5).

For each $\epsilon > 0$ we define $\mathcal{E}_\epsilon$ as the set of sequences $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive real numbers that satisfy $\sum_{n=1}^{\infty} \epsilon_n = \epsilon$. We set $\epsilon > 0$ and choose a sequence $\{\epsilon_n\}_{n\in\mathbb{N}} \in \mathcal{E}_\epsilon$. If we consider a sequence $\{\delta_n\}_{n\in\mathbb{N}} \in \mathcal{D}_\delta$ of positive terms such that $0 < \delta_n < \frac{\epsilon_n}{2}$, according to the robust stability criterion (4.7) of the general conformable fractional differential equation (4.2), we observe that

$$|y(t,r)| \leq \sum_{n=1}^{\infty} |y_n(t)| \leq \frac{R^2}{\pi^2 \kappa_\alpha} \sum_{k=1}^{\infty} \delta_n, \tag{5.1}$$

and

$$\left| D_t^{\alpha,\varphi} y(t,r) \right| \leq \sum_{n=1}^{\infty} |D^{\alpha,\varphi} y_n(t)| \leq 2 \sum_{n=1}^{\infty} \delta_n. \tag{5.2}$$

Similarly, it follows that

$$\left| \frac{\partial y(t,r)}{\partial r} \right| \leq \sum_{n=1}^{\infty} \mu_n \, |y_n(t)| \leq \frac{1}{\kappa_\alpha} \sum_{n=1}^{\infty} \frac{\delta_n}{\mu_n} \leq \frac{R}{\pi \kappa_\alpha} \sum_{n=1}^{\infty} \delta_n. \tag{5.3}$$

Therefore, according to the inequalities (5.1)–(5.3), if we define

$$M = \max\left\{1, \frac{R}{2\pi\kappa_\alpha}, \frac{R^2}{2\pi^2\kappa_\alpha}\right\} = 1,$$

then the estimate $\|y\|_{\mathcal{Y}_\Omega} < \epsilon$ is obtained if we choose

$$\delta < \eta(\epsilon) = \frac{\epsilon}{2M} = \frac{\epsilon}{2}, \tag{5.4}$$

which follows from the restriction of the maximum value of the polar radius R and the thermal diffusivity coefficient κ_α that is described in (3.4).

Remark 2. We note that the robust stability criterion (5.4) for the unperturbed solution $\bar{y}$ of the initial-boundary value problem (3.1)–(3.3) does not depend on the order $\alpha \in (0,1]$ of the general time-conformable fractional derivative in (3.1), or on the conformable fractional function φ that is used to define the general time-conformable fractional derivative, which follows as a consequence of Remark 1.

As a conclusion, we obtain the following result:

Theorem 2. The unperturbed solution $\bar{y} \equiv 0$ of the initial-boundary value problem (3.1)–(3.3) is robustly stable with respect to heat sources $u \in \mathcal{U}_\Omega^\delta$, since for each $\epsilon > 0$ it is enough to choose $\eta(\epsilon) = \frac{1}{2}\sum_{n=1}^{\infty} \epsilon_n$ and $0 < \delta_n < \frac{1}{2}\epsilon$ assuming that $\{\epsilon_n\}_{n\in\mathbb{N}}$ belongs to $\mathcal{E}_\epsilon$.

Remark 3. The inequality (3.4) is a particular case of the two cases that are obtained from the values of the parameter M:

(1) If $R \in (0, \pi)$, then κ_α can satisfy one of the following three subcases:

$$(1a)\ \ 0 < \kappa_\alpha \leq \frac{R^2}{2\pi^2}, \qquad (1b)\ \ \frac{R^2}{2\pi^2} < \kappa_\alpha < \frac{R}{2\pi}, \qquad (1c)\ \ \frac{R}{2\pi} \leq \kappa_\alpha.$$

(2) If $R \in [\pi, +\infty)$, then κ_α can satisfy one of the following three subcases:

$$(2a)\ \ 0 < \kappa_\alpha \leq \frac{R}{2\pi}, \qquad (2b)\ \ \frac{R}{2\pi} < \kappa_\alpha < \frac{R^2}{2\pi^2}, \qquad (2c)\ \ \frac{R^2}{2\pi^2} \leq \kappa_\alpha.$$

The same technique can be applied in each case to show that the unperturbed solution of the initial-boundary value problem (3.1)–(3.3) is robustly stable in the sense of Definition 1. To do this, it is necessary to consider the corresponding value of M. The subcase of study in this paper corresponds to *(1c)*.

The following example shows the validity of the results that have been obtained numerically with GNU Octave 7.2.0.

Example 1. As a particular case, we set $\kappa_\alpha = \frac{1}{4}$ and $R = 1$ in the equation (3.1). We consider the problem of finding solutions of (3.1)–(3.3) such that $\|y\|_{\mathcal{Y}_\Omega} < 1$.

In the set $\mathcal{E}_1$ we choose the sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ such that $\epsilon_n = (1-p)p^{n-1}$ with $p \in (0,1)$. In particular, we consider the case $p = \frac{3}{10}$, which allows us to obtain $\epsilon_n = \frac{9}{10^n}$. According to Theorem 2, we can choose $\{\delta_n\}_{n\in\mathbb{N}}$ such that $\delta_n = \frac{9}{20}\epsilon_n$. As a particular case, we consider the following conformable fractional functions:

$$\varphi^k(t,\alpha) = b_k t^{1-\alpha}, \quad k \in \{1,2\},$$

with $b_1 = 1$ and $b_2 = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)}$. In such a case, we obtain the functions

$$h^k(t,\alpha) = b_k \frac{t^\alpha}{\alpha}, \quad k \in \{1,2\}.$$

We consider the external perturbations for the conformable fractional differential equation (4.2):

$$u_n^k(t) = \delta_n \cos\left(b_k \frac{t^\alpha}{\alpha}\right) \in \mathcal{U}_{\delta_n}, \quad k \in \{1,2\},$$

and with them, we consider the following approximation to the heat sources $u^k(t,r) = \sum_{n=1}^{N} u_n^k(t) j_0(\mu_n r)$ for some $N \in \mathbb{N}$. If we consider the values $\alpha = \beta = \frac{1}{2}, \frac{3}{4}, 1$, then we obtain the approximations of the solutions shown in Figures 1 and 2. These approximations have been obtained by considering the partial sums $y^k(t,r) = \sum_{n=1}^{N} y_n^k(t) j_0(\mu_n r)$. Similar expressions are used for the approximation of the functions $\frac{\partial}{\partial r} y^k$, $D_t^{\alpha,\varphi_1} y^k$ and $D_t^{\alpha,\varphi_2} y^k$.

We finally observe that in any of the two cases we obtain $\|y^k\|_{\mathcal{Y}_\Omega} \leq 0.828235$, $k \in \{1, 2\}$.

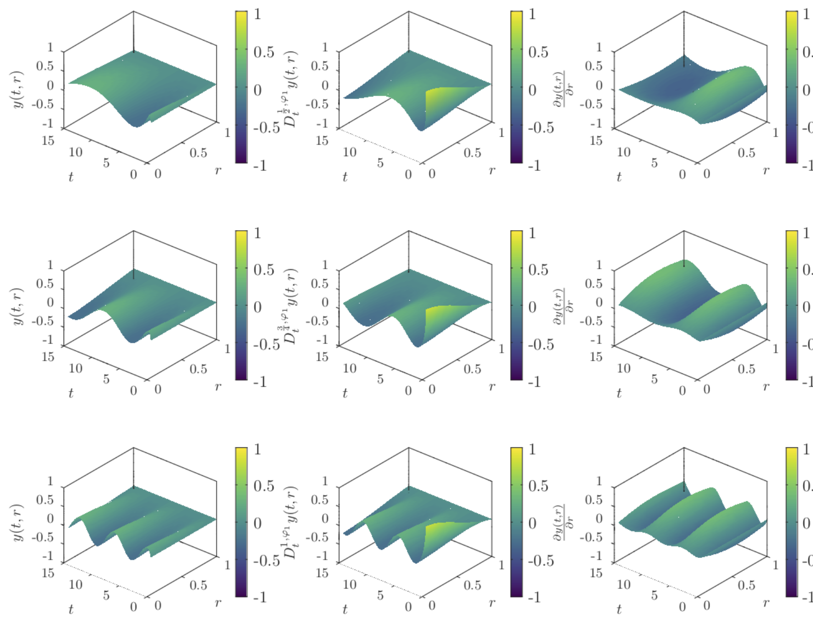

Figure 1: Approximations of the solutions of the initial-boundary value problem (3.1)–(3.3) with conformable fractional function $\varphi_1(t,\alpha)$ for Example 1.

6. Proof of Theorem 1

We set $\delta > 0$ and consider an arbitrary sequence $\{\delta_n\}_{n\in\mathbb{N}} \in \mathcal{D}_\delta$. Subsequently, we choose an external perturbation $u_n \in \mathcal{U}_{\delta_n}$ for each $n \in \mathbb{N}$. According to the representation of the solution of the general conformable fractional differential equations (4.2), the representation (3.5) of the solution of the initial-boundary value problem (3.1)–(3.3) is represented by

$$y(t,r) = \sum_{n=1}^{\infty} j_0(\mu_n r) \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)} \, \mathrm{d}s. \tag{6.1}$$

To prove Theorem 1 it is enough to determine under what conditions the series that define the functions (6.1) and (4.1) converge almost uniformly on Ω.

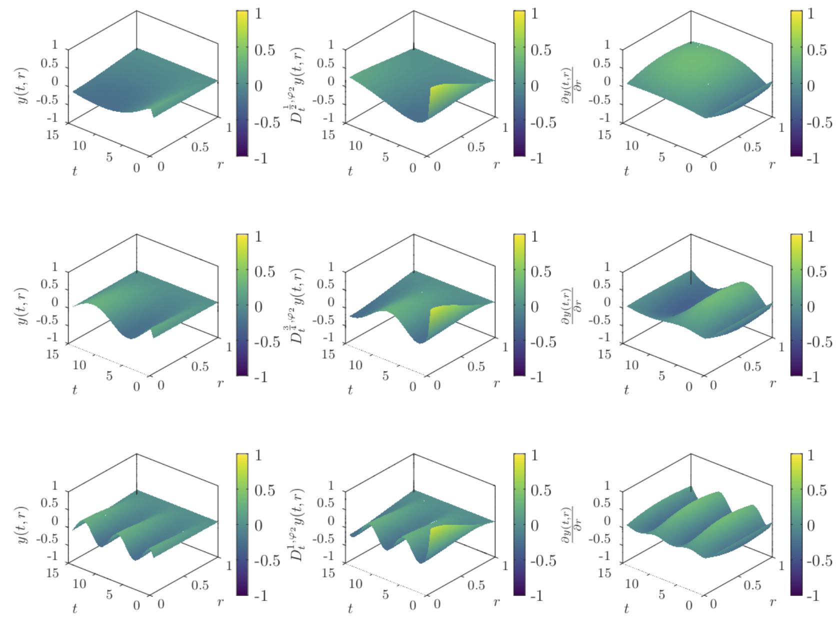

Figure 2: Approximations of the solutions of the initial-boundary value problem (3.1)–(3.3) with conformable fractional function $\varphi_2(t,\alpha)$ for Example 1.

Regarding the series on the right-hand side of (6.1), we observe that

$$|y(t,r)| \leq \sum_{n=1}^{\infty} \left| \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)} \, \mathrm{d}s \right| < \sum_{n=1}^{\infty} \frac{\delta_n}{\kappa_\alpha \mu_n^2} < \frac{R^2}{\kappa_\alpha \pi^2} \sum_{n=1}^{\infty} \delta_n,$$

where we have used the fact that $|j_0(s)| \leq 1$ for all $s \geq 0$, and that $\mu_n^2 > \frac{\pi^2}{R^2}$ for each $n \in \mathbb{N}$. We note that the series on the right-hand side of (6.1) converges almost uniformly to the function y on Ω, provided that $\sum_{n=1}^{\infty} \delta_n$ is convergent.

On the other hand, the general time-conformable fractional derivative of the function (3.5) for $(t,r) \in \Omega$ is

$$D_t^{\alpha,\varphi} y(t,r) = \sum_{n=1}^{\infty} j_0(\mu_n r) \left(u_n(t) - \kappa_\alpha \mu_n^2 \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)} \, \mathrm{d}s \right). \tag{6.2}$$

Therefore, the following inequalities hold for $(t,r) \in \Omega$:

$$\begin{aligned} \left| D_t^{\alpha,\varphi} y(t,r) \right| &\leq \sum_{n=1}^{\infty} \left| u_n(t) - \kappa_\alpha \mu_n^2 \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)} \, \mathrm{d}s \right| \\ &\leq \sum_{n=1}^{\infty} \delta_n \left(2 - e^{-\kappa_\alpha \mu_n^2 (h(t)-h(0^+))} \right) \leq 2 \sum_{n=1}^{\infty} \delta_n. \end{aligned}$$

We note once again that the series on the right-hand side of (6.2) converges almost uniformly to the function $D_t^{\alpha,\varphi} y$ on Ω, provided that $\sum_{n=1}^{\infty} \delta_n$ is convergent.

Similarly, as

$$\frac{1}{r}\frac{\partial y}{\partial r}(t,r) = -\sum_{n=1}^{\infty} \frac{\mu_n}{r} j_1(\mu_n r) \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)}\, \mathrm{d}s, \tag{6.3}$$

and according to the validity of the identity

$$\frac{3 j_1(s)}{s} = j_0(s) + j_2(s), \quad s \geq 0; \tag{6.4}$$

see [20], thus the following inequalities hold:

$$\begin{aligned}
\left| \frac{1}{r}\frac{\partial y}{\partial r}(t,r) \right| &\leq \sum_{n=1}^{\infty} \left| \frac{\mu_n}{r} j_1(\mu_n r) \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)}\, \mathrm{d}s \right| \\
&\leq \sum_{n=1}^{\infty} \left| \frac{2\mu_n^2}{3} \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)}\, \mathrm{d}s \right| \\
&\leq \sum_{n=1}^{\infty} \frac{2\mu_n^2}{3} \cdot \frac{\delta_n}{\kappa_\alpha \mu_n^2} \left(1 - e^{-\kappa_\alpha \mu_n^2 (h(t)-h(0^+))} \right) \leq \frac{2}{3\kappa_\alpha} \sum_{n=1}^{\infty} \delta_n.
\end{aligned}$$

Therefore, the series on the right-hand side of (6.3) converges almost uniformly to the function $\frac{1}{r}\frac{\partial y}{\partial r}$ on Ω provided that $\sum_{n=1}^{\infty} \delta_n$ is convergent.

Finally, for the series

$$\frac{\partial^2 y}{\partial r^2}(t,r) = -\sum_{n=1}^{\infty} \mu_n^2 \left(j_0(\mu_n r) - \frac{2}{\mu_n r} j_1(\mu_n r) \right) \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)}\, \mathrm{d}s, \tag{6.5}$$

and according to the identity (6.4), it follows that

$$\begin{aligned}
\left| \frac{\partial^2 y}{\partial r^2}(t,r) \right| &\leq \sum_{n=1}^{\infty} \mu_n^2 \left| \int_0^t e^{-\kappa_\alpha \mu_n^2 (h(t)-h(s))} \frac{u_n(s)}{\varphi(s,\alpha)}\, \mathrm{d}s \right| \\
&\leq \sum_{n=1}^{\infty} \mu_n^2 \cdot \frac{\delta_n}{\kappa_\alpha \mu_n^2} \left(1 - e^{-\kappa_\alpha \mu_n^2 (h(t)-h(0^+))} \right) \leq \frac{1}{\kappa_\alpha} \sum_{n=1}^{\infty} \delta_n.
\end{aligned}$$

It is concluded that the series on the right-hand side of (6.5) converges almost uniformly to the function $\frac{\partial^2 y}{\partial r^2}$ on Ω, provided that the series $\sum_{n=1}^{\infty} \delta_n$ is convergent.

After substituting the expressions (6.1), (6.2), (6.3) and (6.5) in the heat equation with a general time-conformable fractional derivative (3.1), it can be verified that the function in (6.1) is a unique solution (in the almost everywhere sense) to the initial-boundary value problem (3.1)–(3.3) associated with the heat source $u \in \mathcal{U}_\Omega^\delta$.

7. Conclusion

A method has been described that allows to stabilize the trivial solution of the heat equation with axial symmetry and with time-conformable fractional derivative that is defined on a sphere and that admits heat sources that belong to a prefixed set, assuming zero initial-boundary conditions. With the help of this criterion, the values of its solution have been bounded by a positive constant value, as well as the values of its time-conformable fractional derivative and its first partial derivative with respect to the radial axis. The obtained criterion is independent of the order of the time-conformable fractional derivative as well as of the conformable fractional function that is used in the general time-conformable fractional derivative.

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