Cuadernos de Investigación y Formación en Educación Matemática

e-ISSN en línea: 2215-5627
Impresa: ISSN 1659-2573
Vol: 19 · N°. 2 · 2026
175-196
Recibido: 07/04/2026
Aprobado: 17/06/2026

Artículos de investigación

GENERALIZATION PROCESSES IN A LEARNING SITUATION RELATED TO THE TRIGONOMETRIC FOURIER SERIES

PROCESOS DE GENERALIZACIÓN EN UNA SITUACIÓN DE APRENDIZAJE RELATIVA A LA SERIE TRIGONOMÉTRICA DE FOURIER

Fabián W. Romero Fonseca[1]
ORCID iD: https://orcid.org/0000-0003-4472-963X

Rosa María Farfán Márquez[2]
ORCID iD: https://orcid.org/0000-0003-1229-8521

ABSTRACT

This research deals with the processes of mathematical generalization when constructing the Trigonometric Fourier Series, since, from its epistemology, the importance of these processes for its adequate construction is evidenced. This approach responds to the Theory of Operative Generalization, which allows glimpsing the generalization processes from the students' actions. The analysis of the students' productions and interactions were based on the consideration of analytical questions resulting from the theoretical position to characterize their actions, what they do and how they do it, the elements of the actions and the relationships between these elements, what they do it on, and their symbolization, by means of how they do it. Three organizing categories of the actions and their invariants in the process of generalization emerged from the analysis: 1) those related to fixing the causes that provoke the behaviour of the system, 2) those concerning the recognition that the phenomenon is susceptible to reach a stable, or stationary, state, 3) those related to the relation between the convergence of the series with the stability of the system, and 4) those concerning the calculation of the Fourier coefficients. In

[1] Universidad de Costa Rica, San José, Costa Rica. Correo electrónico: fabian.romero@ucr.ac.cr

[2] Centro de Investigación y de Estudios Avanzados del IPN, Ciudad de México, México. Correo electrónico: rfarfan@cinvestav.mx

addition, relationships are established between these categories that allow the identification of extensional generalizations.

Keywords: Mathematical generalization, Operative generalization, Series convergence.

RESUMEN

Esta investigación aborda los procesos de generalización matemática en la construcción de la Serie de Fourier trigonométrica, ya que, desde su epistemología, se evidencia la importancia de estos procesos para su adecuada construcción. Este enfoque responde a la Teoría de la Generalización Operativa, la cual permite vislumbrar los procesos de generalización a partir de las acciones de los estudiantes. El análisis de las producciones e interacciones de los estudiantes se basó en la consideración de preguntas analíticas derivadas de la postura teórica, con el fin de caracterizar sus acciones, qué hacen y cómo lo hacen, los elementos de dichas acciones y las relaciones entre estos elementos, sobre qué lo hacen, así como su simbolización, mediante qué lo hacen. Del análisis emergieron cuatro categorías organizadoras de las acciones y sus invariantes en el proceso de generalización: 1) aquellas relacionadas con la fijación de las causas que provocan el comportamiento del sistema, 2) aquellas concernientes al reconocimiento de que el fenómeno es susceptible de alcanzar un estado estable (o estacionario), 3) aquellas relacionadas con la relación entre la convergencia de la serie y la estabilidad del sistema, y 4) aquellas relativas al cálculo de los coeficientes de Fourier. Además, se establecen relaciones entre estas categorías que permiten la identificación de generalizaciones extensionales.

Palabras clave: Generalización matemática, Generalización operativa, Convergencia de series.

1. INTRODUCTION

The Trigonometric Fourier Series (TFS) has been the subject of numerous studies in Mathematics Education. Specifically, a research group at the Department of Mathematics Education at Cinvestav-IPN in the field of higher education, which has focused on the study of series and their convergence (Farfán, 2012), as well as on positioning the trigonometric series as the most advanced stage in the development of trigonometric thinking (Montiel, 2011).

Romero (2016), on his study, even though it is not the main theme, highlighted the importance of considering generalization processes that emerge when constructing the TFS. This is because meaning must be derived from the use of the series before generalizing the trigonometric series representation of a function. However, to the best of our knowledge, there has not been an in-depth exploration of the generalization processes associated with the TFS. Despite Romero and Farfán (2016) conducting a synthesis of research related to the TFS, in which they address various aspects such as its historical genesis, the periodicity hypothesis, and notions of heat and convergence, among others, the following paragraphs present additional research findings related to the TFS and discussions of the generalization process associated with the series, both directly and indirectly.

Farfán (2012) showed that the controversy surrounding the vibrating string problem[1] is not about the solution itself, but rather about what constitutes the general solution and the methodology used to find it. It is the desire to generalize results that leads to a debate lasting

[1] The statement of this problem and a detailed explanation can be found in Farfán (2012).

nearly a century and that questions the very foundations of Mathematical Analysis at the time. The problem is solved in Fourier's work (1822), and in an effort to generalize his results, what it's known today as Fourier coefficients emerged driven by the prevailing positivist paradigm, characterized by applied mathematics and phenomenological empiricism, deeply rooted in the French Enlightenment and the École Polytechnique tradition (Friedman, 1977).

Within the context of heat propagation, Fourier's trigonometric series emerged as a mathematical tool for representing temperature distributions in steady-state conditions. Regarding this notion of steady state, Marmolejo (2006) pointed out that there is a generalized idea in school discourse asserting that the steady state is achieved when temperature variations become infinitesimally small over an infinitely long time. The phenomenological environment plays a crucial role here, as "it allows us to pre-establish boundary conditions, which not only serve the purpose of obtaining a particular solution to the heat transfer problem, but also guide or regulate the student's intuition, constraining their thinking." (Marmolejo, 2006, p. 77). In other words, the boundary conditions derived from the phenomenological environment do more than only helping to find particular solutions; for instance, they guide students' thinking towards understanding the general behaviour of the heat propagation phenomenon. This involves an inductive process, seeking to generalize the results from studying particular cases to grasp the overall behaviour of the phenomenon. To enable such generalization, Ulín (1984) asserted that it is impossible to separate the physical phenomenon from its mathematization. However, schools do not provide the necessary theoretical and empirical tools on the physical level for understanding the phenomenon of heat propagation (Morales, 2010), let alone for its mathematization.

With respect to the convergence of the TFS, Farfán (2012) and Albert (1996) highlighted Leibniz's principle of permanence as an epistemological obstacle. This principle leads both, students and teachers, to transfer properties of partial sums (e.g., continuity, oscillatory behaviour) to the limit function. In other words, they overgeneralize results from the finite case to the infinite case, which warrants careful attention when dealing with generalizations when working with the TFS. In addition, it must be considered that "Fourier's generalizations are for intervals of finite length, not over all of $\mathbb{R}$" (Vásquez, 2006, p. 45). Yet, there is a generalized trend in the school discourse regarding the series in which its periodic property dominates within a purely mathematical context. This fosters the overgeneralization of properties since it does not allow for the confrontation of the idea that the convergence value of a function series does not necessarily possess the same properties as the terms of the series (Vásquez, 2006).

Regarding the convergence of numerical series, Flores (1992) argues that its investigation involves five stages, the last two being conjecturing and generalizing. During the conjecturing stage, he highlighted the importance of examining particular cases in order to develop general criteria for assessing convergence. Although Flores (1992) refers to Cauchy's convergence criteria, these emerged within a mathematical framework developed after Fourier's work and focused primarily on numerical series. In the context of the TFS, by contrast, the analysis of particular cases of trigonometric series precedes the calculation of Fourier coefficients (Romero, 2016; Farfán & Romero, 2017). Furthermore, Flores (1992) argues that generalization "consists of the formal statement and rigorous proof of new convergence criteria" (p. 201), conceiving such criteria as sufficient conditions under which a series converges. That is, in this sense, to generalize means to determine and prove a

cc BY NC ND

sufficient condition for convergence. Keeping in mind the epistemological difference between Cauchy's and Fourier's work, the question arises: What it is the meaning of generalizing in Fourier's work, assuming one must start from particular cases to derive meaning? This is one of the key concerns of the present study.

1.1 Problem definition

As previously discussed, the TFS has been problematized in various research studies to understand the functioning of the didactic system and the conceptions held by students and teachers regarding the related mathematical ideas involved in the series. Considering the contributions made by all these researchers, Romero (2016) analyzes the practices associated with the emergence of the TFS in its historical context. He also mentions that, in the context of heat transfer phenomena, constructing the series requires the existence of:

> [As] a context of meaning, the modelling of a stationary phenomenon with periodic and bounded variation, for which the TFS becomes a predictive tool, where the act of interpretation plays a fundamental role by enabling the validation of mathematical argumentation within the physical phenomenon. (Romero, 2016, p. 127)

The above characterizes the necessary phenomena to evolve trigonometric thinking from functionality to formality. On the other hand, referring to the notion of variation in the phenomenological context of heat transfer, as Solís (1993) stated:

> The culmination of handling this notion of variation occurs when, being able to operate with the symbols of variation, an individual is capable of establishing the laws that govern a phenomenon of variation—that is, constructing a model that allows the phenomenon to become predictable. (p. 2)

Therefore, it is through the mathematization of the heat transfer phenomenon that one can deeply understand its behavior and make it predictable, that is, to determine its steady state in Fourier's terms. A general model of the phenomenon can then be expressed when it is mathematized through variation symbols[2]; in other words, when the differential equation that models the phenomenon is established.

For instance, in the vibrating string problem, the differential equation given by D'Alembert (1747) represents the general model of a variation phenomenon.

$$a^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \tag{1}$$

In equation (1), $y(x,t)$ represents the position (height) of each point of the string over time, and a is a constant depending on the tension and the linear mass density of the string. After establishing the model, the discussion centers around what the general solution to the equation is. This posed a question that was not clearly answered by the mathematicians of the 18th century. Bernoulli (1755) proposed a solution as a superposition of waves, arriving at the

[2] According to Solís (1993), variation symbols refer to differentials and partial derivatives, such as: Δx, dy, $\frac{\partial T}{\partial z}$, etc.

CC BY NC ND

idea that the initial shape of the string (an arbitrary function) can be represented by a trigonometric series.

Euler strongly opposed this solution because, according to him, "Bernoulli bases his opinions on no mathematical argument; thus, there is no proof of the generality that a function can be represented in such a way; in addition, there are no indications of any method to compute the coefficients" (Farfán, 2012, p. 66). The controversy around the vibrating string problem was resolved through Fourier's work on heat propagation, whose differential equation is another example of generalizing the behaviour of a variation phenomenon:

$$\frac{\partial v}{\partial t} = \frac{K}{C \cdot D}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \qquad (2)$$

In equation (2), $v(x, y, z, t)$ represents the temperature at point (x, y, z) in a solid body at time t, and the constants K, C, and D represent thermal conductivity, specific heat capacity, and material density, respectively. By using this equation to solve the problem of heat propagation in an infinite plate[3], Fourier, through extensive algebraic mastery, was able to determine the coefficients of a trigonometric series that converged to a specific value. This led him to generalize his results, for which "he stated and proved what the most prominent mathematicians of the 18th century deemed unacceptable: the possibility of representing an arbitrary function as a trigonometric series" (Farfán, 2012, p. 128).

Romero (2016) accounts for the geometric-analytical meaning behind Fourier's reasoning in calculating the coefficients, and proposes two key moments in the construction of the TFS: (i) Moment A seeks to understand that particular trigonometric series may converge to a function; (ii) and Moment B aims to generalize the trigonometric series representation to an arbitrary function, assuming it meets Dirichlet conditions, that means to determine the series' coefficients.

Moment B in the social construction of the TFS arises from Moment A. Given the need for generalization in mathematics, it becomes natural to ask the reverse question: if the function to which the series converges is known, what is the series? Moment B does not aim to determine the conditions under which a function can be represented by a trigonometric series. It rather assumes from the start that such representation is possible and focuses exclusively on how the coefficients are calculated. It is shown that behind Fourier's work lies a foundation of graphic and geometric argumentation that allows the coefficient calculation to be meaningful (Romero, 2016, p. 128).

Therefore, Moment B cannot occur without Moment A; it is necessary to derive meaning from the use of the series before generalizing the trigonometric series representation for an arbitrary function that satisfies Dirichlet conditions. Hence, it is hypothesized that, to achieve the generalization of the representation of a function in a TFS, one must first construct meaning for the series itself through its use, and then construct meaning for the calculation of its coefficients.

[3] The statement of this problem and a detailed explanation can be found in Farfán (2012).

CC BY NC ND

Regarding the calculation of the Fourier coefficients (i.e., the Moment B of the social construction of the TFS), Romero (2016) and Farfán and Romero (2017) show that Fourier's initial arguments for computing the series coefficients are found within the geometric-analytical register (area under the curve and orthogonality of trigonometric functions) before transitioning to algebraic arguments. In other words, the calculation of the coefficients requires the articulation of at least two representation registers: the geometric-analytical and the algebraic, to validate the latter through the former.

This analysis, discussed in greater detail in Romero (2016) and Romero and Farfán (2018), leads us to propose the hypothesis that the generalization process involved in calculating Fourier coefficients demands an entirely mathematical starting point. However, this process is grounded in geometric-analytical concepts, where the symbolic representation of the elements of the actions facilitates the construction of the relationship between the formulas (the integrals) and the area under the curve. This ultimately provides meaning to the former through the latter.

Stemming from prior research and the historical-epistemological reflections presented above on the TFS, we note the importance of considering the generalization processes that accompany its construction. From this perspective, meaning must emerge through the use of the series before its representation as a function can be generalized. Despite the relevance of this issue, the generalization processes associated with the TFS have not been thoroughly studied. Therefore, the objective of the present paper is to identify the components of the generalization process involved in the construction of the TFS. In particular, emphasis is placed on identifying the invariants of action according to the process of operative generation described by Dörfler (1991).

2. THEORETICAL FRAMEWORK: OPERATIVE GENERALIZATION

Dörfler (1991) proposed that in the process of generalization, actions or systems of actions, in the sense of Piaget (2009), are the starting point for the construction of mathematical knowledge. These actions are introduced by the subject in a situation or stimulus that creates a state of disequilibrium. The term "action" should be understood as broadly as necessary; for instance, even the purest mathematical operations must be considered as actions due to the cyclical nature of the generalization process. In the formation of a mathematical concept, at an elementary level, an action may consist of grouping specific material objects, while at a higher level of the same concept, the action may take the form of mathematical operations (e.g., ordering or addition, operating on abstract elements).

Actions or systems of actions can be imaginary, material, or symbolic, but they are always concrete actions. They focus the subject's attention on the elements of those actions, which are certain material or ideal objects. The goals set, the means employed, and the course of such actions direct the subject's attention toward certain relationships and connections among the elements of the actions. These relationships appear as invariant when the actions are repeated and are called invariants of the actions.

Establishing these invariants, however, requires some symbolic description, as symbols must be introduced to represent elements of the actions or the relevant quantities. These

CC BY NC ND

symbols may be verbal, iconic, geometric, arithmetic, or algebraic in nature. In any case, the invariants are described through these symbols and thus are fixed in this way. Initially, the symbols serve as mere representations and play only a descriptive role.

The establishment of invariants and their symbolic description is a process of abstraction, since certain properties and relationships are highlighted and foregrounded. Through this process, they gain a degree of independence from the original elements or objects with which they were associated, resulting in the objectification of the symbols. It is important to note that this process clearly corresponds to constructive abstraction, in which what is abstracted arises from actions and gains meaning and existence through those very actions.

Initially, the concrete symbols have a range of reference defined by the context of the action; in this sense, they are referential variables. The invariants, in their symbolic description, will serve as a search screen for the possibility of identical actions, due to the substitutability and interchangeability of the action's elements (empirical generalization) and of the action itself. Therefore, the structure of the declared actions must be maintained through the scheme and the invariants, constituting a form of extensional generalization. Later, the invariants will become the criteria for selecting objects as elements of action, and the symbols used at the beginning will evolve into variables with substitution properties.

As the symbolic description of the invariants is reflected upon, the symbols gradually begin to replace the elements of the actions. At this point, these symbols take on the character of objects; they become the elements of actions and serve as representatives, or "carriers," of the invariant relationships. The symbol thus shifts from its original range of reference, becoming an independent object whose meaning and significance stem from the relationships (the invariants) and from the operators for which they function as symbolic objects. In this way, the symbols transform into objects, both objectively and within the student's cognition, defined by the invariants described above.

Thus, a new object of thought is developed, a mathematical object whose signifier and intentional meaning resides in the invariants. Since it has a broader range of reference, this mental object is a generality. In this object, symbols now have the primary quality of variables with the status of objects. This reification (objectification) of variables and symbols completes the process of abstraction that began with the fixation of invariants. In this way, the invariants are separated from their original "carriers", gaining independence and forming the abstracted generality. Here, abstraction is how intentional generalization is developed.

The constructed generality, expressed using concrete variables similar to objects (for action elements and actions), later serves for further extensional generalization (both from a subjective-cognitive and an objective-epistemological perspective). By properly interpreting the variables, new systems of actions can be included within the generality whose reference range has now extended. This can result in the construction of new action systems through generality, making the invariants expressed in the generality, present and "valid" there.

The described process (see Figure 1) results in a variable object, both in the subject's cognition and in its quality as a mathematical object. Due to the operative character of the constructed generality, one can design tasks that allow for the gathering of evidence on the degree of success students have achieved in constructing a general concept. Such evidence will appear, first and foremost, through the students' own productions, as expressed in written or verbal explanations and expressions used for calculations, which will be how key elements of the process can be assessed. These elements are: the actions, the invariants of the actions, and the results of the process (i.e., the constructed generalizations).

Figure 1. Diagram of the process of generalization of actions and invariants of actions.

Source: Dörfler (1991, p. 74).

We should emphasize the cyclical nature of the generalization process. Every constructed generalization can become the object of a new action by the subject and serve as the starting point for a new generalization. In this sense, the last two stages of the theoretical framework gain importance, where the focus may shift from the conditions of the action to its results. During these processes, students use conceptual schemas (Piaget, 2009), through which they construct new schemas and establish relationships between the newly constructed and already existing schemas, reinforcing the consistency of the latter.

It is worth mentioning that Dörfler (1991) does not specify how the initial situation that triggers the subject's action (or system of actions) should be constructed. He assumes that it is a matter of the designer's insight to pose appropriate tasks for the subject according to the generalization process one aims to provoke.

3. METHOD

This research follows a qualitative approach with a case study design (Yin, 2017), employing a detailed and in-depth focus to understand a specific phenomenon within its natural context. This design allows for the exploration of the experiences, perceptions, and processes involved in the phenomenon under study through multiple sources of data, such as direct observations and analysis of relevant documents. According to Hernández et al. (2014), the case study focuses on the description and in-depth analysis or examination of one or more units and their context in a systemic and holistic manner.

Thus, studying the process of generalization required a starting situation for the construction of the TFS. For this purpose, it is used a learning situation based on a physical-geometric model of planetary motion proposed by Farfán and Romero (2019). Although this situation was not originally designed to address the generalization process, the theoretical framework used considers action, in the Piagetian sense (2009), as an observable practice in the subject's activity, which creates an appropriate environment for identifying action invariants relevant to the TFS, which is the object of this research.

3.1 Participants

The study involved the voluntary participation of 13 individuals (3 women and 10 men), aged between 20 and 27 years, all fifth-semester students from a higher education institution in central Mexico. The participants were labelled M1 to M3 (for the women) and H1 to H10 (for the men). It was agreed with the participants that they would complete the learning tasks in three sessions, two tasks per session, three hours per task. Additionally, all participants signed a consent form for the use of the collected records and their productions for research purposes.

3.2 Implementation and data collection

The implementation was carried out in the computer lab of the Department of Mathematics Education at the Center for Research and Advanced Studies of IPN. A total of 10 individuals actively participated in the sessions, H5 and H10 did not participate in any session, and H8 did not participate in the third session. The tasks were carried out in four teams of up to three students each.

Generally, for each part, at first, students worked in teams. However, despite being grouped, most students worked individually, sparsely interacting with their teammates, and limited themselves to validating their responses. Then, a plenary session followed, where different answers to each question were shared and discussed, aiming for consensus or explicitly acknowledging when there wasn't any.

During implementation, the researcher acted as the instructor. In addition, a non-participant observer collaborated in the process. This observer considered aspects ranging from the organization of space and time to the interaction between students and the mathematical knowledge involved.

The primary data source consisted of the students' final products, which were their responses to the tasks of the learning situation. The non-participant observer's notes served as a secondary data source, along with audio recordings of student team interactions and video recordings of the plenary discussions.

3.3 Data analysis

To identify the invariants of actions, the data analysis was divided into stages to clearly show how the theoretical model was applied. Below are the stages in which the data analysis was organized:

Stage 0: Sample selection and data organization

To select the participants whose productions and interactions would be analyzed, the following criteria were used:

1) Their engagement during the implementation phase, which include active participation and discussion with team members.
2) Their consistency in working with the same team across the three sessions allowed for analysis not only at the individual level but also at the team level.

Based on these criteria, and in collaboration with the observer, the following students were selected: M2, M3, H1, H3, and H4. This allowed for case analysis at three levels: individual, team, and group (considering the group video recordings of the plenary sessions).

After the sample selection, the data was organized by transferring the task responses into a table that included: task number, task objective, task parts, and questions along with their instructional intentions (Figure 2)[4].

[4] The initial data organization for the six tasks of the learning situation can be consulted in Romero (2020, Appendix 9.6).

Figure 2. Initial data organization table for Task #1

TASK #1					
Task Objective:	**To characterize the behavior of the system in a qualitative way, which will allow a deeper understanding of the phenomenon, beyond what can be perceived through direct sensory observation.**				
Part I.	**What does this model allow us to explain?**				
Intention:	To understand why the Alexandrian model can explain what the Greek model could not. The aim is to promote a deep understanding of the system, in the same way that Fourier understood the behavior of the heat propagation phenomenon through empirical observation before proposing his mathematical model of the phenomenon (Farfán & Romero, 2017).				
Student	**M2**	**M3**	**H1**	**H3**	**H4**
Question a					
Question b					
Question c					
Intentionality	In items a, b, and c, students are expected to relate the change in luminosity and the seasons of the year to the distance between T and P. Therefore, a circular model does not allow explaining these phenomena, since this distance never changes, unlike the model with 2, 3, and 4 circles, where the distance from P to T changes throughout the trajectory. Regarding the retrograde motion phenomenon, its explanation lies in the loops generated in the model with 4 circles.				
Part II.	**What if we increase the number of epicycles?**				
Intention:	With the support of an applet, the aim is to gain insight into the notion of system stability. For this purpose, the applet allows increasing the number of circles up to thirty, in order to analyze the changes in the system's behavior.				
Student	**M2**	**M3**	**H1**	**H3**	**H4**
Question a					
Intentionality	The aim is for students to conclude qualitatively that the radii of the circles tend to zero. For this, students may use expressions such as "they decrease," "they become smaller," among others.				
Question b					
Intentionality	Through the study of change, students are expected to infer that the speed of points on each circle keeps increasing without bound. They may use expressions such as "it goes faster," "in the same time it travels increasingly greater distances," among others. According to the pilot test, it is possible that for question b students may interpret the graph as representing the trajectory of the planet when adding circles, when in fact it refers to the motion of each point on each circle.				
Question c					
Intentionality	The intention is for students to identify the qualitative stable character of the system, using expressions such as "when there are many circles, the trajectory shape hardly changes as more are added," "it tends to resemble…," etc. To do so, they are expected to analyze changes in the trajectory shape as more circles are added. It is expected that they express that "something strange was happening on the left side of the trajectory," which is important since not the entire trajectory is stable.				
Question d					
Intentionality	The aim is to confront how the planet's trajectory changes (stability) with what causes this change. However, determining how it changes and what produces the change is a cognitively very complex task in the study of steady-state phenomena (Farfán, 2012; Romero, 2016). It is expected that, with the teacher's support and based on students' ideas, they approach the notion that as the radii of the circles tend to zero and the velocity of the point tends to infinity, stability is achieved. That is, adding a very small circle with a very fast-moving point does not produce significant changes in the overall system.				

Source: Authors' own elaboration.

To continue with the next stages, it was necessary to identify the components of the generalization process as it relates to the construction of the TFS. It is important to note that, based on the task intentions from Farfán and Romero (2019) and the generalization framework adopted in this research, Tasks #1, #2, #3, and #5 correspond to variations of the action system for Moment A of the social construction of the TFS. However, Task #4 aims to analyze a different phenomenon (the Gibbs phenomenon) without a direct relation to the situational context. Thus, even though the situation was implemented in its entirety, Task #4 was not analyzed.

Furthermore, since Tasks #2 and #3 shared the same objectives and intentions, the initial organization of the data allowed for a comparison of students' productions and revealed that Task #3 did not provide additional information beyond Task #2. Therefore, Task #3 was excluded from the analysis. Consequently, Tasks #1, #2, #5, and #6 were analyzed through the following two stages.

Stage 1: Identification of actions

Since action is the starting point in the generalization model adopted for this research, its identification and initial characterization are essential. Thus, "[action] can be associated with the questions: What do they do? and How do they do it?" (Cruz, 2019, p. 82). These questions served as analytical prompts for identifying and characterizing the students' actions. For this, three rows were added to the initial data table under each task intention:

1) What do they do?

CC BY NC ND

2) How do they do it?
3) Arguments and confrontations presented.

Additionally, two columns were added: Team 3 and Plenary Session. These helped conduct the analysis at the three levels mentioned earlier: individual, team, and group.

Stage 2: Identification of invariant of actions

After identifying actions in the previous stage (What do they do? and how do they do it?), the next step involves determining the elements of the actions, the relationships among those elements, and their symbolization, according to the adopted generalization model:

- Since action elements correspond to material or ideal objects the subject focuses on while acting, they were identified with the question: Upon which objects is the action carried out?
- As action invariants refer to relationships among these elements that remain invariant when the actions are repeated, identifying these relationships first involved the question: Upon which relationships is the action carried out?
- The question: By means of what is the action carried out? was used to identify the symbolic descriptions of both the action elements and their relationships, which was a necessary step for establishing invariants.

These questions functioned as analytical tools for identifying and characterizing the invariants of the actions. Accordingly, four rows were added to the initial data table, again considering the columns Team 3 and Plenary Session, under each task intention:

1) Upon which objects is it carried out?
2) Upon which relationships is it carried out?
3) By means of what is it carried out?
4) Invariants of actions.

For the first three points, the questions refer to the actions identified in Stage 1. The last point involves a horizontal perspective, examining relationships among the action elements, since invariants are identified through the repetition of actions. In this research, frequency was analyzed at the individual, team, and plenary levels.

You can find the complete analysis tables for Stages 1 and 2 in Romero (2020, Appendix 9.7). Due to the volume of information, the next section presents a synthesis of the most important findings.

4. RESULTS

This section discusses the generalization processes developed by the students when engaging with the situation. While the broader study identified a wide range of invariant relationships, this section focuses specifically on those that most clearly illuminate the generalization schema underpinning the present analysis.

When organizing the invariants of the actions, three organizing categories of actions and their invariants were identified:

1) those related to identifying the causes that generate the system's behaviour,

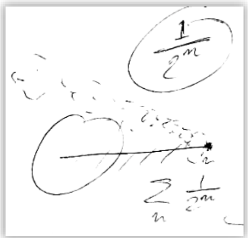
2) those concerning the recognition that the phenomenon is capable of reaching a stable or stationary state,
3) those regarding the relationship between the convergence of the series and the stability of the system, and
4) those related to the calculation of Fourier coefficients.

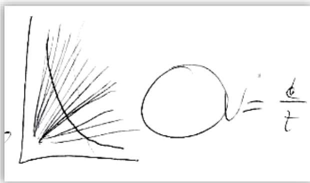
Regarding the invariants connected to focusing on the causes behind the system's behaviour, such as the variation in the sequence of radii and in the sequence of angular velocities, the results from Task #1 of the situation reveal interesting observations. Specifically, students used phrases like "they get smaller and smaller" to describe the sequence of radii and "they go faster and faster" to describe the sequence of angular velocities (see Table 1). However, this was not enough to conclude that the first sequence converges to zero and the second to infinity; instead, the invariant relationships established remained in terms of the monotonicity of the sequences. Mainly, since the radii sequence was decreasing, and the angular velocity sequence was increasing.

It was necessary, during the class-wide discussion in Task #5, for the instructor to question whether monotonicity alone a strong enough condition was to ensure system stability. For instance, the instructor asked what would happen if the sequence of radii decreased but was always bounded below by some positive number. This prompted students to identify the intended relationship.

Perhaps this preference for monotonicity over limits in sequences stemmed from the nature of the mathematical objects being studied. Since the radii are always positive, assuming a decreasing behaviour might have seemed sufficient to imply convergence to zero (a similar assumption may have been made for the angular velocities). It is important to pay attention to this phenomenon for future research or to consider redesigning the learning situation.

Table 1. Beginning of the generalization process related to the causes of the system's behaviour.

Actions What is done and how	**Reflection on the action system and establishment of invariants** What is it applied to?	**Symbolisation*** By means of what?
Observe Enables the perception of changes in the radii directly from the applet, without making a direct comparison between one radius and another — it is observed from the applet that the radii decrease. **Estimate**	*Elements of the action* The sequence of radii. *Invariant relationship* The measurement of the radii decreases.	Verbal: The radii. Algebraic: r_n or A_n. 

Attempts to quantify the variation based on perception from the applet/figure, without directly comparing one state to another — it is observed that the radii decrease by half; the closer to the axis, the faster the point moves.		Verbal: From the figure, it can be seen that the radii decrease. Algebraic: $r_n > r_{n+1}$ for all natural n. Algebraic: $\frac{R}{2^n}$, where R is the radius of the first circle.
Measure Directly measures the radii in order to compare them. **Compare** Compares the radii, either based on their measurements or by using the first radius as a reference. In the case of slopes, compares the slopes of the lines.	*Elements of the action* The sequence of angular velocities. *Invariant relationships* The speed of the points increases.	Verbal: The velocities. Algebraic: ω_n.  Verbal: The slope in the graph for the points on larger circles is greater than for those on circles with fewer subdivisions. Algebraic: $\omega_n < \omega_{n+1}$ for all natural n.

***Note:** This column contains the English translation of the participating students' responses and dialogues, with an effort to remain as faithful as possible to the original wording.
Source: Authors' own elaboration.

Regarding the recognition that the phenomenon is likely to reach a stable (or stationary) state, the results show that students were able to establish as an invariant relationship that, when many circles are added, the trajectory takes on a defined shape (see Table 2). In fact, as early as Task #1, students mention the notion of convergence in relation to this fact, even though the instructional design does not directly reference series convergence until Task #2 – Part II (specifically in question f).

We can observe how these two categories, focusing on the causes of the phenomenon's behaviour and recognizing that the phenomenon reaches a stable state, are organized in a way that facilitates what Dörfler (1991) refers to as an initial extensional generalization. This occurs at the end of Task #1, as the structure of the previously established invariants is preserved. This type of generalization consists of linking the causes of the phenomenon with its behaviour. Specifically, the students identified the following as an invariant relationship: *adding a very small circle and a point on it that moves very quickly does not significantly change the shape of the trajectory*. This relationship is associated with the action of comparing to infinity, which has a predictive character, as it does not only consider variation, but also the causes of that variation in order to describe its future behaviour, in this case, at infinity.

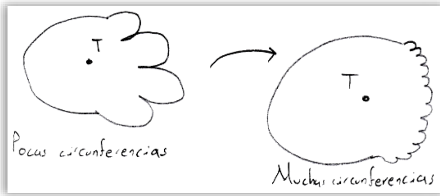
Later, in Task #5, the relationships concerning the causes of the phenomenon's behaviour appear as elements of the action of establishing necessary conditions for the system's stability, which gives them the status of object, thereby concluding the abstraction process that began with the identification of invariants.

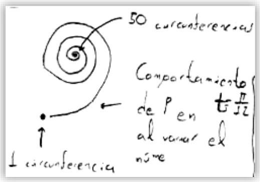
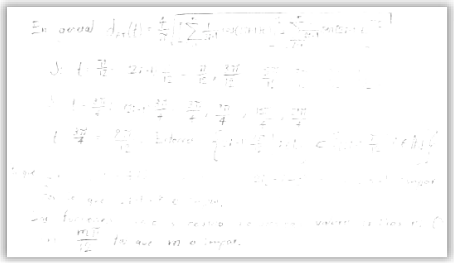
Table 2. Beginning of the generalization process related to the behaviour of the system.

<table>
<tr><th>Actions
What is done and how</th><th>Reflection on the action system and establishment of invariants
What is it applied to?</th><th>Symbolization*
By means of what?</th></tr>
<tr><td rowspan="2">Observe
Allows the perception of local changes in the trajectory directly from the applet, without making a direct comparison between one trajectory and another — it is observed from the applet that as the number of circles increases, there are more oscillations in the trajectory.

Compare locally
Allows the perception of local changes in the trajectory directly from the applet, making comparisons between one trajectory and another — the oscillations become more frequent and increasingly smaller.</td><td>Elements of the action
Trajectories with different numbers of circles.

Invariant relationship
The number of oscillations in the trajectory depends on the number of circles.</td><td>Verbal: The trajectory.
Task #1
Verbal: The number of oscillations increases as the number of circles increases.
Verbal: The number of loops increases, with one being larger and the rest decreasing in size.</td></tr>
<tr><td>Elements of the action
Trajectories with different numbers of circles.

Invariant relationships
The trajectory takes on a defined shape as more circles are added.</td><td>Verbal: The trajectory.
Task #1
Verbal: It seems that the trajectory tends towards an ellipse with the Earth at one of the foci.

Verbal: I observed an apparent convergence towards a very well-defined trajectory.
Verbal: Smaller and smaller arcs are added, which will model a smooth curve when a large number of circles is added.
Task #2
Verbal: The trajectory starts to take on a particular shape.
Verbal: In the central part, the loops seem to come closer together until they form two parallel lines.
Verbal: As more circles are added, the central part of the trajectory approaches a straight line, while the ends resemble circles.</td></tr>
</table>

Compare with infinity Enables the relation between the convergence of the series and the stability of the system. **Compare** Compares partial sums at specific moments in time, either using the applet or algebraically. This makes it possible to identify points in the trajectory where the system is not stable.	*Elements of the action* The planet, the Earth, and a given moment in time. *Invariant relationships* The distance between the planet and the Earth at a given moment in time.	Algebraic: Let P be the planet and T the Earth.  **Task #2** Verbal: For $t = \frac{\pi}{12}$, we observe a tendency towards stability, with the distance to the planet converging to a specific value.  Verbal: For $t = \pi$ y $t = 2\pi$, as more circles are added, the distance PT increases. Algebraic: 

***Note:** This column contains the English translation of the participating students' responses and dialogues, with an effort to remain as faithful as possible to the original wording.
Source: Authors' own elaboration.

For the category concerning the relationship between the convergence of the series and the stability of the system, the identified invariants provide evidence that the students were able to interpret the convergence of the series. Such interpretation was constructed by mathematizing the given model, in terms of the system's stability at specific time values. Then, they identified convergence through the study of the graphs of the sequence of partial sums (see Table 3). In the specific case of Task #1, it had been hypothesized that examining the partial sums might lead students to believe the series was divergent as it oscillates between the values 1 and -1 (Romero & Farfán, 2019). This indeed occurred during the implementation. However, identifying the nature of the convergence value (a function) enabled them to distinguish between the limit of functions (or sequences) and the convergence of a series of functions.

Once again, the categories of recognizing that the phenomenon reaches a stable state and linking the convergence of the series with the system's stability are structured in such a way that they support another initial extensional generalization in Task #2: "the series converges to a function." This is carried out through the action of comparing to infinity, which

CC BY NC ND

has a predictive character, as it enables the prediction of the behaviour of the physical phenomenon via its representation in terms of the TFS. The students indicated that its meaning lies in the fact that the formula describes the planet's trajectory.

Later, in Task #5, both the system's stability and the convergence value of the series (the function) appear as elements of the action recognizing periodic behaviour through the regularity of its repetition, which gives them the status of object, thus concluding the abstraction process that began with the identification of invariants.

It is important to emphasize that the generalization processes described thus far did not occur in a linear fashion; rather, they unfolded simultaneously during the development of the situation. Separating them using the categories as a guiding thread was a strategy to present them in an organized manner and to clearly reveal the fixation of invariants up to the point where symbols are considered as objects.

Table 3. Beginning of the generalization process related to the convergence of the series and the stability of the system.

<table>
<tr><th>Actions
What is done and how</th><th>Reflection on the action system and establishment of invariants
What is it applied to?</th><th>Symbolization*
By means of what?</th></tr>
<tr><td rowspan="2">Distinguish
Allows the perception of the relationship between the convergence of the series and the stability of the system for specific time values, that is, by studying particular numerical series.
Compare with infinity
Enables the connection between the convergence of the series and the convergence of the sequence of partial sums through their graphs.
Identify
Allows for differentiation between the notion of series convergence and that of the limit of functions, in order to accept that convergence occurs.</td><td>Elements of the action
Formula evaluated at specific time values.

Invariant relationship
The convergence of the series takes significance through the stability of the system.</td><td>Verbal: Plug t into the formula and check for convergence.

Task #2
Verbal:
H1: Ah::: it's that the formula explains::: the shape that the figure takes.
P: Right! And regarding what you said earlier about it taking on a defined shape?
H4: We'd need to look at the convergence values. Plug t into the formula and check for convergence.</td></tr>
<tr><td>Elements of the action
The convergence value.

Invariant relationships
The graph converges to a function.</td><td>Verbal: The convergence value.

Task #2
Verbal: It converges to a step function.
Verbal: It converges to a function.</td></tr>
</table>

***Note:** This column contains the English translation of the participating students' responses and dialogues, with an effort to remain as faithful as possible to the original wording.
Source: Authors' own elaboration.

CC BY NC ND

Regarding the calculation of the Fourier coefficients (Task #6 of the situation), there is no clear variation in the system of actions; only the initial part of the generalization process can be considered for each of the characterized actions (see Table 4).

Table 4. Beginning of the generalization process related to the calculation of the Fourier coefficients.

Actions What is done and how	**Reflection on the action system and establishment of invariants** What is it applied to?	**Symbolization*** By means of what?
Compare Compares the areas of the shaded regions. **Geometrise** Determines the area under the curve using known geometric figures.	*Elements of the action* The shaded regions. *Invariant relationship* The shaded regions have the same area.	Verbal: The shaded region. Verbal: The value of each shaded area is the same. Verbal: Both regions in the interval p have an area of 0, since they are symmetrical. Verbal: The sum of the areas of the double-shaded regions is equal to the sum of the areas of the single-shaded regions.
Calculate Determines the value of the areas under the curve as requested.	*Elements of the action* The region under the curve $y = \frac{a_0}{2}$ in the interval $[0, p]$. *Invariant relationships* The area under the curve $y = \frac{a_0}{2}$ in the interval $[0, p]$ is equal to the area of the rectangle whose sides measure p and $\frac{a_0}{2}$.	Verbal: The shaded region. Verbal: [VG]-6-|01:22:28|- $y = \frac{a_0}{2}$ the teacher asked who first used the rectangle argument, and H1 raised their hand. 2. Verbal: [VG]-6-|02:45:54| P: Did everyone do it that way? H9: Well::: yes, if it's by symmetry, the area ends up being the same, so you just calculate the area of the rectangle and divide it by 2.
	Elements of the action The region under the curve $y = a_1 \cos^2\left(\frac{2\pi}{p} t\right)$ in the interval $[0, p]$. *Invariant relationships* The value of the area under the curve over an interval of length p is equal to the value of the definite integral of that curve over the same interval.	Algebraic:

		$A_1 = \int_0^P y\,dt = \int_0^P \frac{a_0}{2}\,dt = \frac{a_0}{2}P$ $A_2 = \int_0^P a_1 \cos\left(\frac{2\pi}{P}t\right)dt = a_1 \frac{P}{2\pi} \operatorname{Sen}\left(\frac{2\pi}{P}t\right)\Big\vert_0^P = a_1\frac{P}{2\pi}\left(\operatorname{Sen} 2\pi - \operatorname{Sen} 0\right) = 0.$ $A_3 = \int_0^P b_1 \operatorname{Sen}\left(\frac{2\pi}{P}t\right)dt = -b_1\frac{P}{2\pi}\cos\left(\frac{2\pi t}{P}\right)\Big\vert_0^P = -b_1\frac{P}{2\pi}(1-1)=0$ $A_4 = \int_0^P a_2 \cos\left(\frac{4\pi}{P}t\right)dt = 0$ $A_5 = \int_0^P b_2 \operatorname{Sen}\left(\frac{4\pi}{P}t\right)dt = 0$ $A_6 = \int_0^P a_3 \cos\left(\frac{6\pi}{P}t\right)dt = 0$ $A_7 = \int_0^P b_3 \operatorname{Sen}\left(\frac{6\pi}{P}t\right)dt = 0$ Algebraic: $\cos^2\left(\frac{2\pi}{P}t\right) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{4\pi}{P}t\right).$ $\Rightarrow a_1\int_0^P \cos^2\left(\frac{2\pi}{P}t\right)dt = a_1\int_0^P \frac{1}{2}\,dt + \frac{a_1}{2}\int_0^P \cos\left(\frac{4\pi}{P}t\right)dt$ $= \frac{a_1 P}{2}.$

***Note:** This column contains the English translation of the participating students' responses and dialogues, with an effort to remain as faithful as possible to the original wording.
Source: Authors' own elaboration.

It is worth mentioning that this task aims to articulate the geometric–analytic and algebraic registers in order to validate the latter through the former. It was observed that, although this was achieved through the work on the situation, students still show a strong attachment to the algebraic register, since their initial strategies in most cases were algebraic (e.g., calculating integrals). This may be due to the fact that the situation attempted to handle both registers simultaneously. For instance, the graph was always presented along with its analytical expression, which should be taken into account in a redesign of the learning situation.

In this regard, a redesign could consider a purely geometric–analytic approach, that is, working with graphs without having access to the analytical expressions, and then moving on to algebraic work as a variation in the system of actions. This might help broaden the analysis within the generalization process.

Undoubtedly, more empirical research is needed to delve deeper into the object of study. The generalization processes related to the TFS seem to open a promising line of inquiry. It will be necessary to explore other variations in the system of actions and a diversity of mathematical knowledge. What lies ahead appears to be promising.

5. CONCLUSION

In line with the objective of the present paper, this study identified the components of the generalization process involved in the construction of the TFS through a learning situation grounded in a physical–geometric model of planetary motion. The qualitative, case-study-based approach enabled an in-depth analysis of student actions, their invariants, and the underlying reasoning structures developed throughout the tasks.

The results show that generalization does not occur linearly but rather emerges progressively and in parallel across different conceptual dimensions. The identification of action invariants allowed us to trace key transitions in the abstraction process. These were categorized into four domains: (1) identifying the causes behind system behaviour, (2) recognizing the potential for a stable or stationary state, (3) relating series convergence to system stability, and (4) understanding the role of Fourier coefficients.

In particular, the students' focus on monotonicity rather than limits when analyzing sequences in Task #1 revealed both intuitive reasoning and epistemological obstacles. Although they initially failed to infer convergence toward zero or infinity, class discussions (especially during Task #5) enabled refinement of their conceptual tools, supported by teacher questioning. This suggests the need for didactic designs that explicitly problematize such assumptions.

Moreover, students demonstrated an early understanding of system stability by recognizing that adding more circles in the geometric model produced a defined trajectory shape. This led to a first instance of what Dörfler (1991) describes as extensional generalization: the structure of the invariants is preserved while extending the system toward infinity. The convergence of the TFS (initially misunderstood in Task #1) was later correctly interpreted in Task #2 as convergence to a function, which the students linked to the trajectory of the planet. This was a significant step in abstracting mathematical objects from operational actions.

Task #6, focused on calculating Fourier coefficients, showed that students remained predominantly algebraic in their strategies, despite the situation's attempt to integrate geometric–analytic and algebraic representations. This preference for algebraic manipulation points to a need for more deliberate separation and sequencing of representational registers in future designs. A redesign that initially restricts access to analytic expressions, allowing students to rely solely on graphical information, might foster deeper engagement with geometric reasoning and support a richer process of generalization.

The study also underscores the potential of TFS-based analyses to illuminate subtle conceptual developments and support more robust modelling of generalization in mathematics education. Although this research was limited to a single case and specific context, the findings suggest fruitful directions for further empirical investigation. Future work could explore additional variations in the system of actions and extend the framework to other areas of mathematical knowledge.

In conclusion, the investigation highlights that fostering generalization requires more than exposing students to diverse mathematical content. It rather needs environments that allow for meaningful action, reflection, and abstraction. The promising results obtained here invite further research into TFS as both an analytical tool and a pedagogical objective.

BY NC ND

AUTHOR CONTRIBUTION STATEMENTS

FRF and RFM conceived the presented idea. FRF developed the theory. FRF adapted the methodology, carried out the activities, collected, and analyzed the data. All authors actively participated in the discussion of the results and reviewed and approved the final manuscript.

DATA AVAILABILITY STATEMENT

The data that support the findings of this study will be available from the corresponding authors, FRF and RFM, upon reasonable request.

5. REFERENCES

Albert, J. (1996). *La convergencia de series en el nivel superior: Una aproximación sistémica* [Unpublished doctoral dissertation]. Centro de Investigación y de Estudios Avanzados del IPN, México.

Bernoulli, D. (1755). Réflexions et éclaircissemens sur les nouvelles vibrations des cordes exposées dans les mémoires de l'Académie de 1747 & 1748. *Mémoires de l'Académie Royale des Sciences et Belles-Lettres, 9*, 147–172.

Cruz, M. (2019). *Linealidad y angularidad en la esfera: un nuevo escenario de trabajo geométrico* [Master's thesis]. Centro de Investigación y de Estudios Avanzados del IPN, México. doi:10.13140/RG.2.2.25114.49604

D'Alembert, J. (1747). Recherches sur la courbe que forme une corde tendue mise en vibration. *Histoire de l'Académie Royale des Sciences et des Belles-Lettres de Berlin, 3*, 214–219.

Dörfler, W. (1991). Forms and means of generalization in mathematics. In A. Bishop, S. Mellin-Olsen y J. Dormolen (Eds.), *Mathematical knowledge: its growth through teaching* (pp. 63-85). Kluwer Academic Publishers.

Farfán, R. (2012). *Socioepistemología y ciencia: el caso del estado estacionario y su matematización*. Gedisa S. A.

Farfán, R., & Romero, F. (2017). Construcción social del conocimiento matemático: La serie trigonométrica de Fourier desde la Socioepistemología. *Perspectivas da Educação Matemática – INMA/UFMS, 10*(23), 483–503.

Farfán, R., & Romero, F. (2019). Learning situation for the trigonometric Fourier series from a Socio-epistemological standpoint. *Acta Scientiae, 21*(2), 28–48. https://doi.org/10.17648/acta.scientiae.v21iss2id5019

Flores, R. (1992). *Sobre la construcción del concepto de convergencia en relación al manejo heurístico de los criterios* [Unpublished master's thesis]. Centro de Investigación y de Estudios Avanzados del IPN, México.

Fourier, J. (1822). *Théorie analytique de la chaleur* (Reimpressions Editions Jacques Gabay (1988) ed.). París: Chez Firmin Didot, père et fils. Libraires pour les mathématiques, l'architecture hydraulique et la marine. Rue Jacop. No. 24.

CC BY NC ND

Friedman, R. (1977). The Creation of a New Science: Joseph Fourier's Analytical Theory of Heat. *Historical Studies in the Physical Sciences, 8*, 73–99. https://doi.proxyucr.elogim.com/10.2307/27757368

Hernández, R., Fernández, C., & Baptista, M. (2014). *Metodología de la investigación* (6ta ed.). McGraw Hill.

Marmolejo, R. (2006). *Estudio de la noción de estado estacionario en el ámbito fenomenológico de la transferencia de calor* [Unpublished master's thesis]. Centro de Investigación y de Estudios Avanzados del IPN, México.

Montiel, G. (2011). *Construcción de conocimiento trigonométrico: Un estudio socioepistemológico*. Díaz de Santos.

Morales, F. (2010). *Causas y efectos de la ambigüedad en el tratamiento didáctico de la noción de calor. Una caracterización del pensamiento fisicomatemático* [Unpublished doctoral dissertation]. Centro de Investigación y de Estudios Avanzados del IPN, México.

Piaget, J. (2009). *Psicología de la inteligencia* (Juan C. Foix, trad.). Editorial Crítica. (Original work published in 1947).

Romero, F. (2016). *Construcción social de la serie trigonométrica de Fourier: Pautas para un diseño de intervención en el aula* [Master's thesis]. Centro de Investigación y de Estudios Avanzados del IPN, México. doi:10.13140/RG.2.2.14118.63048

Romero, F. (2020). *Sobre los Procesos de Generalización en Entornos de Construcción Social de Conocimiento: el caso de la serie trigonométrica de Fourier* [Doctoral dissertation]. Centro de Investigación y de Estudios Avanzados del IPN, México. doi: 10.13140/RG.2.2.11916.54405

Romero, F., & Farfán, R. (2016). Estado actual de la investigación alrededor de la serie trigonométrica de Fourier. In F. Rodríguez, R. Rodríguez, y L. Sosa (Eds.), *Investigación e Innovación en Matemática Educativa, 1*, pp. 275–282. Oaxaca, México: Red de Centros de Investigación en Matemática Educativa A. C.

Romero, F., & Farfán, R. (2018). Cálculo de los coeficientes de Fourier. *Acta Latinoamericana de Matemática Educativa, 31*(2), 1567–1575.

Solís, M. (1993). *Estudio de la noción de variación en contextos físicos: El fenómeno de la propagación de calor* [Unpublished master's thesis]. Centro de Investigación y de Estudios Avanzados del IPN, México.

Ulín, C. (1984). *Análisis histórico-crítico de la difusión de calor: el trabajo de Fourier* [Unpublished master's thesis]. Centro de Investigación y de Estudios Avanzados del IPN, México.

Vásquez, R. (2006). *Sobre el papel de la hipótesis de periodicidad en las series de Fourier* [Unpublished master's thesis]. Centro de Investigación y de Estudios Avanzados del IPN, México.

Yin, R. (2017). *Case Study Research and Applications: Design and Methods*. Washington DC: Sage Publications.