Revista de Matemática: Teoría y Aplicaciones **33**(2): 107-156
CIMPA - UCR ISSN: 1409-2433(Print), 2215-3373(Online)
DOI: https://doi.org/10.15517/vv48fd28

CHARACTER VARIETIES AND THEIR RELATION WITH PRINCIPAL HIGGS BUNDLES

VARIEDADES DE CARACTERES Y SU RELACIÓN CON LOS FIBRADOS PRINCIPALES DE HIGGS

ANA CRISTINA CASIMIRO[1]

Received: 26/Nov/2025; Accepted: 12/Jun/2026

---

[1]NOVA School of Science and Technology (NOVA FCT), Center for Mathematics and Applications (NOVA Math) and Department of Mathematics, Caparica, Portugal. E-Mail: amc@fct.unl.pt

Abstract

We begin by reviewing the notions of Lie groups and algebraic groups, both real and complex, as well as finitely generated groups, which are essential components in the definition of a character variety. We then analyze the space of representations of a finitely generated group into a Lie or an algebraic group, focusing on its topology, variety structure, tangent space, and the conjugation action by the target group. Next, we construct the quotient with respect to this action, namely the character variety, and present different approaches to this construction. We also examine the existence of deformation retractions between character varieties when considering maximal compact subgroups of the Lie group. Finally, we discuss the correspondence between representations of surface groups and principal bundles, with particular emphasis on Schottky representations, and relate these constructions to the geometric setting of principal Higgs bundles and branes. This perspective suggests further directions connecting representation theory, non-Abelian Hodge correspondence, and the geometry of moduli spaces.

Keywords: character varieties; deformation retraction; principal bundles; principal Higgs bundles; branes; Schottky representations.

Resumen

En primer lugar, se revisan las nociones de grupos de Lie y grupos algebraicos, tanto reales como complejos, así como de grupos finitamente generados, que constituyen ingredientes esenciales en la definición de una variedad de caracteres. A continuación, se analiza el espacio de representaciones de un grupo finitamente generado en un grupo de Lie o en un grupo algebraico, atendiendo a su topología, su estructura de variedad, su espacio tangente y la acción por conjugación de este último grupo. Posteriormente, se construye el cociente respecto de esta acción, la variedad de caracteres, presentando distintos enfoques para dicha construcción. Asimismo, se estudia la existencia de retracciones por deformación entre variedades de caracteres al considerar subgrupos compactos máximos del grupo de Lie. Finalmente, se aborda la correspondencia entre representaciones de grupos de superficies y fibrados principales, con especial atención a las representaciones de Schottky, y se relacionan estas construcciones con el marco geométrico de los fibrados principales de Higgs y las branas. Esta perspectiva sugiere nuevas direcciones que conectan la teoría de representaciones, la correspondencia de Hodge no abeliana y la geometría de los espacios de módulos.

Palabras clave: variedades de caracteres; retracción de deformación; fibrados principales; fibrados principales de Higgs; branas; representaciones de Schottky.

Mathematics Subject Classification: Primary 14M35; Secondary 20F38, 14R20, 14P25, 14H70, 30F10.

1. Introduction

The study of *character varieties* lies at the crossroads of several areas of modern geometry, including algebraic geometry, differential geometry, and representation theory. Roughly speaking, a character variety is a space that parametrizes group representations of the fundamental group of a topological space—typically a compact Riemann surface—into a Lie group G, modulo conjugation. These spaces naturally appear in geometry, topology, and mathematical physics, and play a central role in gauge theory, moduli problems, and non-Abelian Hodge theory (see, e.g., [11, 14, 22, 49]). A first motivation comes from the correspondence between representations and flat principal bundles. Given a compact Riemann surface X and a complex Lie group G, every representation of $\pi_1(X)$ into G defines a flat G-bundle on X, and conversely, every flat G-bundle determines such a representation up to conjugation. This correspondence provides a bridge between the algebraic viewpoint (via homomorphisms of the fundamental group) and the geometric viewpoint (via bundles with connection). This assignment offers a powerful geometric framework for studying moduli spaces of representations and their deformation theory. The next step in this geometric interpretation is the introduction of *principal Higgs bundles*, which enrich the structure of a principal bundle with a Higgs field, providing a holomorphic analogue of a flat connection. The moduli space of G-Higgs bundles carries a natural hyperkähler structure, as shown by Hitchin [26], and serves as one of the most remarkable examples of such spaces. Through the celebrated non-Abelian Hodge theorem ([11, 14, 26, 49]), one establishes a diffeomorphism between the moduli space of G-Higgs bundles and the moduli space of reductive representations of $\pi_1(X)$ in G. This correspondence unifies seemingly distinct moduli problems and allows techniques from complex geometry, topology, and differential equations to interact fruitfully. Within this setting, new geometric and physical phenomena emerge. In particular, certain submanifolds of the hyperkähler moduli space, known as *branes*, play a central role in mirror symmetry and in the geometric Langlands program. Following ideas of Kapustin and Witten [29], Baraglia and Schaposnik [1] constructed families of such branes using anti-holomorphic involutions of the Riemann surface X, revealing deep connections between real structures, moduli spaces, and 3-dimensional topology.

The present work aims to outline the geometric relations between character varieties, principal bundles, and principal Higgs bundles, emphasizing how these objects interact through topological and analytical constructions. After reviewing the correspondence between representations and principal bundles, we describe the moduli of principal Higgs bundles and the notion of branes, and finally discuss how Schottky representations fit naturally into this framework, leading to concrete examples of (A, B, A)-branes in the sense of Baraglia and Schaposnik.

Structure of the paper. The paper is organized into five main sections, each building on the previous one to guide the reader from the algebraic and differential foundations of Lie theory to the geometric interpretation of representation and character varieties, and finally to their connections with principal bundles, principal Higgs bundles, and branes. Section 2 gathers the necessary background material. It begins with an overview of Lie groups and their algebraic and topological properties, followed by a discussion of adjoint representations and the classification of Lie algebras. This section also introduces algebraic groups and finitely generated groups, providing the foundational framework for the constructions developed later on. Section 3 is devoted to representation varieties. It defines the space of representations $\mathsf{Hom}(\Gamma, G)$ of a finitely generated group Γ into a Lie group G and explores its main structural properties. The conjugation action of G on this space is analyzed in detail, together with the description of tangent spaces, the identification of smooth points, and the characterization of orbits. Several important subsets of representations—such as irreducible, reductive, or stable ones—are introduced and studied in this context. Section 4 concerns the construction of character varieties as quotients of representation spaces. We begin with the discussion of topological quotients, including Hausdorff and $\mathcal{T}_1$ quotients, and then pass to the affine (GIT) quotient $\mathsf{Hom}(\Gamma, G)/\!/G$. This section also examines strong deformation retractions that connect the complex and compact settings, highlighting the topological relationship between the analytic and algebraic viewpoints. Section 5 establishes the link between representations and differential geometry through the study of principal G-bundles. After recalling basic notions on principal bundles and their connections, we explain how flat G-bundles can be parametrized by representations of $\pi_1(X)$. We then introduce Schottky representations as a particularly tractable family, leading to their geometric interpretation within the moduli of principal bundles. The section concludes with the theory of G-Higgs bundles and the study of the Baraglia–Schaposnik branes, culminating in the inclusion of Schottky representations inside these branes. This final discussion reveals a deep interplay between three-dimensional topology, brane geometry, and representation theory, pointing to further avenues for research within the framework of non-Abelian Hodge theory.

2. Preliminaries

In this section, we recall several foundational concepts that will be used throughout the paper. Our aim is not to develop the full theory, but rather to establish notation and emphasize the structures that play a central role in the study of representation and character varieties. We begin with Lie groups and their Lie algebras, proceed to algebraic groups, and conclude with finitely generated and surface groups. Together, these notions provide the algebraic and topological backbone for the constructions developed in the following sections. The exposition follows the clear survey of Arnaut Maret [37], complemented by classical references such as [5, 25, 27, 33, 39, 40].

2.1. Lie groups.

We start with the basic object linking differential geometry and group theory.

Definition 1. A *Lie group* G is a real (or complex) smooth manifold equipped with a group structure such that the group operations, called multiplication and inversion,

$$m : G \times G \to G, \quad (g, h) \mapsto gh, \qquad \text{and} \qquad i : G \to G, \quad g \mapsto g^{-1},$$

are smooth (resp. holomorphic) maps.

Every Lie group admits a unique (up to diffeomorphism) analytic structure for which the group operations are analytic. Let G° denote the identity component of a Lie group G. For a subset $S \subseteq G$, the *centralizer* of S in G is the Lie subgroup

$$Z(S) := \{g \in G : gsg^{-1} = s \ \text{ for all } s \in S\}.$$

In particular, the *center* of G is $Z(G)$.

To build intuition, we recall some of the main examples of matrix Lie groups that will appear repeatedly in what follows.

Example 1. For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $n,\ p,\ q \in \mathbb{N}$,

1. The *general linear groups*:

$$\mathrm{GL}(n, \mathbb{K}) = \{M \in \mathrm{Mat}_{n\times n}(\mathbb{K}) : \det(M) \neq 0\},$$

consisting of all invertible real (resp. complex) matrices, i.e. all linear automorphisms of $\mathbb{R}^n$ (resp. $\mathbb{C}^n$).

2. The *special linear group*,

$$\mathrm{SL}(n, \mathbb{K}) = \{M \in \mathrm{GL}(n, \mathbb{K}) : \det(M) = 1\},$$

formed by the volume-preserving linear transformations.

3. The *orthogonal* and *special orthogonal groups*,

$$\mathrm{O}(n, \mathbb{R}) = \{M \in \mathrm{GL}(n, \mathbb{R}) : M^\top M = I_n\}, \qquad \mathrm{SO}(n, \mathbb{R}) = \mathrm{O}(n, \mathbb{R}) \cap \mathrm{SL}(n, \mathbb{R}),$$

consisting respectively of the transformations preserving the Euclidean inner product and its connected component of determinant one. Here $M^\top$ denotes the transpose of M.

4. The *special unitary* and *symplectic* groups

$$\begin{aligned} \mathrm{SU}(p, q) &= \{M \in \mathrm{SL}(p+q, \mathbb{C}) : M^* I_{p,q} M = I_{p,q}\}, \\ \mathrm{Sp}(2n, \mathbb{R}) &= \{M \in \mathrm{GL}(2n, \mathbb{R}) : M^\top J M = J\}, \end{aligned}$$

where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ define the standard Hermitian or symplectic form, respectively. The group $\mathrm{SU}(p, q)$ preserves a Hermitian form of signature (p, q), while $\mathrm{Sp}(2n, \mathbb{R})$ consists of linear maps preserving a symplectic structure. Here M^* denotes the conjugate transpose of M.

5. Finally, the *complex orthogonal group*,

$$\mathrm{SO}(n,\mathbb{C}) = \{M \in \mathrm{SL}(n,\mathbb{C}) : M^\top M = I_n\},$$

provides a complex counterpart of the real orthogonal group.

Not all Lie groups are linear: for instance, the universal covering group of $\mathrm{SL}(2,\mathbb{R})$ cannot be realized as a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ for any n. This illustrates how the analytic viewpoint extends beyond purely algebraic realizations.

Passing from a Lie group to its quotient by the center $Z(G)$ leads to the *adjoint Lie group* of G. For matrix groups, this is often denoted by prefixing with a "P", such as $\mathrm{PGL}(n,\mathbb{R})$ or $\mathrm{PSL}(n,\mathbb{R})$.

At the infinitesimal level, the *Lie algebra* $\mathfrak{g}$ of G is defined as the tangent space at the identity, endowed with the Lie bracket arising from the commutator of left-invariant vector fields (see Milne [39, Chapter 5]). A key link between the algebraic and differential structures of G is given by the exponential map

$$\exp : \mathfrak{g} \longrightarrow G,$$

which locally provides a diffeomorphism near the identity.

The *adjoint representation* of G on $\mathfrak{g}$,

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g}), \qquad \mathrm{Ad}(g)(v) = \left.\frac{d}{dt}\right|_{t=0} g\exp(tv)g^{-1},$$

plays a central role in representation theory and differential geometry. Differentiating Ad at the identity yields the *adjoint representation of the Lie algebra*,

$$\mathrm{ad} : \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}), \qquad \mathrm{ad}(v_1)(v_2) = [v_1, v_2].$$

The kernel of ad is the center $\mathfrak{z}(\mathfrak{g})$, which coincides with the Lie algebra of $Z(G)$.

Lie algebras admit a well-understood structural classification, essential for understanding the corresponding Lie groups (see, for instance, [27, §5–6] and [39, Ch. 6]):

1. A *simple* Lie algebra $\mathfrak{g}$ is a non-abelian Lie algebra with no nontrivial ideals. Equivalently, its adjoint representation is irreducible and $\dim\mathfrak{g} > 1$.

2. A *semisimple* Lie algebra $\mathfrak{g}$ is one without nonzero abelian ideals; equivalently, it is a direct sum of simple Lie algebras. The *Killing form*

$$K(v_1, v_2) := \mathrm{Tr}(\mathrm{ad}(v_1)\,\mathrm{ad}(v_2))$$

for v_1, $v_2 \in \mathfrak{g}$, where Tr is the endomorphism trace, provides a symmetric bilinear form that is non-degenerate precisely when $\mathfrak{g}$ is semisimple.

3. A *reductive* Lie algebra is a direct sum of a semisimple and an abelian algebra, or equivalently, one whose adjoint representation is completely reducible.

A Lie group is said to be simple, semisimple or reductive if its Lie algebra is simple, semisimple or reductive, respectively.

Example 2. 1. The groups $\mathrm{SL}(n,\mathbb{R})$ $(n \geqslant 2)$, $\mathrm{Sp}(2n,\mathbb{R})$, and $\mathrm{SU}(p,q)$ $(p+q \geqslant 2)$ are simple. The identity component of $\mathrm{SO}(n,\mathbb{R})$ is simple for $n \geqslant 3$, except $n = 4$, where it is semisimple since its Lie algebra $\mathfrak{so}(4) \cong \mathfrak{so}(3) \oplus \mathfrak{so}(3)$.

2. The identity component of $\mathrm{GL}(n,\mathbb{R})$ is reductive but not semisimple; its Lie algebra decomposes as $\mathfrak{gl}(n,\mathbb{R}) = \mathfrak{sl}(n,\mathbb{R}) \oplus \mathbb{R} \cdot I_n$, where $\mathfrak{sl}(n,\mathbb{R})$ is the Lie algebra of $\mathrm{SL}(n,\mathbb{R})$.

3. For $\mathrm{SL}(2,\mathbb{R})$, the trace form $\mathrm{Tr}(v_1 v_2)$ is Ad-invariant and non-degenerate, of signature $(2,1)$ with respect to the standard basis of $\mathfrak{sl}(2,\mathbb{R})$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

2.2. Algebraic groups.

We now move to the algebraic category. Algebraic groups arise naturally in representation theory, providing the algebraic-geometric framework for studying character varieties.

Recall that the Zariski topology on an algebraic variety is defined by taking as closed sets the common zero loci of collections of polynomial functions. This topology is much coarser than the (standard) analytic topology and is the natural topology in algebraic geometry. In particular, every Zariski-open set is open in the analytic topology, although the converse does not hold. The analytic topology is Hausdorff, meaning that any two distinct points admit disjoint open neighbourhoods, whereas the Zariski topology is typically not. Moreover, on an irreducible algebraic variety, every nonempty Zariski-open subset is dense, both in the Zariski and in the analytic topology.

A map $f : X \to Y$ between algebraic varieties is called a *regular morphism* (or simply a *morphism*) if, locally on X, its coordinate functions are given by polynomial functions.

Definition 2. An *affine algebraic group* G is an algebraic variety (the common zero set of finitely many polynomials) equipped with a group structure such that multiplication and inversion,

$$m : G \times G \to G, \quad (g,h) \mapsto gh, \qquad \text{and} \qquad i : G \to G, \quad g \mapsto g^{-1},$$

are regular morphisms of varieties.

The Zariski closure of a subgroup of an algebraic group is again an algebraic group, hence algebraic subgroups are Zariski closed ([5, Ch. I, §1]). Every real or complex algebraic group also has a natural Lie group structure.

Example 3. 1. The classical matrix groups in Example 1 are affine algebraic groups, called *linear algebraic groups*.

2. Elliptic curves, that is, Riemann surfaces of genus 1, with a choice of a base point, provide examples of projective (non-affine and non-linear) algebraic groups.

3. $\mathrm{PGL}(n,\mathbb{R})$, $\mathrm{PGL}(n,\mathbb{C}) = \mathrm{PSL}(n,\mathbb{C})$ and, for odd n, $\mathrm{PSL}(n,\mathbb{R}) = \mathrm{PGL}(n,\mathbb{R})$ are algebraic groups. Nevertheless, for even n, $\mathrm{PSL}(n,\mathbb{R})$ is only semialgebraic (defined also by polynomial inequalities).

Proposition 1 ([5, 27])**.** *Every algebraic group contains a unique maximal connected solvable normal subgroup, called the* radical*. A connected complex algebraic group is reductive if and only if its radical is isomorphic to* $(\mathbb{C}^*)^n$ *for some* $n \geqslant 0$*, and semisimple if and only if its radical is trivial.*

Example 4. The group $\mathrm{SL}(2,\mathbb{C})$ is a semisimple algebraic group of complex dimension 3, simple as a Lie group, and irreducible as a variety, with center $Z(\mathrm{SL}(2,\mathbb{C})) = \{\pm I_2\}$.

2.3. Finitely generated groups.

Topological and geometric contexts often lead to finitely generated (and finitely presented) groups, which will serve as the source of representations into Lie or algebraic groups.

Definition 3. A group Γ is *finitely generated* if it admits a finite set of generators S such that every element of Γ is a finite product of elements of S and their inverses.

Example 5. 1. Finite groups, e.g. the cyclic groups $\mathbb{Z}/n\mathbb{Z}$.

2. The infinite cyclic group $\mathbb{Z}$.

3. The symmetric and dihedral groups, and more generally, Coxeter groups.

4. Free groups F_n on n generators.

A particularly important family are the *surface groups*, which capture the topology of Riemann surfaces.

Definition 4 (Surface Groups)**.** Let $g, n \geqslant 0$. The *surface group* of genus g with n punctures is

$$\pi_{g,n} := \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_n \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] = \prod_{j=1}^{n} c_j \right\rangle,$$

where $[a_i, b_i] = a_i b_i a_i^{-1} b_i^{-1}$ for $i = 1, \ldots, g$.

When $n = 0$, this corresponds to the fundamental group of a closed surface of genus g. For $n \geqslant 1$, the relation implies $\pi_{g,n}$ is a free group of rank $2g + n - 1$. The groups $\pi_{0,0}$ and $\pi_{0,1}$ are trivial. Only $\pi_{0,2} \cong \mathbb{Z}$ and the torus group $\pi_{1,0} \cong \mathbb{Z}^2$ are abelian; higher genus groups are non-abelian.

Theorem 1. *Let $\Sigma_{g,n}$ be an orientable surface of genus g with n punctures. Then $\pi_1(\Sigma_{g,n}) \cong \pi_{g,n}$.*

Proof. For $n = 0$, see Theorem 2.3.15 of [33]. For $n \geqslant 1$, a sphere with n punctures is homotopy equivalent to a wedge of $n-1$ circles, while a genus-g surface with one puncture retracts to a wedge of $2g$ circles. Using the decomposition $\Sigma_{g,n} = \Sigma_{g,1} \cup \Sigma_{0,n+1}$ and Van Kampen's theorem (see [25]), the claimed presentation follows. □

The Lie or algebraic groups G, together with the finitely generated groups Γ, introduced in this section, will interact in the next chapters through the study of homomorphisms $\rho : \Gamma \to G$, whose geometric and algebraic quotients define the representation and character varieties central to this work.

3. Representation Varieties

Let Γ be a finitely generated group, and let G be a (real or complex) Lie group. The main object of study in this section is the set of homomorphisms from Γ to G.

Definition 5. For a finitely generated group Γ and a Lie group G, the *representation variety* associated with Γ and G, denoted by $\mathsf{Hom}(\Gamma, G)$, is defined as

$$\mathsf{Hom}(\Gamma, G) = \{\rho : \Gamma \to G \mid \rho \text{ is a group homomorphism}\}.$$

Topology. The space $\mathsf{Hom}(\Gamma, G)$ carries the subspace topology induced from the compact-open topology on the space of continuous maps $\Gamma \to G$, where Γ is endowed with the discrete topology. A sub-basis for this topology is given by sets of the form

$$V(K, U) := \{f : \Gamma \to G \mid K \subset \Gamma \text{ finite},\; U \subset G \text{ open},\; f(K) \subset U\}.$$

Fixing a generating set $\{\gamma_1, \ldots, \gamma_n\}$ of Γ, one considers the following subset of G^n:

$$X(\Gamma, G) = \{(\rho(\gamma_1), \ldots, \rho(\gamma_n)) \mid \rho \in \mathsf{Hom}(\Gamma, G)\} \subset G^n.$$

Proposition 2. *If G is a Lie group endowed with an analytic atlas, then $X(\Gamma, G)$ is an analytic subvariety of G^n, homeomorphic to $\mathsf{Hom}(\Gamma, G)$. In particular, $\mathsf{Hom}(\Gamma, G)$ inherits a natural structure of analytic variety, which is independent of the choice of generators of Γ.*

Proof. Let $\{\gamma_1, \ldots, \gamma_n\}$ be generators of Γ, and let $\{r_i\}$ denote a set of relations (possibly infinite). Each relation r_i can be interpreted as an analytic map $r_i : G^n \to G$, since it is obtained using multiplication and inversion in G, which are analytic operations. Then $X(\Gamma, G)$ is precisely the subset of $(g_1, \ldots, g_n) \in G^n$ defined by the conditions $r_i(g_1, \ldots, g_n) = 1$, hence an analytic subvariety. Every homomorphism $\rho : \Gamma \to G$ is uniquely determined by its values in the chosen generators. This yields a bijection

$$\Psi : \mathsf{Hom}(\Gamma, G) \longrightarrow X(\Gamma, G), \qquad \rho \longmapsto (\rho(\gamma_1), \ldots, \rho(\gamma_n)).$$

To see that Ψ is a homeomorphism, consider open sets $U_1, \ldots, U_n \subset G$. Then

$$\Psi^{-1}(X(\Gamma, G) \cap (U_1 \times \cdots \times U_n)) = \mathsf{Hom}(\Gamma, G) \cap \bigcap_{i=1}^{n} V(\{\gamma_i\}, U_i).$$

Conversely, for a finite set $K \subset \Gamma$ and $U \subset G$ open, any $\gamma \in K$ can be expressed as a word in the generators, so the evaluation at γ defines an analytic map $\gamma : G^n \to G$. Hence,

$$\Psi(\mathsf{Hom}(\Gamma, G) \cap V(K, U)) = X(\Gamma, G) \cap \bigcap_{\gamma \in K} \gamma^{-1}(U).$$

This shows that both Ψ and Ψ^{-1} are continuous, and therefore Ψ is a homeomorphism.

Finally, suppose $\{\gamma'_1, \ldots, \gamma'_m\}$ is another generating set of Γ, and let $X'(\Gamma, G)$ be defined analogously. The natural composition

$$X(\Gamma, G) \to \mathsf{Hom}(\Gamma, G) \to X'(\Gamma, G)$$

is an isomorphism of analytic varieties, since each γ'_i is a word in γ_j, and thus $\rho(\gamma'_i)$ is an analytic expression in $\rho(\gamma_j)$. □

Proposition 3. *If G is a real or complex algebraic group, then $X(\Gamma, G)$ is an algebraic subvariety of G^n. In particular, $\mathsf{Hom}(\Gamma, G)$ carries a natural structure of real or complex algebraic variety, independent of the choice of generators of Γ.*

Remark 1.
1. Since Γ may be defined by infinitely many relations, the defining equations for $X(\Gamma, G)$ may also be infinite in number. However, by Hilbert's basis theorem, every algebraic variety over a field can be described as the common zero locus of finitely many polynomials. Thus, $X(\Gamma, G)$ is cut out by finitely many equations.
2. If G is a real or complex algebraic group, then it is also a Lie group. Hence, $\mathsf{Hom}(\Gamma, G)$ acquires two natural structures: as an analytic variety (with the standard topology) and as an algebraic variety (with the Zariski topology).

3.1. Symmetries: conjugation action of G.

We now describe natural symmetries of $\mathsf{Hom}(\Gamma, G)$. Two obvious actions are: the *right action* of $\mathrm{Aut}(\Gamma)$ by pre-composition and the *left action* of $\mathrm{Aut}(G)$ by post-composition. Both actions preserve the analytic or algebraic structure of $\mathsf{Hom}(\Gamma, G)$, as follows from the next general statement.

Proposition 4. *Let $\Gamma, \Gamma_1, \Gamma_2$ be finitely generated groups, and let G, G_1, G_2 be Lie or algebraic groups.*

1. *If $f : \Gamma_1 \to \Gamma_2$ is a homomorphism, then the induced map*
$$f^* : \mathsf{Hom}(\Gamma_2, G) \to \mathsf{Hom}(\Gamma_1, G), \qquad \rho \mapsto \rho \circ f$$
is analytic (or regular).
2. *If $\phi : G_1 \to G_2$ is a homomorphism, then the induced map*
$$\phi_* : \mathsf{Hom}(\Gamma, G_1) \to \mathsf{Hom}(\Gamma, G_2), \qquad \rho \mapsto \phi \circ \rho$$
is analytic (or regular).

Among automorphisms of G, a particularly important normal subgroup is formed by the *inner automorphisms*, defined by conjugation with elements of G and denoted by $\mathrm{Inn}(G)$. Explicitly, given $\rho \in \mathsf{Hom}(\Gamma, G)$ and $g \in G$, one defines
$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1}, \qquad \gamma \in \Gamma.$$
Conjugation by elements of the center $Z(G)$ acts trivially, so $\mathrm{Inn}(G) \cong G/Z(G)$.

When G is semisimple, $\mathrm{Inn}(G)$ has finite index in $\mathrm{Aut}(G)$, the automorphism group of G. Indeed, if G is simply connected, the differential map induces an isomorphism $\mathrm{Aut}(G) \to \mathrm{Aut}(\mathfrak{g})$ at the level of Lie algebras. Since for a semisimple Lie algebra $\mathfrak{g}$ one has $\mathrm{Lie}(\mathrm{Inn}(\mathfrak{g})) \cong \mathrm{Lie}(\mathrm{Aut}(\mathfrak{g}))$, it follows that $\mathrm{Inn}(\mathfrak{g})$ is of finite index in $\mathrm{Aut}(\mathfrak{g})$, and hence the same holds for the groups. If G is not simply connected, one reduces to this case via the universal cover of G.

The action of $\mathrm{Inn}(G)$ on $\mathsf{Hom}(\Gamma, G)$ is fundamental in many geometric situations: for example, holonomy representations of surface structures are only well-defined up to conjugation by G.

Example 6. ([23, Ch. 1]) Let X be a manifold and let G be a Lie group acting transitively on X. A (G, X)-structure on a manifold M is an atlas with values in X whose transition maps are restrictions of elements of G. Such structures naturally give rise to representations of surface groups into Lie groups. Indeed, if $\Sigma_{g,n}$ is endowed with a (G, X)-structure, then there exists a developing map
$$\mathrm{dev} : \widetilde{\Sigma}_{g,n} \to X$$

which is a local diffeomorphism from the universal cover, $\widetilde{\Sigma}_{g,n}$, of $\Sigma_{g,n}$ to X, and a holonomy representation

$$\mathrm{hol} : \pi_1(\Sigma_{g,n}) \to G,$$

obtained by transporting the local charts along loops in $\Sigma_{g,n}$. These satisfy the equivariance relation

$$\mathrm{dev}(\gamma \cdot x) = \mathrm{hol}(\gamma) \cdot \mathrm{dev}(x), \qquad \gamma \in \pi_1(\Sigma_{g,n}).$$

The holonomy representation is well defined up to conjugation in G.

For

$$(G, X) = (\mathrm{PSL}(2, \mathbb{R}), \mathbb{H}^2),$$

the corresponding (G, X)-structures are precisely hyperbolic structures. If $g \geqslant 2$, the holonomies of hyperbolic structures on the closed surface $\Sigma_{g,0}$ are exactly the discrete and faithful representations in

$$\mathsf{Hom}(\pi_1(\Sigma_{g,0}), \mathrm{PSL}(2, \mathbb{R})).$$

This motivates the study of the quotient

$$\mathsf{Hom}(\Gamma, G)/\mathrm{Inn}(G),$$

which is a precursor of the character variety associated with Γ and G.

Example 7. $\Gamma = \mathbb{Z}^2$, $G = \mathrm{SL}(2, \mathbb{C})$. Let $\Gamma = \langle a, b \mid ab = ba \rangle \cong \mathbb{Z}^2$ and $G = \mathrm{SL}(2, \mathbb{C})$. Then $\mathsf{Hom}(\Gamma, G) = \{(\rho(a), \rho(b)) \in G^2 \mid \rho(a)\rho(b) = \rho(b)\rho(a)\}$.

Since a and b commute, $\rho(a)$ and $\rho(b)$ must be simultaneously triangularizable. Up to conjugation, we may assume

$$\rho(a) = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \rho(b) = \begin{pmatrix} \mu & \beta \\ 0 & \mu^{-1} \end{pmatrix}, \quad \lambda, \mu \in \mathbb{C}^*,\ \alpha, \beta \in \mathbb{C}$$

with $(\lambda - \lambda^{-1})\beta = (\mu - \mu^{-1})\alpha$. The conjugation action identifies $(\rho(a), \rho(b))$ up to simultaneous conjugation. The quotient

$$\mathsf{Hom}(\mathbb{Z}^2, G)/\mathrm{Inn}(G)$$

can be described algebraically as $\{(\lambda + \lambda^{-1}, \mu + \mu^{-1}, \lambda\mu + (\lambda\mu)^{-1}) \in \mathbb{C}^3\}$.

This example shows how the representation variety $\mathsf{Hom}(\Gamma, G)$ can be embedded in G^n and then quotiented by conjugation.

3.2. Tangent spaces.

In this subsection, we study the tangent spaces of representation varieties. Recall that these varieties are not smooth in general. Hence, we must fix a notion of tangent space suitable for this context, independent of the choice of generators of the finitely generated group Γ. To achieve such a definition, we use the framework of ringed spaces (see [30, 34]).

Recall that a *sheaf* on a topological space X assigns to each open subset $U \subseteq X$ a collection $\mathscr{O}(U)$ of objects (in this context, real-valued functions on U) together with restriction maps

$$\mathscr{O}(U) \to \mathscr{O}(V), \qquad V \subseteq U,$$

satisfying the usual locality and gluing conditions: functions that agree locally agree globally, and compatible local functions glue to a global one. For $x \in X$, the *stalk* $\mathscr{O}_x$ is the set of germs of sections near x, that is, equivalence classes of local functions defined in neighbourhoods of x, where two such functions are identified if they agree on some neighbourhood of x.

Definition 6. A *real-valued ringed space* is a topological space X equipped with a sheaf $\mathscr{O}$ of real-valued functions, called *admissible functions*. The *Zariski tangent space* to X at $x \in X$ is the vector space

$$T_xX := \mathrm{Der}(\mathscr{O}_x, \mathbb{R}),$$

where $\mathrm{Der}(\mathscr{O}_x, \mathbb{R})$ denotes the vector space of $\mathbb{R}$-linear maps

$$D : \mathscr{O}_x \to \mathbb{R}$$

satisfying the Leibniz rule

$$D(fg) = f(x)D(g) + g(x)D(f).$$

This definition generalizes the usual tangent space for smooth manifolds, as well as the Zariski tangent space for analytic and algebraic varieties.

Example 8.
1. Smooth manifolds with the sheaf of smooth real-valued functions.
2. Analytic varieties with the sheaf of analytic functions.
3. Algebraic varieties with the sheaf of regular functions.

We regard $\mathsf{Hom}(\Gamma, G)$ as a subspace of the infinite product G^Γ, of maps from Γ to G. In this way, the embedding does not depend on a choice of generators for Γ. Define the sheaf $C^\infty(G^\Gamma)$ of locally smooth functions on G^Γ as the direct limit of the sheaves $C^\infty(G^I)$ of smooth functions on finite-dimensional manifolds G^I, where $I \subset \Gamma$ is a finite subset (and for $I \subset J \subset \Gamma$ finite, we have $C^\infty(G^I) \hookrightarrow C^\infty(G^J)$). A function $G^\Gamma \to \mathbb{R}$ is said to be *locally smooth* if it is locally a smooth function of finitely many coordinates. With this structure, G^Γ becomes a real-valued ringed space.

Proposition 5. *The Zariski tangent space to G^Γ at any point is isomorphic to $\mathfrak{g}^\Gamma$.*

Proof. The Zariski tangent space to G^Γ at a point f, denoted $T_f G^\Gamma$, can be characterized as the space of tangent vectors to smooth deformations of f. Let $\exp : \mathfrak{g} \to G$ be the Lie exponential map. For $u \in \mathfrak{g}^\Gamma$, consider the 1-parameter family $\exp(tu) f$. This defines a deformation of f. One checks that the map $u \mapsto \frac{d}{dt}\big|_{t=0} \exp(tu) f$ is an isomorphism $T_f G^\Gamma \simeq \mathfrak{g}^\Gamma$. □

The representation variety can be defined by the relations

$$F_{\alpha,\beta}(f) := f(\alpha\beta) f(\beta)^{-1} f(\alpha)^{-1} = 1, \qquad \forall \alpha,\beta \in \Gamma, \quad f \in G^\Gamma.$$

The sheaf of smooth functions on $\mathsf{Hom}(\Gamma, G)$ is defined as follows: for an open subset $U \subset G^\Gamma$, consider $\mathsf{Hom}(\Gamma, G) \cap U$ and assign to it the quotient ring

$$C^\infty(U)/(\phi \circ F_{\alpha,\beta} : \alpha,\beta \in \Gamma),$$

where $\phi : G \to \mathbb{R}$ is a smooth function with $\phi(1) = 0$. In this way, $\mathsf{Hom}(\Gamma, G)$ is naturally a ringed space. We now verify that choosing generators for Γ induces the same structure.

Proposition 6. *If Γ is generated by n elements and F_n is the free group on n generators, then the following diagram of ringed spaces commutes:*

$$\begin{tikzcd} \mathsf{Hom}(\Gamma, G) \arrow[r] \arrow[d] & G^n \arrow[d] \\ G^\Gamma \arrow[r] & G^{F_n} \end{tikzcd}.$$

The inclusion $\mathsf{Hom}(\Gamma, G) \subset G^\Gamma$ induces an inclusion of Zariski tangent spaces

$$T_\rho \mathsf{Hom}(\Gamma, G) \subset \mathfrak{g}^\Gamma, \qquad \rho \in \mathsf{Hom}(\Gamma, G).$$

Explicitly, $T_\rho \mathsf{Hom}(\Gamma, G)$ is the intersection of the kernels of the linear maps $D_\rho F_{\alpha,\beta} : \mathfrak{g}^\Gamma \to \mathfrak{g}$, for all $\alpha, \beta \in \Gamma$. By definition, for $v \in \mathfrak{g}^\Gamma$,

$$\begin{aligned} D_\rho F_{\alpha,\beta}(v) &= \frac{d}{dt}\bigg|_{t=0} F_{\alpha,\beta}(\exp(tv)\rho) \\ &= \frac{d}{dt}\bigg|_{t=0} \exp(tv(\alpha\beta))\rho(\alpha\beta)\rho(\beta)^{-1}\exp(-tv(\beta))\rho(\alpha)^{-1}\exp(-tv(\alpha)) \\ &= v(\alpha\beta) - v(\alpha) - \mathrm{Ad}(\rho(\alpha))v(\beta). \end{aligned}$$

Hence, the Zariski tangent space at ρ is

$$T_\rho \mathsf{Hom}(\Gamma, G) = \{v \in \mathfrak{g}^\Gamma : v(\alpha\beta) = v(\alpha) + \mathrm{Ad}(\rho(\alpha))v(\beta), \quad \forall \alpha,\beta \in \Gamma\}.$$

Tangent space as cocycles. Another characterization of the tangent space is given via group cohomology (see [7, Ch. I]). An element $\rho \in \mathsf{Hom}(\Gamma, G)$ endows $\mathfrak{g}$ with a Γ-module structure via $\Gamma \xrightarrow{\rho} G \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(\mathfrak{g})$. We denote this Γ-module by $\mathfrak{g}_\rho$. Define

$$C^k(\Gamma, \mathfrak{g}_\rho) := \mathrm{Map}(\Gamma^k, \mathfrak{g}_\rho), \quad k \geqslant 0,$$

the space of k-cochains. The differential $\partial^k : C^{k-1}(\Gamma, \mathfrak{g}_\rho) \to C^k(\Gamma, \mathfrak{g}_\rho)$ is given by, for $u \in C^{k-1}(\Gamma, \mathfrak{g}_\rho)$ and $(\gamma_1, \dots, \gamma_k) \in \Gamma^k$,

$$\begin{aligned}(\partial^k u)(\gamma_1, \dots, \gamma_k) := \gamma_1 \cdot u(\gamma_2, \dots, \gamma_k) + & \\ & + \sum_{i=1}^{k-1} (-1)^i\, u(\gamma_1, \dots, \gamma_{i-1}, \gamma_i\gamma_{i+1}, \gamma_{i+2}, \dots, \gamma_k) + \\ & + (-1)^k u(\gamma_1, \dots, \gamma_{k-1}).\end{aligned}$$

It is straightforward to check that $\partial^k \partial^{k-1} = 0$ for all $k \geqslant 1$. Denote by $Z^k(\Gamma, \mathfrak{g}_\rho)$ and $B^k(\Gamma, \mathfrak{g}_\rho)$ the spaces of k-cocycles and k-coboundaries, respectively. The space of 1-cocycles is

$$Z^1(\Gamma, \mathfrak{g}_\rho) = \{v \in \mathfrak{g}^\Gamma : v(\gamma_1\gamma_2) = v(\gamma_1) + \mathrm{Ad}(\rho(\gamma_1))v(\gamma_2), \quad \forall \gamma_1, \gamma_2 \in \Gamma\},$$

which coincides with $T_\rho \mathsf{Hom}(\Gamma, G)$. The space of 1-coboundaries is

$$B^1(\Gamma, \mathfrak{g}_\rho) = \{v \in \mathfrak{g}^\Gamma : \exists \zeta \in \mathfrak{g},\ v(\gamma) = \zeta - \mathrm{Ad}(\rho(\gamma))\zeta, \quad \forall \gamma \in \Gamma\}.$$

3.3. Smooth points.

Let $X \subset \mathbb{R}^n$ be an analytic variety. A point $x \in X$ is called a *smooth point* if there exists an open neighborhood $U \subset X$ of x such that U is an embedded submanifold of $\mathbb{R}^n$. By the implicit function theorem, smooth points can be characterized as those at which the Jacobian matrix has maximal rank. Equivalently, these are the points where the dimension of the Zariski tangent space of X is minimal. If every point of an analytic variety is smooth, then the variety is an analytic manifold. Representation varieties are analytic varieties. If Γ is a free group, then $\mathsf{Hom}(\Gamma, G)$ is an analytic manifold, since no relations are imposed in its definition. The following property is straightforward:

Proposition 7. *The set of smooth points of* $\mathsf{Hom}(\Gamma, G)$ *is invariant under the conjugation action of* G*.*

Proof. The conjugation action of G is analytic, and therefore preserves smooth neighborhoods of points in $\mathsf{Hom}(\Gamma, G)$. Equivalently, one can observe that the Zariski tangent spaces at ρ and $g\rho g^{-1}$ are isomorphic as Γ-modules:

$$v \in Z^1(\Gamma, \mathfrak{g}_\rho) \longmapsto \mathrm{Ad}(g)v \in Z^1(\Gamma, \mathfrak{g}_{g\rho g^{-1}}).$$

In particular, these tangent spaces have the same dimension. □

For the case where $\Gamma = \pi_{g,0}$ is the fundamental group of a closed surface of genus g, and G is a reductive Lie group, it is possible to characterize smooth points explicitly (see [22]). Denote by $Z(\rho)$ the centralizer of $\rho(\Gamma)$ in G, also called the *stabilizer* of ρ under the conjugation action.

Theorem 2. *Let G be a reductive Lie group. Then the dimension of the Zariski tangent space to $\mathsf{Hom}(\pi_{g,0}, G)$ at ρ is*

$$\dim Z^1(\pi_{g,0}, \mathfrak{g}_\rho) \ = \ (2g-1)\dim G + \dim Z(\rho).$$

Moreover, the representations ρ for which $\dim Z(\rho) = \dim Z(G)$ are precisely those that minimize the dimension of their Zariski tangent space.

Proof. As we saw in Subsection 3.2, the Zariski tangent space at ρ can be identified with $Z^1(\pi_{g,0}, \mathfrak{g}_\rho)$. The group cohomology of $\pi_{g,0}$ with coefficients in $\mathfrak{g}_\rho$ is naturally isomorphic to the de Rham cohomology of the surface $\Sigma_{g,0}$ with coefficients in the adjoint bundle of the associated principal G-bundle (see Section 5 or [22]). In particular, it vanishes in degrees greater than 2. The Euler characteristic

$$\dim H^0(\pi_{g,0}, \mathfrak{g}_\rho) \ - \ \dim H^1(\pi_{g,0}, \mathfrak{g}_\rho) \ + \ \dim H^2(\pi_{g,0}, \mathfrak{g}_\rho)$$

does not depend on ρ, since it is the alternating sum of the dimensions of the spaces of cochains. In simplicial cohomology with local coefficients, the cochain spaces are finite-dimensional and independent of ρ, as the $\pi_{g,0}$-module structure of $\mathfrak{g}_\rho$ only enters into the differential. Taking ρ to be the trivial representation, $\mathfrak{g}_\rho$ becomes a trivial $\pi_{g,0}$-module, so the Euler characteristic above equals the Euler characteristic of $\Sigma_{g,0}$ multiplied by $\dim G$. Hence,

$$\dim H^1(\pi_{g,0}, \mathfrak{g}_\rho) \ = \ (2g-2)\dim G + \dim H^0(\pi_{g,0}, \mathfrak{g}_\rho) + \dim H^2(\pi_{g,0}, \mathfrak{g}_\rho).$$

By Poincaré duality, $H^2(\pi_{g,0}, \mathfrak{g}_\rho) \ \cong \ H^0(\pi_{g,0}, \mathfrak{g}_\rho^*)^*$. Since G is reductive, there exists a non-degenerate, Ad-invariant bilinear form on $\mathfrak{g}$, yielding a $\pi_{g,0}$-module isomorphism $\mathfrak{g}_\rho \cong \mathfrak{g}_\rho^*$. It follows that

$$\dim H^2(\pi_{g,0}, \mathfrak{g}_\rho) \ = \ \dim H^0(\pi_{g,0}, \mathfrak{g}_\rho).$$

On the other hand, $H^0(\pi_{g,0}, \mathfrak{g}_\rho) = \mathfrak{z}(\rho)$, the Lie algebra of $Z(\rho)$. Thus,

$$\dim H^1(\pi_{g,0}, \mathfrak{g}_\rho) \ = \ (2g-2)\dim G + 2\dim Z(\rho).$$

Finally, since $\dim B^1(\pi_{g,0}, \mathfrak{g}_\rho) \ = \ \dim G - \dim Z(\rho)$, we conclude that

$$\dim Z^1(\pi_{g,0}, \mathfrak{g}_\rho) \ = \ (2g-1)\dim G + \dim Z(\rho).$$

As $Z(G) \subseteq Z(\rho)$, we have $\dim Z(G) \leqslant \dim Z(\rho)$. Hence, ρ minimizes the dimension of its Zariski tangent space if and only if $\dim Z(\rho) = \dim Z(G)$. $\square$

3.4. Characterization of orbits.

Consider the action of $\mathrm{Inn}(G) \cong G/Z(G)$ on $\mathsf{Hom}(\Gamma, G)$. For $\rho \in \mathsf{Hom}(\Gamma, G)$, let O_ρ denote the orbit of ρ under this action. Denote by $Z(\rho)$ the centralizer of $\rho(\Gamma)$ in G, which is the stabilizer of ρ for the conjugation action. This is a closed subgroup of G. The orbit O_ρ is a smooth manifold diffeomorphic to the homogeneous space $G/Z(\rho)$. A smooth deformation of ρ inside O_ρ can be written as

$$\rho_t = g(t)\rho g(t)^{-1},$$

where $g(t)$ is a smooth one-parameter family in G with $g(0) = 1$. The tangent vector to ρ_t at $t = 0$ is the coboundary

$$v(\gamma) = \zeta - \mathrm{Ad}(\rho(\gamma))\zeta, \qquad \gamma \in \Gamma,$$

where $\zeta \in \mathfrak{g}$ is the tangent vector to $g(t)$ at $t = 0$. Conversely, given $\zeta \in \mathfrak{g}$, the coboundary $v(\gamma) = \zeta - \mathrm{Ad}(\rho(\gamma))\zeta$ is tangent to the curve $\exp(t\zeta)\rho\exp(-t\zeta)$ at $t = 0$. Thus, $T_\rho O_\rho = B^1(\Gamma, \mathfrak{g}_\rho)$. The following inclusions of ringed spaces

$$O_\rho \subset \mathsf{Hom}(\Gamma, G) \subset G^\Gamma$$

induce a chain of inclusions on Zariski tangent spaces:

$$\begin{tikzcd}
T_\rho O_\rho \arrow[r, hook] \arrow[d, "\simeq"'] & T_\rho \mathsf{Hom}(\Gamma, G) \arrow[r, hook] \arrow[d, "\simeq"] & T_\rho G^\Gamma \arrow[d, "\simeq"] \\
B^1(\Gamma, \mathfrak{g}_\rho) \arrow[r, hook] & Z^1(\Gamma, \mathfrak{g}_\rho) \arrow[r, hook] & C^1(\Gamma, \mathfrak{g}_\rho)
\end{tikzcd}.$$

If $\mathfrak{z}(\rho)$ denotes the Lie algebra of $Z(\rho)$, then the space of 1-coboundaries $B^1(\Gamma, \mathfrak{g}_\rho)$ can be identified with the quotient $\mathfrak{g}/\mathfrak{z}(\rho)$. Consequently,

$$\dim B^1(\Gamma, \mathfrak{g}_\rho) = \dim O_\rho = \dim G - \dim Z(\rho).$$

It is immediate that the conjugation action of $\mathrm{Inn}(G) \cong G/Z(G)$ on $\mathsf{Hom}(\Gamma, G)$ is not free: for instance, the trivial representation is a fixed point. More generally, for every $\rho \in \mathsf{Hom}(\Gamma, G)$ we have $Z(G) \subset Z(\rho)$. Consider the subset

$$\{\rho \in \mathsf{Hom}(\Gamma, G) : Z(\rho) = Z(G)\}.$$

This set is invariant under conjugation, and on it the action is free. Moreover:

Proposition 8. *The conjugation action on* $\mathsf{Hom}(\Gamma, G)$ *is locally free (i.e. the stabilizer of every element is discrete) if and only if* $\dim Z(\rho) = \dim Z(G)$ *for every* $\rho \in \mathsf{Hom}(\Gamma, G)$.

Proof. The conjugation action on $\mathsf{Hom}(\Gamma, G)$ induces, for each ρ, a surjective linear map

$$\psi_\rho : \mathfrak{Inn}(G) \longrightarrow T_\rho O_\rho, \qquad \psi_\rho(\zeta) = \frac{d}{dt}\Big|_{t=0} \exp(t\zeta)\cdot\rho,$$

where $\mathfrak{Inn}(G)$ is the Lie algebra of $\mathrm{Inn}(G)$. The action is locally free at ρ if and only if ψ_ρ is injective. Since ψ_ρ is always surjective, this holds precisely when $\dim \mathfrak{Inn}(G) = \dim T_\rho O_\rho$. Now, $\dim \mathfrak{Inn}(G) = \dim G - \dim Z(G)$, while $\dim T_\rho O_\rho = \dim G - \dim Z(\rho)$. These are equal if and only if $\dim Z(\rho) = \dim Z(G)$. □

By Theorem 2, in the case $\Gamma = \pi_{g,0}$ and G a reductive Lie group, the smooth points of $\mathsf{Hom}(\pi_{g,0}, G)$ are exactly those for which the conjugation action is locally free.

3.5. Subsets of representations.

The last proposition of the previous subsection motivates the introduction of certain natural subsets of $\mathsf{Hom}(\Gamma, G)$, defined in terms of the structure of the image and its centralizer in G. These subsets will play an important role in describing the local and global properties of the representation and character varieties.

Definition 7. A representation $\rho \in \mathsf{Hom}(\Gamma, G)$ is said to be *regular* if

$$\dim Z(G) = \dim Z(\rho),$$

and *very regular* if $Z(G) = Z(\rho)$. Here $Z(\rho)$ denotes the centralizer of $\rho(\Gamma)$ in G.

We denote by $\mathsf{Hom}^{\mathrm{reg}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subset of regular representations, and by $\mathsf{Hom}^{\mathrm{vreg}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subset of very regular representations.

The regularity condition measures how much the centralizer of $\rho(\Gamma)$ differs from the center of the group G. Regular representations are those for which this centralizer has minimal possible dimension, while very regular representations achieve the minimal possible centralizer, namely the center of G itself.

Example 9. Consider representations $\rho : \Gamma \to \mathrm{PSL}(2, \mathbb{R})$. The non-regular ones are precisely those whose image fixes a point in the hyperbolic plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ or on its boundary. In this case, the centralizer $Z(\rho)$ has dimension strictly greater than that of $Z(G)$. This follows from the classification of abelian subgroups of $\mathrm{PSL}(2, \mathbb{R})$.

Understanding how orbits of representations behave under conjugation by G is fundamental when studying the topological and algebraic structure of the quotient $\mathsf{Hom}(\Gamma, G)/\mathrm{Inn}(G)$. In particular, to determine whether two orbits can be separated by disjoint open subsets, a key property for the quotient to be Hausdorff or a manifold, it is necessary to introduce special subgroups of G, such as Borel, parabolic, and Levi subgroups.

Definition 8. A *Borel subgroup* of a complex algebraic group G is a maximal connected solvable Zariski-closed subgroup of G. A *parabolic subgroup* of a real or complex algebraic group G is a Zariski-closed subgroup containing a Borel subgroup (over $\mathbb{C}$). A *Levi subgroup* of a real or complex algebraic group G is a connected subgroup isomorphic to $G/R_u(G)$, where $R_u(G)$ denotes the unipotent radical of G. A subgroup of an algebraic group is called *irreducible* if it is not contained in any proper parabolic subgroup of G.

These notions are central in the structure theory of linear algebraic groups: every parabolic subgroup has a canonical Levi decomposition as a semidirect product of its Levi subgroup and its unipotent radical. The Levi subgroups behave, in some sense, like the "reductive parts" of the parabolic subgroups.

Example 10. Let $G = \mathrm{GL}(n, \mathbb{C})$. The Borel subgroups of G are precisely those that preserve a full flag in $\mathbb{C}^n$; for instance, the subgroup of upper triangular matrices is a Borel subgroup. The parabolic subgroups are those that preserve partial flags in $\mathbb{C}^n$, and their Levi subgroups correspond to block-diagonal matrices respecting the same decomposition.

The concept of irreducibility for subgroups naturally extends to representations:

Definition 9. Let G be an algebraic group. A representation $\rho : \Gamma \to G$ is called *irreducible* if its image $\rho(\Gamma)$ is not contained in any proper parabolic subgroup of G. We denote by $\mathsf{Hom}^{\mathrm{irr}}(\Gamma, G)$ the $\mathrm{Inn}(G)$-invariant subset of irreducible representations.

This definition generalizes the classical notion of irreducibility for linear representations. Indeed, in the case $G = \mathrm{GL}(n, \mathbb{C})$, a representation ρ is irreducible if and only if the corresponding Γ-module $\mathbb{C}^n$ admits no nontrivial Γ-invariant subspace.

Remark 2.

1. A representation $\rho : \Gamma \to G$ is irreducible if and only if $\rho(\Gamma)$ is an irreducible subgroup of G.
2. A representation can be irreducible over $\mathbb{R}$ but reducible over $\mathbb{C}$; this distinction will be relevant when comparing real and complex character varieties.
3. If $G = \mathrm{GL}(n, \mathbb{C})$, then $\rho : \Gamma \to G$ is irreducible if and only if $\mathbb{C}^n$ is an irreducible Γ-module (see Example 10).
4. If $G = \mathrm{SL}(2, \mathbb{C})$, irreducibility can be characterized in terms of traces: $\rho : \Gamma \to G$ is irreducible if and only if there exists $\gamma \in [\Gamma, \Gamma]$, a commutator in Γ, such that $\mathrm{Tr}(\rho(\gamma)) \neq 2$ (see Lemma 1.2.2 of [12]).

For reductive algebraic groups, irreducible representations possess particularly useful geometric properties: they form an open subset in the representation variety and behave well under conjugation.

Proposition 9. *[47, Propositions 27 and 28] Let G be a reductive algebraic group. Then:*

1. $\mathsf{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathsf{Hom}^{\mathrm{reg}}(\Gamma, G)$ *(the centralizer of an irreducible subgroup of G is a finite extension of $Z(G)$);*
2. $\mathsf{Hom}^{\mathrm{irr}}(\Gamma, G)$ *is Zariski open in* $\mathsf{Hom}(\Gamma, G)$*;*
3. *If $\Gamma = \pi_{g,n}$ is a surface group, then* $\mathsf{Hom}^{\mathrm{irr}}(\pi_{g,n}, G)$ *is dense in a nonempty union of irreducible components of* $\mathsf{Hom}(\pi_{g,n}, G)$*.*

Thus, for surface groups, irreducible representations capture a "generic" behavior in $\mathsf{Hom}(\pi_{g,n}, G)$. This makes them a natural locus on which to define the smooth part of the character variety.

Theorem 3 ([28])**.** *Let G be a reductive algebraic group. The conjugation action on* $\mathsf{Hom}^{\mathrm{irr}}(\Gamma, G)$ *is proper.*

Properness of the conjugation action guarantees that the quotient by $\mathrm{Inn}(G)$ behaves well topologically: orbits are closed and the resulting quotient space is Hausdorff. When combined with the regularity condition, this leads to the notion of "good" representations. Representations that are both irreducible (Definition 9) and very regular (Definition 7) are called *good representations*, a notion introduced by Johnson and Millson in [28]. We denote by $\mathsf{Hom}^{\mathrm{gd}}(\Gamma, G)$ the conjugation-invariant subset of good representations. By Proposition 9 and Theorem 3, the conjugation action on $\mathsf{Hom}^{\mathrm{gd}}(\Gamma, G)$ is both free and proper, ensuring that the quotient by conjugation inherits the structure of a smooth manifold.

Proposition 10 ([22, 47])**.** *Let G be a reductive algebraic group.*

1. $\mathsf{Hom}^{\mathrm{gd}}(\Gamma, G)$ *is Zariski open in* $\mathsf{Hom}(\Gamma, G)$*.*
2. *If $\Gamma = \pi_{g,0}$, then:*
   - *(a)* $\mathsf{Hom}^{\mathrm{gd}}(\pi_{g,0}, G)$ *is an analytic manifold of dimension* $(2g-1)\dim G + \dim Z(G)$*;*
   - *(b) The conjugation action on* $\mathsf{Hom}^{\mathrm{gd}}(\pi_{g,0}, G)$ *is free and proper;*
   - *(c) The quotient* $\mathsf{Hom}^{\mathrm{gd}}(\pi_{g,0}, G)/\mathrm{Inn}(G)$ *is an analytic manifold of dimension* $(2g-2)\dim G + 2\dim Z(G)$*, which is always even.*

The notion of irreducibility can be relaxed to that of *reductivity* (or *complete reducibility*), which encompasses representations whose image behaves algebraically like that of a reductive group.

Definition 10. 1. An algebraic group is called *linearly reductive* if all its finite-dimensional representations are completely reducible, i.e., they decompose as direct sums of simple subrepresentations.

2. A subgroup of an algebraic group is called *completely reducible* if its Zariski closure is linearly reductive.

Over $\mathbb{R}$ or $\mathbb{C}$, an algebraic group is linearly reductive if and only if its Zariski identity component is reductive (see Corollary 22.43 of [40]). This equivalence allows one to work interchangeably with these two notions in most geometric contexts.

Definition 11. Let G be an algebraic group. A representation $\rho : \Gamma \to G$ is called *reductive* or *completely reducible* if $\rho(\Gamma) \subset G$ is completely reducible. The conjugation-invariant subset of reductive representations is denoted by $\mathsf{Hom}^{\mathrm{red}}(\Gamma, G)$.

Example 11. A representation $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ is reductive if and only if $\mathbb{C}^n$ is a completely reducible Γ-module, i.e., a direct sum of irreducible Γ-modules. Equivalently, every Γ-invariant subspace of $\mathbb{C}^n$ admits a Γ-invariant complement. This is the standard characterization from classical representation theory.

The following proposition provides a useful intrinsic characterization of reductive representations in terms of parabolic and Levi subgroups.

Proposition 11 ([5, Ch. IV, §14])**.** *Let G be an algebraic group. A representation $\rho : \Gamma \to G$ is reductive if and only if, whenever $\rho(\Gamma)$ is contained in a parabolic subgroup P of G, it is also contained in a Levi subgroup of P.*

For reductive algebraic groups, these various notions are closely related, as summarized below.

Proposition 12. *Let G be a reductive algebraic group. Then:*

1. $\mathsf{Hom}^{\mathrm{irr}}(\Gamma, G) \subset \mathsf{Hom}^{\mathrm{red}}(\Gamma, G)$*;*

2. $\mathsf{Hom}^{\mathrm{irr}}(\Gamma, G) = \mathsf{Hom}^{\mathrm{red}}(\Gamma, G) \cap \mathsf{Hom}^{\mathrm{reg}}(\Gamma, G)$*.*

Finally, reductive representations admit a geometric characterization in terms of orbit closures under conjugation.

Proposition 13 ([17, 47])**.** *Let G be a reductive algebraic group. A representation $\rho : \Gamma \to G$ is reductive if and only if its conjugation orbit O_ρ is closed in* $\mathsf{Hom}(\Gamma, G)$.

As a consequence of Proposition 13, the topological quotient

$$\mathsf{Hom}^{\mathrm{red}}(\Gamma, G)/\mathrm{Inn}(G)$$

is a $\mathcal{T}_1$-space. Moreover, it can be shown that this quotient is, in fact, Hausdorff (see [45]). This provides a natural and geometrically meaningful setting in which to define the *character variety* associated with the pair (Γ, G).

4. Character Varieties

A first and most natural example of what is called a *character variety* is the quotient

$$\mathsf{Hom}(\Gamma, G)/\mathrm{Inn}(G),$$

where $\mathsf{Hom}(\Gamma, G)$ denotes the set of all group homomorphisms from a finitely generated group Γ into a Lie group G, and $\mathrm{Inn}(G)$ acts on it by conjugation,

$$(g \cdot \rho)(\gamma) = g\,\rho(\gamma)\,g^{-1}.$$

This quotient is endowed with the quotient topology. Each point of this space corresponds to a conjugacy class of representations of Γ in G, and thus the character variety provides a natural way of organizing all such representations up to equivalence. The construction is conceptually simple, but topologically it is often rather ill-behaved. Indeed, the conjugation action of $\mathrm{Inn}(G)$ is typically neither free nor proper, and consequently, the quotient may fail to have desirable separation properties. From a geometric standpoint, one seeks a refinement of this quotient that reflects the structure of the representation space while providing a reasonable geometric object—ideally a smooth manifold or an algebraic variety. Before describing more sophisticated constructions (for example, through Geometric Invariant Theory), it is useful to recall some basic notions from topology.

Definition 12. A topological space X is said to be $\mathcal{T}_1$ if every pair of distinct points can be separated by an open set containing one but not the other. Equivalently, in a $\mathcal{T}_1$-space, all singletons are closed. A stronger property, often desirable in geometry, is the *Hausdorff property*: X is $\mathcal{T}_2$ if every pair of distinct points admits disjoint open neighborhoods (see Subsection 2.2). In Hausdorff spaces, limits are unique, sequences behave well, and quotient maps are easier to handle.

These separation properties are therefore minimal requirements for a space that aspires to have geometric meaning. The problem is that, for general G, the quotient $\mathsf{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ is rarely even $\mathcal{T}_1$. For example, distinct conjugacy classes of representations may have closures that intersect, preventing the points of the quotient from being closed. This motivates the search for a "better" quotient, one that retains the idea of identifying conjugate representations but smooths out the topological pathologies of the naive construction. From a categorical viewpoint, one would like to define a space $X(\Gamma, G)$, the *character variety*, which is the largest best-behaved quotient of $\mathsf{Hom}(\Gamma, G)$ by conjugation, equipped with richer algebraic or geometric structure. This space should agree with the naive quotient in simple cases but possess much better topological or algebraic properties in general.

4.1. Examples.

We now illustrate these ideas with a few elementary examples that highlight different aspects of the construction.

1. **The abelian case.** If G is an abelian algebraic group, then conjugation acts trivially. In this case,

$$\mathsf{Hom}(\Gamma, G)/\mathrm{Inn}(G) = \mathsf{Hom}(\Gamma, G).$$

The set of all group homomorphisms from Γ to G already classifies representations up to equivalence, so the naive quotient has no further identifications to make. Moreover, every representation $\rho : \Gamma \to G$ factors through the abelianization $\Gamma^{\mathrm{ab}} := \Gamma/[\Gamma, \Gamma]$, the largest abelian quotient of Γ. Consequently,

$$\mathsf{Hom}(\Gamma, G) \cong \mathsf{Hom}(\Gamma^{\mathrm{ab}}, G),$$

and the character variety is entirely determined by the abelianization of the group. This example shows that in the abelian setting, the naive and refined quotients coincide.

2. **Relation with cohomology.** A particularly instructive case arises when $G = \mathbb{R}$ and $\Gamma = \pi_1(X)$, the fundamental group of a connected topological space X. By the Hurewicz theorem, the abelianization of $\pi_1(X)$ is isomorphic to the first homology group $H_1(X, \mathbb{Z})$. Therefore,

$$\mathsf{Hom}(\pi_1(X), \mathbb{R}) \cong \mathsf{Hom}(H_1(X, \mathbb{Z}), \mathbb{R}) \cong H^1(X, \mathbb{R}),$$

where the last identification follows from the universal coefficient theorem ([25]). Thus, in this case, the space of homomorphisms from the fundamental group into $\mathbb{R}$ is canonically isomorphic to the first cohomology group of X with real coefficients. This provides a direct link between the algebraic notion of representation and the topological notion of cohomology, illustrating how character varieties can be viewed as generalizations of cohomological invariants.

3. **Conjugacy classes of elements.** When $\Gamma = \mathbb{Z}$, the free group on one generator, we have that

$$\mathsf{Hom}(\mathbb{Z}, G) \cong G,$$

since a homomorphism is determined by the image of the generator. The conjugation action of $\mathrm{Inn}(G)$ on G corresponds exactly to conjugation by elements of G, so that

$$\mathsf{Hom}(\mathbb{Z}, G)/\mathrm{Inn}(G) \cong \{\text{conjugacy classes of elements in } G\}.$$

Thus, in this elementary situation, the character variety $X(\mathbb{Z}, G)$ is simply the set of conjugacy classes in G. However, this quotient can be quite singular. For instance, when $G = \mathsf{PSL}(2, \mathbb{R})$, the space of conjugacy classes fails to be Hausdorff and is not even $\mathcal{T}_1$: different conjugacy classes may have closures that overlap. This pathology shows that even for very simple groups, the naive quotient does not behave well topologically, and therefore motivates the introduction of more sophisticated quotients, such as those provided by Geometric Invariant Theory.

These examples illustrate the range of behaviors that can occur. In the abelian case, the quotient is smooth and well-behaved; for nonabelian groups, especially those with nontrivial conjugacy classes, one immediately encounters singularities and non-Hausdorff phenomena. The next step is to refine this construction using tools from algebraic geometry, replacing the naive topological quotient by a more controlled algebraic one.

4.2. Hausdorff and $\mathcal{T}_1$ quotients.

In the previous discussion, we saw that the naive quotient $\mathsf{Hom}(\Gamma, G)/\mathrm{Inn}(G)$ often fails to be Hausdorff, and even to satisfy the weaker separation property of being $\mathcal{T}_1$. We now describe general constructions that associate to any topological space (or, more generally, to a group action) a canonical quotient that restores these separation properties as much as possible. These constructions provide the theoretical foundation for defining refined versions of the character variety.

The Hausdorff quotient. We first recall a general procedure to associate to any topological space its "largest Hausdorff quotient," known as the *Hausdorffization*. Let X be a topological space. Consider all equivalence relations $\approx$ on X such that the quotient $X/\approx$ is Hausdorff. Such relations always exist—for instance, the trivial relation identifying all points of X into one. Define a new equivalence relation $\sim$ on X by declaring

$$x \sim y \iff x \approx y \text{ for all equivalence relations } \approx \text{ such that } X/\approx \text{ is Hausdorff.}$$

In other words, two points of X are identified if they cannot be separated by any Hausdorff quotient. The resulting quotient space

$$\mathrm{Haus}(X) := X/\sim$$

is called the *Hausdorffization* of X. It is itself a Hausdorff topological space and satisfies the following universal property: If Y is a Hausdorff topological space, then every continuous map $f : X \to Y$ factors uniquely through the canonical projection $p : X \to \mathrm{Haus}(X)$. That is, there exists a unique continuous map $\tilde{f} : \mathrm{Haus}(X) \to Y$ such that $f = \tilde{f} \circ p$. We denote by $[x]$ the equivalence class of a point $x \in X$ under the relation $\sim$.

Proposition 14. *Let $x, y \in X$ be such that the closures of their equivalence classes intersect, that is, $\overline{[x]} \cap \overline{[y]} \neq \emptyset$. Then $x \sim y$.*

Proof. Since $\mathrm{Haus}(X)$ is Hausdorff, all its points are closed. Therefore, the equivalence classes for $\sim$ are closed subsets of X. If $x \nsim y$, then the corresponding equivalence classes $[x]$ and $[y]$ are closed and disjoint, which contradicts the hypothesis that their closures intersect. ◻

This construction can now be applied to the space of representations. The *Hausdorff character variety* of a finitely generated group Γ in a Lie group G is defined as

$$\mathrm{Rep}^{\mathrm{Haus}}(\Gamma, G) := \mathrm{Haus}(\mathsf{Hom}(\Gamma, G)/\mathrm{Inn}(G)).$$

By construction, it provides the maximal Hausdorff quotient of the naive topological quotient $\mathsf{Hom}(\Gamma, G)/\mathrm{Inn}(G)$. A detailed discussion of this approach can be found in [41].

The $\mathcal{T}_1$ quotient. A weaker, yet often more flexible, refinement can be constructed under an additional mild assumption on the group action. This version was introduced in [45] and leads to the notion of a $\mathcal{T}_1$-quotient. Let G be a topological group acting continuously on a topological space X. For each $x \in X$, denote by $\mathcal{O}_x := G \cdot x$ the orbit of x. We assume the following property:

$$\text{For every } x \in X, \text{ the closure } \overline{\mathcal{O}_x} \subset X \text{ contains a unique closed } G\text{-orbit.} \tag{4.1}$$

This hypothesis holds for many natural actions, including those arising in representation theory of reductive groups. Let $X/\!/G$ denote the set of all closed orbits of the action. We define a map

$$\pi : X \longrightarrow X/\!/G, \qquad x \longmapsto \text{the unique closed orbit contained in } \overline{\mathcal{O}_x}.$$

The topology on $X/\!/G$ is defined so that π is a quotient map, i.e., a subset $Z \subset X/\!/G$ is closed if and only if its preimage $\pi^{-1}(Z) \subset X$ is closed. Equivalently, one may define an equivalence relation on X by

$$x \sim y \quad \Longleftrightarrow \quad \overline{\mathcal{O}_x} \cap \overline{\mathcal{O}_y} \neq \emptyset,$$

and then $X/\!/G$ is homeomorphic to the quotient $X/\sim$.

Proposition 15. *The space $X/\!/G$ satisfies the following properties:*

1. *$X/\!/G$ is homeomorphic to $X/\sim$.*
2. *$X/\!/G$ satisfies the universal property of the $\mathcal{T}_1$-quotient: for every $\mathcal{T}_1$ topological space Y, any continuous map $f : X \to Y$ that is constant on G-orbits factors uniquely through $\pi : X \to X/\!/G$.*
3. *There exists a natural surjective continuous map $X/\!/G \twoheadrightarrow \mathrm{Haus}(X/G)$ that fits into the commutative diagram*

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[d, two heads, "\pi"'] & X/G \arrow[d, two heads] \\ X/\!/G \arrow[r, two heads] & \mathrm{Haus}(X/G) \end{tikzcd},$$

where all arrows denote canonical quotient maps.

In particular, if $X/\!/G$ is $\mathcal{T}_1$, then it coincides with the largest $\mathcal{T}_1$-quotient of X. Moreover, if $X/\!/G$ happens to be Hausdorff, then it is homeomorphic to the Hausdorffization $\mathrm{Haus}(X/G)$.

Application to representation spaces. Returning to the conjugation action of G on $\mathsf{Hom}(\Gamma, G)$, and assuming that condition (4.1) holds, we define the $\mathcal{T}_1$ character variety of Γ in G by

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) := \mathsf{Hom}(\Gamma, G)/\!/\mathrm{Inn}(G).$$

Although $\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G)$ need not itself be a $\mathcal{T}_1$-space, it always maps onto any $\mathcal{T}_1$-quotient of $\mathsf{Hom}(\Gamma, G)$ as guaranteed by Proposition 15. Furthermore, there exists a canonical continuous surjection

$$\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) \twoheadrightarrow \mathrm{Rep}^{\mathrm{Haus}}(\Gamma, G),$$

which is a homeomorphism whenever $\mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G)$ is Hausdorff.

These two constructions, Hausdorffization and $\mathcal{T}_1$-quotient, provide the necessary topological framework for defining refined versions of character varieties, bridging the gap between the naive quotient and the algebraic quotient that arises in Geometric Invariant Theory.

4.3. Algebraic quotient.

The algebraic quotient arises from the algebraic structure of the group G, through the framework of *Geometric Invariant Theory* (GIT). Its construction is based on the algebra of invariant regular functions. Classical and modern references for this material include [15], [35], and [47], where more details and proofs can be found. We begin by recalling some basic notions needed to describe this construction.

Definition 13. A function $f : \mathsf{Hom}(\Gamma, G) \to \mathbb{K}$, where $\mathbb{K}$ is a field, is said to be an *invariant function* if it is invariant under conjugation by G; that is,

$$f(g\rho g^{-1}) = f(\rho) \quad \text{for all } g \in G,\ \rho \in \mathsf{Hom}(\Gamma, G).$$

In what follows, we consider $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. The set $\mathsf{Hom}(\Gamma, G)$ carries a natural structure of an algebraic variety. Its algebra of regular functions, also called the *coordinate ring*, is denoted by $\mathbb{K}[\mathsf{Hom}(\Gamma, G)]$. The subalgebra of invariant regular functions is written $\mathbb{K}[\mathsf{Hom}(\Gamma, G)]^G$.

The main goal of this construction is to identify a collection of generating invariant functions such that any other invariant regular function can be expressed as a polynomial in these generators. To produce invariant functions $\mathsf{Hom}(\Gamma, G) \to \mathbb{K}$ from invariant functions $f : G \to \mathbb{K}$ under conjugation, choose an element $\gamma \in \Gamma$ and define the function

$$f_\gamma : \mathsf{Hom}(\Gamma, G) \to \mathbb{K}, \qquad f_\gamma(\rho) := f(\rho(\gamma)).$$

When G is a linear algebraic group, it can be realized as a closed subgroup of $\mathrm{GL}(m,\mathbb{K})$. Classical examples of conjugation-invariant functions include the *trace* and the *determinant*. The subalgebra of $\mathbb{C}[\mathsf{Hom}(\Gamma,G)]^G$ generated by trace functions is denoted by $\mathcal{T}(\Gamma,G)$.

Example 12. 1. Fricke and Vogt in [21] proved that $\mathbb{C}[\mathsf{Hom}(\Gamma,\mathrm{SL}(2,\mathbb{C}))]^{\mathrm{SL}(2,\mathbb{C})}$ is (linearly) generated by trace functions Tr_γ, for $\gamma\in\Gamma$. Each trace function $\mathrm{Tr}_\gamma:\mathsf{Hom}(\Gamma,\mathrm{SL}(2,\mathbb{C}))\to\mathbb{C}$ is defined by $\mathrm{Tr}_\gamma(\rho):=\mathrm{Tr}(\rho(\gamma))$. In $\mathrm{SL}(2,\mathbb{C})$, the trace satisfies the classical relation

$$\mathrm{Tr}_{\gamma_1\gamma_2}+\mathrm{Tr}_{\gamma_1\gamma_2^{-1}}=\mathrm{Tr}_{\gamma_1}\cdot\mathrm{Tr}_{\gamma_2},\qquad \gamma_1,\gamma_2\in\Gamma.$$

2. Consider now the case $\Gamma=F_n$, a free group with n generators. Then

$$\mathsf{Hom}(F_n,\mathrm{M}_m(\mathbb{K}))\cong\mathrm{M}_m(\mathbb{K})^n.$$

The function $\mathrm{M}_m(\mathbb{K})^2\to\mathbb{K}$ given by $(X,Y)\mapsto\mathrm{Tr}(XY^\top)$ is invariant under $\mathrm{SO}(m,\mathbb{K})$-conjugation but not under $\mathrm{SL}(m,\mathbb{K})$-conjugation.

Procesi showed in [43] that, in several cases for G, the algebra $\mathbb{K}[\mathrm{M}_m(\mathbb{K})^n]^G$ is finitely generated by trace polynomials. If we replace $\mathrm{M}_m(\mathbb{K})$ by another linear algebraic group, the result depends on the chosen group. For example, the function $\det^{-1}:\mathrm{GL}(m,\mathbb{C})\to\mathbb{C}$ is non-trivial and invariant under $\mathrm{GL}(m,\mathbb{C})$-conjugation, but its restriction to $\mathrm{SL}(m,\mathbb{C})$ is constant equal to 1. By the Cayley–Hamilton theorem, the inverse determinant can be expressed as a rational function in traces. It follows that $\mathbb{C}[\mathsf{Hom}(F_n,\mathrm{GL}(m,\mathbb{C}))]^{\mathrm{GL}(m,\mathbb{C})}$ is generated by Tr_γ and $\det^{-1}$, for $\gamma\in F_n$, whereas $\mathbb{C}[\mathsf{Hom}(F_n,\mathrm{SL}(m,\mathbb{C}))]^{\mathrm{SL}(m,\mathbb{C})}$ is generated solely by trace functions (see also [36]).

Let now Γ be a finitely generated group with generators $\{\gamma_1,\dots,\gamma_n\}$. The embedding

$$\iota:\mathsf{Hom}(\Gamma,G)\hookrightarrow G^n=\mathsf{Hom}(F_n,G)$$

induces a surjective morphism

$$\iota^*:\mathbb{C}[G^n]\twoheadrightarrow\mathbb{C}[\mathsf{Hom}(\Gamma,G)].$$

Since ι^* maps invariant functions to invariant functions, it restricts to

$$(\iota^*)^G:\mathbb{C}[G^n]^G\twoheadrightarrow\mathbb{C}[\mathsf{Hom}(\Gamma,G)]^G.$$

If G is a reductive linear algebraic group, the map $(\iota^*)^G$ is surjective thanks to the existence of *Reynolds operators* ([48]). Moreover, if the algebra $\mathbb{C}[G^n]^G$ is generated by trace functions, then $\mathbb{C}[\mathsf{Hom}(\Gamma,G)]^G=\mathcal{T}(\Gamma,G)$. This holds, for example, when $G=\mathrm{SL}(m,\mathbb{C})$.

A fundamental result is *Nagata's theorem*, which asserts that when G is a reductive algebraic group over $\mathbb{C}$, the algebra $\mathbb{C}[\mathsf{Hom}(\Gamma,G)]^G$ is finitely generated (see e.g. [13], Thm. 3.3). In this case, one can define the algebraic variety

$$\mathsf{Rep}^{\mathrm{GIT}}(\Gamma,G):=\mathsf{Spec}(\mathbb{C}[\mathsf{Hom}(\Gamma,G)]^G),$$

the spectrum of the ring $\mathbb{C}[\mathsf{Hom}(\Gamma, G)]^G$, called the *GIT character variety* of Γ in G. This variety can also be denoted by $\mathsf{Hom}(\Gamma, G)/\!/G$. Its algebra of regular functions is precisely $\mathbb{C}[\mathsf{Hom}(\Gamma, G)]^G$, and its points correspond to the images of representations $\rho \in \mathsf{Hom}(\Gamma, G)$ under a system of generators of this invariant algebra. The GIT character variety has the structure of a complex algebraic variety, and is Hausdorff in the Euclidean topology. The inclusion $\mathbb{C}[\mathsf{Hom}(\Gamma, G)]^G \subset \mathbb{C}[\mathsf{Hom}(\Gamma, G)]$ induces a natural surjective morphism $p : \mathsf{Hom}(\Gamma, G) \twoheadrightarrow \mathsf{Hom}(\Gamma, G)/\!/G$. We summarize some of its fundamental properties.

Proposition 16. *1. (**Universal property**) For every algebraic variety Y, any morphism $\mathsf{Hom}(\Gamma, G) \to Y$ that is constant on G-orbits factors uniquely through p.*

2. For any $\rho_1, \rho_2 \in \mathsf{Hom}(\Gamma, G)$, $p(\rho_1) = p(\rho_2)$ if and only if $\overline{O_{\rho_1}} \cap \overline{O_{\rho_2}} \neq \emptyset$.

3. Each fiber of p contains a unique closed orbit.

From these properties, it follows that the GIT quotient coincides with both the $\mathcal{T}_1$ and Hausdorff character varieties:

$$\mathsf{Hom}(\Gamma, G)/\!/G \;\cong\; \mathrm{Rep}^{\mathcal{T}_1}(\Gamma, G) \;\cong\; \mathrm{Rep}^{\mathrm{Haus}}(\Gamma, G).$$

An alternative description of the GIT character variety involves the notion of *stability* for representations.

Definition 14. Let G be an algebraic group and Γ a finitely generated group. A representation $\rho : \Gamma \to G$ is called

1. *polystable* if its orbit O_ρ is closed;
2. *stable* if it is polystable and regular.

We denote by $\mathsf{Hom}^{ps}(\Gamma, G)$ (resp. $\mathsf{Hom}^{s}(\Gamma, G)$) the subsets of polystable (resp. stable) representations, which are invariant under conjugation. When G is a reductive algebraic group, a representation $\rho \in \mathsf{Hom}(\Gamma, G)$ is polystable if and only if it is *reductive*, and it is stable if and only if it is *irreducible*. The following result then holds.

Theorem 4. *Let G be a reductive complex algebraic group. Then there is a homeomorphism*

$$\mathsf{Hom}^{ps}(\Gamma, G)/\mathrm{Inn}(G) = \mathsf{Hom}^{red}(\Gamma, G)/\mathrm{Inn}(G) \;\cong\; \mathsf{Hom}(\Gamma, G)/\!/G,$$

and $\mathsf{Hom}(\Gamma, G)/\!/G$ contains the Zariski open subset $\mathsf{Hom}^{s}(\Gamma, G)/\mathrm{Inn}(G)$ $= \mathsf{Hom}^{irr}(\Gamma, G)/\mathrm{Inn}(G)$.

Proof. By definition, polystable representations have closed orbits under the $\mathrm{Inn}(G)$-action. Therefore, the projection p factors through the injective map

$$\mathsf{Hom}^{ps}(\Gamma, G)/\mathrm{Inn}(G) \longrightarrow \mathsf{Hom}(\Gamma, G)/\!\!/G,$$

which is also surjective. □

4.4. Strong Deformation Retractions.

In this subsection, G will always be a connected, reductive, linear algebraic group over $\mathbb{C}$, which can be realized as a closed subgroup of some $\mathrm{GL}(n, \mathbb{C})$. We will also consider compact Lie groups:

Definition 15. A *compact Lie group* is a topological group K that is also a compact smooth manifold, such that the group operations

$$\mu\colon K \times K \to K, \quad \mu(g, h) = gh, \quad \text{and} \quad \iota\colon K \to K, \quad \iota(g) = g^{-1}$$

are smooth maps (of class C^∞).

In other words, a compact Lie group K is a finite-dimensional Lie group whose underlying topological space is compact and whose group structure is compatible with the differentiable structure. By the Peter–Weyl theorem, it is also a real algebraic group, embedding in some $O(n, \mathbb{R})$. Its points are defined by an ideal in the coordinate ring $\mathbb{R}[O(n, \mathbb{R})]$. The *complexification* of K, denoted $K_\mathbb{C}$, is defined as the set of complex zeros of this ideal. It forms a complex linear subgroup of $O(n, \mathbb{C})$, with coordinate ring $\mathbb{C}[K_\mathbb{C}] = \mathbb{R}[K] \otimes_\mathbb{R} \mathbb{C}$.

There is an equivalent characterization of reductive groups (see [6]):

Proposition 17. *A connected linear algebraic group over $\mathbb{C}$ is reductive if and only if it is the complexification of a compact Lie group.*

Example 13. Consider the unitary group $U(n) = \{M \in \mathrm{GL}(n, \mathbb{C}) \mid M\overline{M}^\top = I_n\}$, a compact Lie group. If $M = A + \sqrt{-1}B$, then

$$U(n) \cong \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in GL(2n, \mathbb{R}) \mid A^tA + B^tB = I,\ A^tB - B^tA = 0 \right\},$$

which in $\mathrm{GL}(2n, \mathbb{C})$ is isomorphic to

$$\left\{ \begin{pmatrix} k & 0 \\ 0 & (k^{-1})^\top \end{pmatrix} \in GL(2n, \mathbb{C}) \mid k \in U(n) \right\}.$$

More generally, for arbitrary $k \in \mathrm{GL}(n, \mathbb{C})$, we have $U(n)_\mathbb{C} = \mathrm{GL}(n, \mathbb{C})$. Thus $U(n)$ is the real locus of $\mathrm{GL}(n, \mathbb{C})$, and similarly $SU(n)_\mathbb{C} = SL(n, \mathbb{C})$.

Let K be a compact Lie group. Consider the representation variety $\mathsf{Hom}(\Gamma, K)$ and the conjugation action of K on it. Since K is compact, all orbits are compact and hence closed, so the quotient

$$\mathsf{Hom}(\Gamma, K)/K$$

is Hausdorff. We call this the *character variety* of Γ in K. It is not an algebraic variety, but rather a *semi-algebraic* variety: a finite union of sets, each defined by finitely many polynomial inequalities. More generally, if S is a real affine algebraic K-variety, there exists an equivariant closed embedding $S \hookrightarrow W$, with W a real representation of K. The invariant ring $\mathbb{R}[W]^K$ is finitely generated by polynomials $p_1, \ldots, p_d$, and the map $P = (p_1, \ldots, p_d) : S \to \mathbb{R}^d$ is proper, inducing a homeomorphism $S/K \cong P(S)$. Let I be the ideal such that $\mathbb{R}[S]^K \cong \mathbb{R}[p_1, \ldots, p_d]/I$, and let $Z_{\mathbb{R}}(I)$ be the real zeros of I in $\mathbb{R}^d$. Then S/K is a closed semi-algebraic subset of $Z_{\mathbb{R}}(I)$ (see [44, 46]).

Let G be the complexification of K, a reductive linear algebraic group. Then K is a maximal compact subgroup of G, and by the Peter–Weyl theorem we may assume $K \subset O(n, \mathbb{R})$, hence $G \subset O(n, \mathbb{C})$. The inclusion $K \subset G$ gives $\mathsf{Hom}(\Gamma, K) \subset \mathsf{Hom}(\Gamma, G)$, and for G reductive we have

$$\mathbb{C}[\mathsf{Hom}(\Gamma, G)]^G = \mathbb{C}[\mathsf{Hom}(\Gamma, G)]^K.$$

Consequently, the real and imaginary parts of a set of generators of $\mathbb{C}[\mathsf{Hom}(\Gamma, G)]^G$ generate $\mathbb{R}[\mathsf{Hom}(\Gamma, K)]^K$.

Example 14. Let $\Gamma = F_r$ and $K = \mathrm{SU}(n)$. Then $\mathbb{R}[\mathsf{Hom}(F_r, \mathrm{SU}(n))]^{\mathrm{SU}(n)}$ is generated by the real and imaginary parts of trace functions.

Every semi-algebraic set admits a cellular decomposition:

Theorem 5. *([4], p. 214) Let X be a closed and bounded semi-algebraic set. For any family $\{X_i\}$ of semi-algebraic subsets of X, there exists a cellular decomposition of X such that each X_i is a subcomplex.*

There is an inclusion of CW-complexes ([18], Proposition 4.5):

$$\iota_G : \mathsf{Hom}(\Gamma, K)/K \hookrightarrow \mathsf{Hom}(\Gamma, G)/\!\!/G.$$

For simplicity, we denote $\mathsf{Hom}(\Gamma, G)/\!\!/G$ (resp. $\mathsf{Hom}(\Gamma, K)/K$) by $\mathfrak{X}_\Gamma(G)$ (resp. $\mathfrak{X}_\Gamma(K)$). We now address the key question: when does $\mathfrak{X}_\Gamma(G)$ *strongly deformation retract* onto $\mathfrak{X}_\Gamma(K)$? Equivalently, when is $\mathfrak{X}_\Gamma(K)$ a *strong deformation retract* of $\mathfrak{X}_\Gamma(G)$?

Definition 16. Let X be a topological space and $A \subset X$ a subspace. A is a *strong deformation retract* of X if there exists a continuous map $H : [0, 1] \times X \to X$ such that

1. $H(0, x) = x$ for all $x \in X$,
2. $H(t, a) = a$ for all $a \in A$ and $t \in [0, 1]$,
3. $H(1, x) \in A$ for all $x \in X$.

The map H is called a *strong deformation retraction* (SDR) of X onto A.

A strong deformation retraction of X onto A is a homotopy between the identity on X and $j \circ r$, where r is a retraction from X onto A and j the inclusion. Thus, A and X have the same homotopy type.

Definition 17 ([20])**.** A finitely generated group Γ is *flawed* if $\iota_G : \mathfrak{X}_\Gamma(K) \hookrightarrow \mathfrak{X}_\Gamma(G)$ is an SDR for all choices of G. It is *flawless* if ι_G is not an SDR for any non-abelian G.

Proposition 18. *If Γ is finite, then ι_G is a homeomorphism.*

Proof. Each $\rho \in \mathsf{Hom}(\Gamma, G)$ has finite image, so is polystable, giving $\mathfrak{X}_\Gamma(G) = \mathsf{Hom}(\Gamma, G)/G$. Since the image is compact, it is contained in a maximal compact subgroup K of G. All maximal compact subgroups are conjugate, so for each ρ there exists $g \in G$ with $g\rho(\Gamma)g^{-1} \subset K$, giving surjectivity. Compactness of $\mathfrak{X}_\Gamma(K)$ then implies ι_G is a homeomorphism. □

Example 15. 1. Finite groups are flawed.

2. Finitely generated free groups are flawed ([17]).
3. Finitely generated abelian, nilpotent, or virtually nilpotent Kähler groups are flawed ([2, 3, 19]).
4. There exist finitely generated nilpotent groups Γ and non-reductive G with maximal compact K for which $\mathfrak{X}_\Gamma(G)$ is not homotopic to $\mathfrak{X}_\Gamma(K)$ ([2]), showing the necessity of reductivity.
5. Hyperbolic surface groups $\Gamma = \pi_1(\Sigma)$ (genus $\geqslant 2$) are flawless, by the Non-abelian Hodge correspondence ([16]).
6. Groups isomorphic to free products of nilpotent groups, e.g., $\mathsf{PSL}(2, \mathbb{Z})$, are flawed ([20]).

Next, we describe a criterion using Kempf-Ness theory ([31, 42, 46]) and Whitehead's theorem ([25, 50]) to determine whether Γ is flawed. Let V be an affine variety with a rational G-action, and let $V/\!/G$ be its GIT quotient. By [32, Lemma 1.1], V can be G-equivariantly embedded as a closed subvariety of a finite-dimensional complex vector space $\mathbb{V}$, with G acting linearly. Equip $\mathbb{V}$ with a K-invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. For $v \in \mathbb{V}$, define $p_v : G \to \mathbb{R}$ by $p_v(g) = \|g \cdot v\|^2$.

Definition 18. A vector $X \in \mathbb{V}$ is *minimal* if $\|X\| \leqslant \|g \cdot X\|$ for all $g \in G$. The set of minimal vectors is denoted $\mathcal{KN}_G = \mathcal{KN}(G, \mathbb{V})$ and is called the *Kempf-Ness set* of V with respect to G. It is K-stable.

Theorem 6 ([46])**.** *The map $\mathcal{KN}_G \hookrightarrow V \to V/\!/G$ is proper and induces a homeomorphism*

$$\mathcal{KN}_G/K \to V/\!/G.$$

Furthermore, $\mathcal{KN}_G \hookrightarrow V$ is a K-invariant SDR.

Applied to character varieties, with $V = \mathsf{Hom}(\Gamma, G)$ and G acting by conjugation, we have an embedding

$$\mathsf{Hom}(\Gamma, G) \subset G^r \subset \mathbb{V},$$

where r is the number of generators of Γ and $\mathbb{V}$ is a suitable affine space.

Proposition 19 ([10])**.** *The Kempf-Ness set is*

$$\mathcal{KN}_G = \left\{ (g_1, \ldots, g_r) \in \mathsf{Hom}(\Gamma, G) : \sum_{i=1}^{r} g_i^* g_i = \sum_{i=1}^{r} g_i g_i^* \right\},$$

with g_i^* is the conjugate transpose via a Cartan involution. We have $\mathsf{Hom}(\Gamma, K) \subset \mathcal{KN}_G$, which is K-stable under conjugation.

Define $\mathfrak{Y}_\Gamma(G) := \mathsf{Hom}(\Gamma, G)/K$, which is also a finite CW complex. Then, from the above theorem:

Theorem 7. *$\mathfrak{X}_\Gamma(G) \cong \mathcal{KN}_G/K$, and the inclusion $\mathcal{KN}_G/K \subset \mathfrak{Y}_\Gamma(G)$ is an SDR.*

The previous theorem establishes a fundamental topological link between the G-character variety and the Kempf–Ness set $\mathcal{KN}_G$. In particular, it shows that the quotient $\mathcal{KN}_G/K$ provides a strong deformation retraction (SDR) of the analytic quotient $\mathfrak{Y}_\Gamma(G)$. This result, originally due to Kempf and Ness in the context of geometric invariant theory, ensures that many topological invariants of the representation space can be recovered from its compact counterpart. Consequently, one can study homotopy or cohomology groups of $\mathfrak{Y}_\Gamma(G)$ by restricting attention to $\mathcal{KN}_G/K$. To better understand the relationship between the compact and complex quotients, we now examine conditions under which their homotopy types coincide. The next theorem provides a characterization in terms of the group Γ, connecting algebraic properties of Γ with topological properties of the representation spaces.

Theorem 8. *Let $\eta : \mathfrak{X}_\Gamma(K) \to \mathfrak{Y}_\Gamma(G)$ be the natural inclusion. The following are equivalent:*

1. *Γ is flawed.*

2. *η induces isomorphisms $\pi_n(\mathfrak{X}_\Gamma(K)) \cong \pi_n(\mathfrak{Y}_\Gamma(G))$ for all n.*

3. *The inclusion $\mathfrak{X}_\Gamma(K) \subset \mathcal{KN}_G/K$ induces isomorphisms $\pi_n(\mathfrak{X}_\Gamma(K)) \cong \pi_n(\mathcal{KN}_G/K)$ for all n.*

$$\begin{tikzcd}
\mathsf{Hom}(\Gamma, K) \arrow[r, hook, "i"] \arrow[d, two heads, "p_K"'] & \mathsf{Hom}(\Gamma, G) \arrow[d, two heads, "p'_K"] \arrow[dr, bend left, two heads, "p"] & \\
\mathfrak{X}_\Gamma(K) \arrow[r, hook, "\eta"] & \mathfrak{Y}_\Gamma(G) \arrow[r, two heads, "\rho"'] & \mathfrak{X}_\Gamma(G) \cong \mathcal{KN}_G/K
\end{tikzcd}$$

Figure 1: Representation spaces and quotients; Kempf–Ness theory provides a strong deformation retraction ρ from $\mathfrak{Y}_\Gamma(G)$ onto $\mathfrak{X}_\Gamma(G)$.

This equivalence highlights a remarkable rigidity phenomenon: when Γ satisfies the flawedness condition, the topology of the compact character variety $\mathfrak{X}_\Gamma(K)$ completely captures that of the complex one $\mathfrak{Y}_\Gamma(G)$. In such cases, the inclusion maps preserve all homotopy groups, implying that the deformation retraction from the non-compact to the compact setting is, in fact, a homotopy equivalence. Computing Kempf-Ness sets explicitly can be difficult, as are the homotopy groups in general. This theory extends to real groups and real character varieties:

Definition 19. A Lie group G is *real K-reductive* if:

1. K is a maximal compact subgroup of G;

2. There exists a complex reductive algebraic group $\mathbf{G}$ with real polynomials such that
$$\mathbf{G}(\mathbb{R})_0 \subseteq G \subseteq \mathbf{G}(\mathbb{R}),$$
where $\mathbf{G}(\mathbb{R})$ denotes the real points and $\mathbf{G}(\mathbb{R})_0$ its identity component.

3. G is Zariski dense in $\mathbf{G}$.

Remark 3. 1. If $G \neq \mathbf{G}(\mathbb{R})$, then G is not necessarily an algebraic group (for example $G = \mathrm{GL}(n, \mathbb{R})_0$).

2. One can think of both $\mathbf{G}$ and G as Lie groups of matrices. We consider on them the usual Euclidean topology, induced from an embedding into some $\mathrm{GL}(m, \mathbb{C})$, which is independent of the embedding.

3. $\mathbf{G}(\mathbb{R})$ is isomorphic to a closed subgroup of some $\mathrm{GL}(n, \mathbb{R})$, i.e., it is a linear algebraic group.

4. If $\mathbf{G}(\mathbb{R})$ is connected, then $G = \mathbf{G}(\mathbb{R})$ is algebraic and Zariski dense in $\mathbf{G}$. In this case, certain conditions in the definition of a real reductive group hold automatically.

Example 16. 1. All classical real matrix groups are included in this setting.

2. G can also be any complex reductive Lie group viewed as a real reductive Lie group.

3. As an example outside this framework, the universal covering group $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ has no faithful finite-dimensional linear representation, and hence is not a matrix group.

Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^{\mathbb{C}}$ the Lie algebra of $\mathbf{G}$. Fix a Cartan involution $\theta : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ which restricts to a Cartan involution on $\mathfrak{g}$:

$$\theta := \sigma\tau : \mathfrak{g} \to \mathfrak{g},$$

where σ and τ are commuting involutions of $\mathfrak{g}^{\mathbb{C}}$. The Cartan involution θ lifts to a Lie group involution $\Theta : G \to G$, whose differential is θ. By embedding G in some $\mathrm{GL}(n, \mathbb{C})$ as a closed subgroup, the involutions become explicit:

$$\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R}), \quad \mathfrak{g}^{\mathbb{C}} \subset \mathfrak{gl}(n, \mathbb{C}), \quad G \subset \mathrm{GL}(n, \mathbb{R}),$$

with $\tau(A) = -A^*$, $\sigma(A) = \bar{A}$, $\theta(A) = -A^\top$, $\Theta(g) = (g^{-1})^\top$, where $*$ denotes conjugate transpose. We also have commuting involutions T and S corresponding to τ and σ, after minor modifications. Let $\mathrm{Fix}(\alpha)$ denote the fixed points of an involution α. Then $\mathfrak{g} = \mathrm{Fix}(\sigma)$, $\mathfrak{k}' := \mathrm{Fix}(\tau)$, where $\mathfrak{k}'$ is the Lie algebra of a maximal compact subgroup K' of $\mathbf{G}$. The involution σ is called a *real structure* of G. The Cartan involution θ gives a decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{k} = \mathfrak{g} \cap \mathfrak{k}', \quad \mathfrak{p} = \mathfrak{g} \cap i\mathfrak{k}',$$

with $\theta|_{\mathfrak{k}} = 1$, $\theta|_{\mathfrak{p}} = -1$. The subalgebra $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup

$$K = \mathrm{Fix}(\Theta) = K' \cap G,$$

satisfying $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$. Similarly, we have a Cartan decomposition of the complexified algebra

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{p}^{\mathbb{C}}, \quad \theta|_{\mathfrak{k}^{\mathbb{C}}} = 1, \quad \theta|_{\mathfrak{p}^{\mathbb{C}}} = -1.$$

Now we consider representation spaces of finitely generated groups. For a group Γ, we have the natural inclusion $\mathsf{Hom}(\Gamma, K) \hookrightarrow \mathsf{Hom}(\Gamma, G)$, inducing a map

$$\iota_G : \mathfrak{X}_\Gamma(K) \to \mathfrak{X}_\Gamma(G),$$

which is injective (see Remark 4.7 of [19] and Section 3.2 of [10]).

Definition 20. Let G be a reductive Lie group and Γ a finitely generated group.

1. Γ is *strongly flawed* if there exists a K-invariant SDR from $\mathsf{Hom}(\Gamma, G)$ onto $\mathsf{Hom}(\Gamma, K)$ for all G.

2. Γ is *G-flawed* if $\mathfrak{X}_\Gamma(G)$ admits an SDR onto $\iota_G(\mathfrak{X}_\Gamma(K))$ for every maximal compact subgroup $K \subset G$.

3. Γ is *real flawed* if it is G-flawed for all real reductive groups G.

Theorem 9 (Theorem 3.16 of [20]). *If Γ is real flawed, then it is flawed. Conversely, if Γ is strongly flawed and the SDR commutes with a real structure on G, then Γ is real flawed.*

5. Representations and Principal (Higgs) Bundles

In this section, after recalling basic notions on principal bundles and their connections, we explain how flat principal bundles can be parametrized by representations of the fundamental group of a closed oriented smooth surface. We then introduce Schottky representations as a particularly tractable family, leading to their geometric interpretation within the moduli of principal bundles. The section concludes with the theory of principal Higgs bundles and the study of the Baraglia–Schaposnik branes, culminating in the inclusion of Schottky representations inside these branes, which reveals a deep interplay between three-dimensional topology, brane geometry, and representation theory.

5.1. Principal G-bundles.

In this subsection, we recall the basic notions of principal G-bundles, associated bundles, and connections, establishing the geometric framework that will later relate to representation varieties and principal Higgs bundles.

Definition 21. Let Σ be a closed oriented smooth surface, E a smooth complex manifold, and G a Lie group. A *principal G-bundle* is a triple (E, Σ, π), where $\pi : E \to \Sigma$ is a smooth submersion endowed with a smooth right G-action such that:

1. The action of G is free and transitive on each fiber of π;

2. There exists an open cover $\{U_i\}$ of Σ and G-equivariant diffeomorphisms
$$\varphi_i : \pi^{-1}(U_i) \longrightarrow U_i \times G$$
such that $(p_1 \circ \varphi_i)(\pi^{-1}(U_i)) = \pi(\pi^{-1}(U_i))$, where $p_1 : U_i \times G \to U_i$ denotes projection onto the first factor.

For any point $x \in \Sigma$, the fiber $E_x := \pi^{-1}(x)$ is a G-torsor. Upon choosing an element $e \in E_x$, there is a canonical identification $E_x \simeq G$ via the right action.

Example 17. 1. Let $V \to \Sigma$ be a complex vector bundle of rank n. Its *frame bundle* is a principal $\mathrm{GL}(n, \mathbb{C})$-bundle whose fiber over x is
$$E_x = \{f : \mathbb{C}^n \to V_x \mid f \text{ is a linear isomorphism}\}.$$

2. The universal cover $\widetilde{\Sigma} \to \Sigma$ is a principal $\pi_1(\Sigma)$-bundle, with the fundamental group acting on the left by deck transformations.

Let G be a Lie group and V a smooth manifold endowed with a left smooth G-action. Given a principal G-bundle $\pi : E \to \Sigma$, we can form the associated bundle

$$E(V) := (E \times V)/G = E \times_G V,$$

where the points (y, v) and $(y \cdot g, g^{-1} \cdot v)$ are identified for all $y \in E$, $v \in V$, and $g \in G$. The induced projection $\overline{\pi} : E(V) \to \Sigma$, given by $\overline{\pi}[y, v] = \pi(y)$, makes $E(V)$ into a smooth fiber bundle over Σ. We call $E(V)$ the *associated bundle* to E with fiber V. If G is a linear algebraic group and V a finite-dimensional G-representation, then $E(V)$ inherits the structure of a vector bundle with fiber V.

Example 18. 1. Consider a representation $\rho : \pi_1(\Sigma) \to G$, where G is a complex algebraic group. The universal cover $\widetilde{\Sigma}$ defines a principal $\pi_1(\Sigma)$-bundle, and we can construct the associated principal G-bundle

$$E_\rho := \widetilde{\Sigma} \times_\rho G = (\widetilde{\Sigma} \times G)/\pi_1(\Sigma),$$

where $(\tilde{x}, g) \cdot \gamma = (\tilde{x} \cdot \gamma, \rho(\gamma)^{-1} \cdot g)$ for all $\gamma \in \pi_1(\Sigma)$. If $G = \mathrm{GL}(n, \mathbb{C})$, then E_ρ is a rank-n complex vector bundle over Σ.

2. For a principal G-bundle E and the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, one can form the *adjoint bundle*

$$\mathrm{Ad}(E) := E \times_{\mathrm{Ad}} \mathfrak{g}.$$

Its fiber is isomorphic to the Lie algebra $\mathfrak{g}$, and the equivalence relation is

$$(y, Y) \sim (y \cdot g, \mathrm{Ad}(g)^{-1} Y), \qquad y \in E,\ g \in G.$$

Sections of associated bundles can be interpreted equivariantly. Indeed, for a principal G-bundle E over Σ and a left G-manifold V, the space of smooth sections $\Omega^0(\Sigma, E(V))$ is in bijection with the space of smooth maps $f : E \to V$ satisfying

$$f(y \cdot g) = g^{-1} \cdot f(y), \qquad \forall\, y \in E,\ g \in G.$$

Given a section $s : \Sigma \to E(V)$, the corresponding G-equivariant map f is defined by $s(x) = [e, f(e)]$ for any $e \in E_x$.

Connections: A *connection* on a principal G-bundle E is a $\mathrm{Lie}(G)$-valued 1-form $\omega \in \Omega^1(E, \mathrm{Lie}(G))$ satisfying:

1. $\omega_y(\widetilde{Y}(y)) = Y$ for all $Y \in \mathrm{Lie}(G)$ and $y \in E$, where $\widetilde{Y}(y) = \frac{d}{dt}\Big|_{t=0}(y \exp(tY))$ is the fundamental vector field;

2. $R_g^*\omega = \mathrm{Ad}(g^{-1})\omega$ for all $g \in G$, where $R_g : E \to E$ is the right action by g.

Equivalently, a connection can be described in terms of the decomposition of tangent spaces. For $y \in E$, the vertical tangent subspace $T_y^v E := \ker(d\pi_y)$ corresponds to the directions along the fibers. A connection is a choice of a complementary subspace $T_y^h E$ at each point, called the *horizontal subspace*, such that $T_y E = T_y^v E \oplus T_y^h E$ and $(R_g)_* T_y^h E = T^h_{R_g(y)} E$ for all $g \in G$.

Given a connection A on E, one obtains a *covariant derivative*

$$d_A : \Omega^0(\Sigma, E(V)) \longrightarrow \Omega^1(\Sigma, E(V))$$

acting on sections of any associated bundle $E(V)$. If $f : E \to V$ is the G-equivariant map corresponding to a section s, then $d_A s$ is defined via the differential df restricted to horizontal directions determined by A.

The horizontal distribution defines a G-invariant subbundle of TE, and the obstruction to its integrability is measured by the *curvature form*:

$$F(A) = dA + \tfrac{1}{2}[A, A] \in \Omega^2(E, \mathrm{Lie}(G)),$$

where $[A, A]$ combines the wedge product on forms with the Lie bracket on $\mathrm{Lie}(G)$. This curvature descends to a 2-form on Σ with values in the adjoint bundle $\mathrm{Ad}(E)$, which we denote by the same symbol $F(A)$.

A connection A is said to be *flat* if $F(A) = 0$, and a principal G-bundle endowed with a flat connection is called a *flat bundle*. Flatness is equivalent to the horizontal distribution being integrable, that is, to having a discrete structure group. By Frobenius' theorem, one obtains:

Proposition 20. *Let $E \to \Sigma$ be a flat principal G-bundle and $e \in E_x$ for some $x \in \Sigma$. Then, for any sufficiently small neighborhood U of x in Σ, there exists a unique section $s \in \Omega^0(U, E_{|U})$ such that $d_A(s) = 0$ and $s(x) = e$.*

Flat principal bundles thus correspond, locally, to parallel sections determined by the connection, and globally they will later relate to representations of the fundamental group through the monodromy of flat connections.

5.2. Principal bundles parameterized by representations.

Flat principal bundles admit a natural description in terms of representations of the fundamental group. We now make this correspondence precise and discuss how it leads to the construction of the representation (or character) variety. Let Σ be a connected smooth manifold with base point $x_0 \in \Sigma$, and let G be a Lie group. Consider a representation $\rho : \pi_1(\Sigma, x_0) \longrightarrow G$. We can associate to ρ a principal G-bundle E_ρ over Σ as follows. Let $\widetilde{\Sigma} \to \Sigma$ be the universal covering space, equipped with the natural left action of $\pi_1(\Sigma, x_0)$ by deck transformations. We define a right action of $\pi_1(\Sigma, x_0)$ on $\widetilde{\Sigma} \times G$ by

$$(\tilde{x}, g) \cdot \gamma := (\tilde{x} \cdot \gamma, \rho(\gamma)^{-1} g), \qquad \tilde{x} \in \widetilde{\Sigma},\ g \in G,\ \gamma \in \pi_1(\Sigma, x_0).$$

The quotient

$$E_\rho := (\widetilde{\Sigma} \times G)/\pi_1(\Sigma, x_0)$$

is then a smooth principal G-bundle over Σ, with projection $[\tilde{x}, g] \mapsto \pi(\tilde{x})$. This construction is functorial in ρ, and any flat principal G-bundle arises in this way.

Remark 4. Changing the base point x_0 leads to a conjugate representation, and hence to an isomorphic bundle. Indeed, if ρ' is conjugate to ρ by some $g \in G$, i.e. $\rho'(\gamma) = g\rho(\gamma)g^{-1}$ for all $\gamma \in \pi_1(\Sigma, x_0)$, then $E_{\rho'} \cong E_\rho$ as principal G-bundles.

The bundle E_ρ carries a canonical flat connection, induced by the trivial connection on $\widetilde{\Sigma} \times G$. Indeed, the product $\widetilde{\Sigma} \times G$ is a trivial principal G-bundle over $\widetilde{\Sigma}$ with the flat connection given by the horizontal distribution $T\widetilde{\Sigma} \times \{0\}$. This distribution is invariant under the diagonal $\pi_1(\Sigma, x_0)$-action described above, hence descends to a well-defined horizontal distribution on the quotient E_ρ. The resulting connection, denoted A_ρ, is flat and satisfies $F(A_\rho) = 0$. Conversely, suppose $E \to \Sigma$ is a principal G-bundle endowed with a flat connection A. Fix a base point $x_0 \in \Sigma$ and choose a point $e_0 \in E_{x_0}$ in the fiber above x_0. For any smooth loop γ based at x_0, parallel transport along γ with respect to the connection A defines an element $\rho_A(\gamma) \in G$ by the condition

$$e_0 \cdot \rho_A(\gamma) = \mathrm{hol}_A(\gamma)(e_0),$$

where $\mathrm{hol}_A(\gamma)$ denotes the holonomy of A along γ. This defines a group homomorphism

$$\rho_A : \pi_1(\Sigma, x_0) \longrightarrow G,$$

called the *holonomy representation* of (E, A).

Thus, the space of representations parameterizes holomorphic G-bundles, and we can view this construction as providing a natural map, that we call the *uniformization map*:

$$\begin{array}{rccc} \mathbf{E}: & \mathsf{Hom}(\pi_1, G)/\!/G & \to & M_G, \\ & [\rho] & \mapsto & [E_\rho]. \end{array} \tag{5.1}$$

Here M_G represents the set of isomorphism classes of G-bundles that admit a holomorphic flat connection. Note that $\mathbf{E}$ is well defined on conjugacy classes, since if ρ and σ are conjugate representations, then $E_\rho \cong E_\sigma$. This map is surjective by the preceding discussion, but it is not injective, as the following example is intended to show.

Example 19. Let $\Sigma = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, with $\operatorname{Im}\tau > 0$, be a complex torus, and consider $G = \mathrm{GL}(2, \mathbb{C})$. Let (α, β) be the generators of $\pi_1(\Sigma)$ corresponding to 1 and τ, respectively. Define two representations

$$\rho_1, \rho_2 \colon \pi_1(\Sigma) \longrightarrow \mathrm{GL}(2, \mathbb{C})$$

by

$$\rho_1(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \rho_1(\beta) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\rho_2(\alpha) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \rho_2(\beta) = \begin{pmatrix} e^{\pi i \tau} & 0 \\ 0 & e^{\pi i \tau} \end{pmatrix}.$$

These representations are not conjugate in G. Indeed, both are scalar representations, and conjugation acts trivially on scalar matrices, while $\rho_1 \neq \rho_2$. Nevertheless, the associated holomorphic vector bundles

$$E_{\rho_1} = \widetilde{\Sigma} \times_{\rho_1} \mathbb{C}^2, \qquad E_{\rho_2} = \widetilde{\Sigma} \times_{\rho_2} \mathbb{C}^2$$

are isomorphic. Clearly, $E_{\rho_1} \cong \Sigma \times \mathbb{C}^2$. For $\gamma = m + n\tau \in \mathbb{Z} + \tau\mathbb{Z}$, one has $\rho_2(\gamma) = e^{\pi i \gamma} I_2$. Using the convention $\gamma \cdot (z, v) = (z + \gamma, \rho_2(\gamma)v)$, consider the holomorphic map

$$\Phi \colon \mathbb{C} \times \mathbb{C}^2 \longrightarrow \mathbb{C} \times \mathbb{C}^2, \qquad \Phi(z, v) = (z, e^{-\pi i z} v).$$

It satisfies

$$\Phi(z + \gamma, \rho_2(\gamma)v) = (z + \gamma, e^{-\pi i (z+\gamma)} e^{\pi i \gamma} v) = (z + \gamma, e^{-\pi i z} v).$$

Therefore, Φ descends to a holomorphic bundle isomorphism

$$E_{\rho_2} \cong \Sigma \times \mathbb{C}^2.$$

Consequently, $E_{\rho_1} \cong E_{\rho_2}$, although $[\rho_1] \neq [\rho_2]$ in $\mathrm{Hom}(\pi_1(\Sigma), G)/\!/G$.

This shows that, in general, the assignment $\rho \mapsto E_\rho$ is *not injective*, even up to isomorphism of bundles. In order to define an injective map we must consider another equivalence notion on representations, the *analytic equivalence* (see [8]).

5.3. Schottky Representations.

We now introduce a special and particularly important class of representations of the surface group $\pi_1 := \pi_1(\Sigma)$. These will later play a central role in connecting the topology of the surface with the geometry of flat bundles and moduli spaces. To begin, fix generators α_i, β_i, for $i = 1, \ldots, g$, of π_1, giving the usual presentation

$$\pi_1 = \left\langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \,\middle|\, \textstyle\prod_{i=1}^{g} \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1 \right\rangle. \tag{5.2}$$

Let G be a complex connected reductive algebraic group, and let F_g be a fixed free group of rank g with generators $\gamma_1, \ldots, \gamma_g$. The reductive group G acts on

$\mathsf{Hom}(\pi_1, G)$ by conjugation. Denote by $e \in G$ the identity element, and consider the short exact sequence of groups

$$1 \to \ker\varphi \hookrightarrow \pi_1 \xrightarrow{\varphi} F_g \to 1,$$

where φ is the natural epimorphism defined on generators by

$$\varphi(\alpha_i) = 1, \quad \varphi(\beta_i) = \gamma_i, \quad \forall i = 1, \dots, g. \tag{5.3}$$

Hence, the kernel $\ker\varphi$ is the normal subgroup of π_1 generated by all α_i. Intuitively, the map φ collapses the "α-loops" of the surface, keeping only the β-loops as free generators. This construction motivates the following definition.

Definition 22. [9] A representation $\rho : \pi_1 \to G$ is called a *Schottky representation* if $\rho(\ker\varphi) = \{e\}$, that is, if all α_i are sent to the identity element of G.

In other words, Schottky representations are precisely those which factor through the free group F_g via φ. Let $\mathcal{S}$ denote the set of all Schottky representations. It is straightforward to verify that

$$\mathcal{S} \cong \mathsf{Hom}(F_g, \{e\} \times G) \cong \mathsf{Hom}(F_g, G) \cong G^g \subset \mathsf{Hom}(\pi_1, G),$$

where the last isomorphism is simply the evaluation map

$$(\sigma : F_g \to G) \longmapsto \big(\sigma(\gamma_1), \dots, \sigma(\gamma_g)\big).$$

Consequently, $\mathcal{S}$ is a smooth and irreducible affine algebraic variety. Since G acts on $\mathsf{Hom}(\pi_1, G)$ by conjugation, this action naturally restricts to $\mathcal{S}$. One can therefore form the affine GIT quotient

$$\mathbb{S} := \mathcal{S}/\!\!/G \cong G^g/\!\!/G \subset \mathbb{B} = \mathsf{Hom}(\pi_1, G)/\!\!/G.$$

The affine algebraic variety $\mathbb{S}$ remains irreducible but is, in general, singular ([8, Proposition 2.4]).

In order to focus on the smooth part of the moduli space, we use the notion of a *good representation*, which allows us to identify the smooth points of the quotient.

Definition 23. A representation $\rho \in \mathcal{S} \subset \mathsf{Hom}(\pi_1, G)$ is said to be *good* if it is good as an element of $\mathsf{Hom}(\pi_1, G)$.

Denote by $\mathsf{Hom}^{\mathrm{gd}}(\pi_1, G)$ the set of all good representations (see subsection 3.5), and by $\mathcal{S}^{\mathrm{gd}}$ the subset of good Schottky representations. Since goodness is invariant under conjugation, we can form the corresponding quotient spaces:

$$\mathbb{B}^{\mathrm{gd}} := \mathsf{Hom}^{\mathrm{gd}}(\pi_1, G)/\!\!/G, \quad \mathbb{S}^{\mathrm{gd}} := \mathcal{S}^{\mathrm{gd}}/\!\!/G,$$

with a natural inclusion $\mathbb{S}^{\mathrm{gd}} \subset \mathbb{B}^{\mathrm{gd}}$. The locus of good representations is Zariski open in $\mathcal{S}$ (see, for example, [47]). Furthermore, by [38, Lemma 4.6], there exists at least one good representation in $\mathsf{Hom}(\pi_1, G)$ whenever the genus $g \geqslant 2$. The case $g = 1$ requires separate consideration (see Section 9 of [8]).

Proposition 21 ([8], Proposition 2.13)**.** *Let $g \geqslant 2$. Then there always exists a good Schottky representation $\rho : \pi_1 \to G$. Moreover, such a representation can be chosen to take values in a maximal compact subgroup of G.*

This ensures that the Schottky locus is not only nonempty but contains points of favorable analytic and geometric behavior.

Theorem 10. *Let $g \geqslant 2$. The subsets of good representations $\mathsf{Hom}^{\mathrm{gd}}(\pi_1, G)$ and $\mathcal{S}^{\mathrm{gd}}$ are Zariski open in $\mathsf{Hom}(\pi_1, G)$ and $\mathcal{S}$, respectively. Each good representation defines a smooth point in the corresponding geometric quotient. Consequently, the quotients $\mathbb{B}^{\mathrm{gd}}$ and $\mathbb{S}^{\mathrm{gd}}$ are complex manifolds, and $\mathbb{S}^{\mathrm{gd}}$ is a complex submanifold of $\mathbb{B}^{\mathrm{gd}}$.*

Proof. By Proposition 21, there exists a good Schottky representation for $g \geqslant 2$. According to [47, Proposition 33], the subsets of good representations in $\mathsf{Hom}(\pi_1, G)$ and $\mathcal{S}$ are Zariski open, proving the first claim. Furthermore, since π_1 and F_g are either surface or free groups, [47, Corollary 50] implies that if ρ is good, then its class $[\rho]$ represents a smooth point in $\mathbb{B}$ (or in $\mathbb{S}$). Hence the result follows. □

Having established the existence and smoothness of good Schottky representations, we now turn to their local structure, described in terms of group cohomology.

Tangent space and dimension. The tangent space to the character variety at a good representation can be interpreted in cohomological terms (see Subsection 3.2). Let Γ be a finitely generated group and fix $\rho \in \mathsf{Hom}(\Gamma, G)$. The adjoint action of G on its Lie algebra $\mathfrak{g} = \mathsf{Lie}(G)$, when composed with ρ,

$$\mathrm{Ad}_\rho : \Gamma \longrightarrow G \longrightarrow GL(\mathfrak{g}),$$

endows $\mathfrak{g}$ with the structure of a Γ-module, denoted by $\mathfrak{g}_{\mathrm{Ad}_\rho}$.

The following classical result, due to Goldman [22] and Martin [38], provides a precise identification between the tangent space and the first cohomology group.

Theorem 11. *For a good representation $\rho \in \mathsf{Hom}(\Gamma, G)$ we have an isomorphism*

$$T_{[\rho]}\left(\mathsf{Hom}(\Gamma, G)/\!/G\right) \cong H^1\left(\Gamma, \mathfrak{g}_{\mathrm{Ad}_\rho}\right).$$

This identification is extremely useful, as it allows cohomological techniques to be applied to the study of deformation spaces of representations. In particular, one can compute the dimension of the complex manifolds $\mathbb{B}^{\mathrm{gd}}$ and $\mathbb{S}^{\mathrm{gd}}$ using group cohomology.

When $\Gamma = \pi_1$ is the fundamental group of a compact Riemann surface of genus g, [38, Lemma 6.2] gives the following explicit dimensions, valid for any good representation ρ:

$$\dim Z^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}) = (2g-1)\dim G + \dim Z, \quad \dim B^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}) = \dim G - \dim Z,$$

where Z denotes the center of G. Hence, for $[\rho] \in \mathbb{B}^{\mathrm{gd}}$,

$$T_{[\rho]}\mathbb{B} \cong H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}), \quad \dim T_{[\rho]}\mathbb{B} = (2g-2)\dim G + 2\dim Z. \tag{5.4}$$

Next, we compute the dimension of $\mathbb{S}$ using the same cohomological methods. By Theorem 10, this computation may be carried out at good representations, which form a dense open subset.

Proposition 22 ([8], Proposition 7.1)**.** *Let $g \geqslant 2$. Then the dimension of $\mathbb{S}$ is given by*

$$\dim \mathbb{S} = (g-1)\dim G + \dim Z.$$

Lagrangian submanifold. We now describe an important geometric property of $\mathbb{S}^{\mathrm{gd}}$: it forms a Lagrangian submanifold inside $\mathbb{B}^{\mathrm{gd}}$ with respect to the natural symplectic form on the latter. Recall that a submanifold $L \subset M$ of a symplectic manifold (M, ω) is Lagrangian if ω vanishes on L and $\dim L = \frac{1}{2}\dim M$. It is well known that the character variety $\mathbb{B}$ carries a natural symplectic structure ([22]). This can be constructed as follows. Let $\langle\, ,\, \rangle$ be an Ad-invariant bilinear form on $\mathfrak{g}$. Then, using the cup product on group cohomology,

$$\cup : H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}) \otimes H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}) \to H^2(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}), \tag{5.5}$$

and composing it with contraction via $\langle\, ,\, \rangle$ and evaluation on the fundamental 2-cycle, we obtain a non-degenerate bilinear pairing

$$H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}) \otimes H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}) \xrightarrow{\cup} H^2(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho}) \xrightarrow{\langle\,,\,\rangle} H^2(\pi_1, \mathbb{C}) \cong \mathbb{C}. \tag{5.6}$$

Under the identification $T_{[\rho]}\mathbb{B}^{\mathrm{gd}} \cong H^1(\pi_1, \mathfrak{g}_{\mathrm{Ad}_\rho})$, this pairing defines a complex symplectic form on $\mathbb{B}^{\mathrm{gd}}$, which is holomorphic with respect to its complex structure. Moreover, the submanifold $\mathbb{S}^{\mathrm{gd}} \subset \mathbb{B}^{\mathrm{gd}}$ is Lagrangian.[1]

Theorem 12. *The good locus of the Schottky space, $\mathbb{S}^{\mathrm{gd}}$, is a Lagrangian submanifold of $\mathbb{B}^{\mathrm{gd}}$.*

Proof. The restriction of the map (5.5) to $H^1(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho})$ is trivial, since free groups have vanishing higher cohomology groups (see [7]):

$$\cup : H^1(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}) \otimes H^1(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}) \to H^2(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho}) = 0.$$

[1] For a general real Lie group, the analogous pairing defines a smooth (C^∞) symplectic structure; see [22].

Since the tangent space to $\mathbb{S}^{\mathrm{gd}}$ at a good point is identified with $H^1(F_g, \mathfrak{g}_{\mathrm{Ad}_\rho})$ (see Theorem 11), the symplectic form vanishes on any two tangent vectors to $\mathbb{S}^{\mathrm{gd}}$. As the dimension of $\mathbb{B}^{\mathrm{gd}}$ is twice that of $\mathbb{S}^{\mathrm{gd}}$ (see (5.4) and Proposition 22), the result follows. □

Relation with flat connections. We conclude this subsection by interpreting Schottky representations in terms of flat G-connections on a three-dimensional manifold. Let X be a compact Riemann surface of genus g, and let M be a compact 3-handlebody of genus g with boundary $\partial M \cong X$. We assume that $\pi_1(M, x_0) = F_g$ for some basepoint $x_0 \in X \subset M$. The inclusion $(X, x_0) \hookrightarrow (M, x_0)$ induces a surjective homomorphism

$$\varphi : \pi_1(X, x_0) \longrightarrow \pi_1(M, x_0),$$

sending $\alpha_i \mapsto 1$ and $\beta_i \mapsto \gamma_i$. Denote by $\mathbb{F}_M(G)$ the moduli space of flat G-connections on M.

Theorem 13. *The moduli space $\mathbb{S}$ of Schottky representations (with respect to φ) coincides with the moduli space $\mathbb{F}_M(G)$ of flat G-connections over M. That is,*

$$\mathbb{S} = \mathsf{Hom}(F_g, G)/\!/G \cong \mathbb{F}_M(G).$$

Proof. By construction, $\pi_1(M, x_0)$ is a free group of rank g, and $\pi_1(X, x_0)$ has a symplectic presentation as in Equation (5.2), so that

$$\varphi(\alpha_i) = 1, \quad \varphi(\beta_i) = \gamma_i, \quad i = 1, \ldots, g,$$

with $\gamma_1, \ldots, \gamma_g$ forming a free basis of $\pi_1(M, x_0)$. Thus, a Schottky representation $\rho : \pi_1(X) \to G$ factors through $F_g \cong \pi_1(M)$ if and only if ρ is the holonomy of a flat connection on M. The quotient by conjugation identifies equivalent flat bundles, giving the desired isomorphism of moduli spaces. □

5.4. Principal Higgs Bundles and Branes.

The theory of Higgs bundles provides a deep geometric counterpart to the study of representation varieties and flat connections discussed in the previous subsections. While representations of the fundamental group describe flat principal G-bundles, the Higgs bundle viewpoint introduces a holomorphic structure together with an additional field, the Higgs field, that encodes infinitesimal deformations of the connection. This correspondence lies at the heart of the celebrated *non-Abelian Hodge correspondence*, which relates complex geometry, algebraic geometry, and gauge theory in a unified framework. From a differential-geometric perspective, Higgs bundles arise as solutions to the *Hitchin equations*, which can be interpreted as a dimensional reduction of the self-dual Yang–Mills equations on a Riemann surface. The resulting moduli space of Higgs bundles carries a natural hyperkähler structure, endowing it with three compatible complex structures and a rich symplectic geometry. Within this space, certain submanifolds can be characterized as either complex or Lagrangian with respect to each of the complex structures. Following ideas of Kapustin and Witten [29], such submanifolds are called *branes*, and they play a central role in the geometric Langlands correspondence and in mirror symmetry. Let X be a compact Riemann surface of genus $g \geqslant 2$ and G a connected complex reductive group.

Definition 24. A pair (E, ϕ) is called a *G-Higgs bundle* on X if E is a holomorphic principal G-bundle on X and ϕ, the *Higgs field*, is a holomorphic section of $Ad(E) \otimes K$, where $Ad(E)$ denotes the adjoint bundle of E and K the canonical bundle of X.

Once a suitable notion of stability is introduced, one can construct the moduli space $\mathcal{H}$ of polystable G-Higgs bundles, which carries a natural hyperkähler structure (Hitchin [26]). Denoting by (I, J, K) the three complex structures defining the hyperkähler structure, one may consider submanifolds of $\mathcal{H}$ that are Lagrangian (type A) or complex (type B) with respect to each of these structures. Kapustin and Witten [29] introduced the terminology of *branes* for these submanifolds, which may thus be of types (B, A, A), (A, B, A), or (A, A, B). Such objects play a fundamental role in the geometric Langlands program and in mirror symmetry. A rich source of examples arises from the study of anti-holomorphic involutions of X.

Anti-holomorphic involutions. Let $f : X \to X$ be an *anti-holomorphic involution*. Fixing a base point $x_0 \in X$, the map f induces an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, f(x_0))$. Choosing a path γ from x_0 to $f(x_0)$, the composition of this isomorphism with the conjugation by γ defines an automorphism of $\pi_1(X, x_0)$. Changing the path γ modifies this automorphism by an inner automorphism, and thus f induces a well-defined outer automorphism of $\pi_1(X, x_0)$. This construction yields an involution on the Betti moduli space $\mathbb{B}$. In particular, it preserves the good locus $\mathbb{B}^{\mathrm{gd}}$, which may be identified with the moduli space of gauge-equivalence classes of flat G-connections on X with reductive holonomy. The involution acts on this space by pullback of connections.

By the non-Abelian Hodge theorem (Hitchin [26], Simpson [49], Donaldson [14], Corlette [11]), the moduli space of such flat connections is diffeomorphic to $\mathcal{H}$, the moduli space of G-Higgs bundles. Hence, the anti-holomorphic involution of X induces a corresponding involution of $\mathcal{H}$. In [1], Baraglia and Schaposnik denoted by $\mathcal{L}_G$ the set of fixed points of this involution in $\mathbb{B}^{\mathrm{gd}}$ (or equivalently in $\mathcal{H}$) and proved the following result.

Proposition 23 ([1, Proposition 10])**.** *If non-empty, the set of good points of $\mathcal{L}_G$ is a smooth Lagrangian submanifold of $\mathbb{B}^{\mathrm{gd}}$.*

Following the ideas of Kapustin and Witten [29], Baraglia and Schaposnik further established the brane interpretation of this locus:

Theorem 14 ([1, Theorem 14]). *The fixed-point locus $\mathcal{L}_G$ defines an (A, B, A)-brane inside the hyperkähler manifold $\mathcal{H}$.*

Higgs bundles and 3-manifolds. Consider now the 3-manifold with boundary

$$\widehat{X} := X \times [-1, 1],$$

and define an orientation-preserving involution $\sigma : \widehat{X} \to \widehat{X}$ by $\sigma(x, t) = (f(x), -t)$. The boundary of $\widehat{X}$ thus consists of two copies of X, and the quotient

$$M := \widehat{X}/\sigma$$

is a compact 3-manifold whose boundary is homeomorphic to X.

Proposition 24 ([1, Proposition 43]). *The representations of X into G that extend to M belong to the (A, B, A)-brane $\mathcal{L}_G$.*

This subspace of representations can equivalently be described as the space $\mathbb{F}_M(G)$ of flat G-connections on X that extend to flat G-connections over M.

5.5. Schottky Representations and Branes.

Let X be a compact Riemann surface equipped with an anti-holomorphic involution $f : X \to X$, which defines a real structure on X. Following the constructions and notation introduced in Subsection 5.4, consider the compact 3-manifold M whose boundary is homeomorphic to X. The interaction between the topology of M and the geometry of the moduli space of principal Higgs bundles on X allows one to relate Schottky representations to the (A, B, A)-brane $\mathcal{L}_G$, inside the hyperkähler moduli space $\mathcal{H}$.

Theorem 15. *Let $f : X \to X$ be an anti-holomorphic involution such that the associated 3-manifold M is a handlebody of genus g, and let $x_0 \in X \subset M$ be a fixed point of f. Then the moduli space $\mathbb{S}$ of Schottky representations with respect to the map φ in (5.3) is contained in the Baraglia–Schaposnik brane $\mathcal{L}_G$.*

Proof. By Proposition 24, there exists an inclusion

$$\mathbb{F}_M(G) \longrightarrow \mathcal{L}_G \subset \mathcal{H}.$$

Since the moduli space $\mathbb{S}$ can be naturally identified with $\mathbb{F}_M(G)$ by Theorem 13, the claim follows immediately. □

Remark 5. The hypotheses of Theorem 15 are satisfied when the anti-holomorphic involution f has as fixed-point locus the disjoint union of $g + 1$ loops, and the corresponding orientation double cover $X/\langle f\rangle$ is disconnected (see [24]). In this situation, Proposition 23 ensures that the locus of smooth points of $\mathcal{L}_G$ is a non-empty Lagrangian submanifold of the hyperkähler manifold $\mathcal{H}$. This particular geometric configuration can be further investigated in subsequent work.

In light of this construction, and since good Schottky representations exist for every genus $g \geqslant 2$, one concludes that the set of smooth points of the Baraglia–Schaposnik brane $\mathcal{L}_G$ is non-empty.

Concluding Remarks. The constructions developed here, ranging from Schottky representations to branes in hyperkähler geometry, demonstrate how topological data encoded by fundamental groups manifests in the rich geometry of holomorphic and flat connections. Further connections with three-dimensional topology and mirror symmetry also remain fertile ground for exploration, suggesting that the interplay between topology, complex geometry, and representation theory is far from fully understood. In particular, a deeper understanding of the Baraglia–Schaposnik branes through the lens of Schottky representations may reveal new insights into the geometry of moduli spaces and their role in the broader landscape of the geometric Langlands correspondence.

Acknowledgements

This work originated from a series of lectures delivered at the *Workshop on Character Varieties and Higgs Bundles*, held in Liberia, Guanacaste, Costa Rica, in August 2025. I am deeply grateful to Ronald Zúñiga-Rojas and Alexander Schmitt for their invitation and for the opportunity to discuss these ideas in such an inspiring and stimulating setting. My thanks also go to the University of Costa Rica, Sede Guanacaste, for its kind hospitality during the workshop and throughout the development of this work. I would like to express my appreciation to Steven Bradlow, Leticia Brambila-Paz, Guillermo Gallego, Jacques Hurtubise, Ruxandra Moraru, Olga Trapeznikova, Mengxue Yang, and Alfonso Zamora, as well as to all students and participants of the workshop, for their insightful comments and constructive suggestions that helped to improve the exposition and clarify several aspects of the presentation. I am also grateful to my collaborators and co-authors Susana Ferreira, Carlos Florentino, Sean Lawton and André Oliveira for many valuable discussions related to the topics of this work. I thank the anonymous referee for carefully reading the manuscript and for suggestions that helped to improve its clarity and presentation. This work is funded by national funds through the FCT – Fundação para a Ciência e a Tecnologia, I.P., under the scope of the projects UID/297/2025 and UID/PRR/297/2025 (Center for Mathematics and Applications – NOVA Math).

References

[1] D. Baraglia, L. P. Schaposnik, *Higgs Bundles and (A,B,A)-Branes*. Communications in Mathematical Physics **331**(2014), 1271–1300. DOI: 10.1007/s00220-014-2053-6

[2] M. Bergeron, *The topology of nilpotent representations in reductive groups and their maximal compact subgroups*. Geometry and Topology **19**(2015), no. 3, 1383–1407. DOI: 10.2140/gt.2015.19.1383

[3] I. Biswas, C. Florentino, *Character varieties of virtually nilpotent Kähler groups and G–Higgs bundles*. Annales de l'Institut Fourier (Grenoble) **65** (2015), no. 6, 2601–2612. DOI: 10.5802/aif.2997

[4] J. Bochnak, M. Coste, M.-F. Roy, *Real Algebraic Geometry*. Vol. 36. Ergebnisse der Mathematik und ihrer Grenzgebiete (3). Springer-Verlag, Berlin, 1998. DOI: 10.1007/978-3-662-03718-8

[5] A. Borel, *Linear Algebraic Groups*. 2nd ed. Vol. 126. Graduate Texts in Mathematics. Springer-Verlag, New York, 1991. DOI: 10.1007/978-1-4612-0941-6

[6] T. Bröcker, T. tom Dieck, *Representations of Compact Lie Groups*. Vol. 98. Graduate Texts in Mathematics. Springer, New York, 1985. DOI: 10.1007/978-3-662-12918-0

[7] K. S. Brown, *Cohomology of Groups*. Graduate Texts in Mathematics 87. Springer, New York, 1994. DOI: 10.1007/978-1-4684-9327-6

[8] A. C. Casimiro, S. Ferreira, C. Florentino, *Principal Schottky Bundles over Riemann Surfaces*. Geometriae Dedicata **201**(2019), 379–409. DOI: 10.1007/s10711-018-0398-2

[9] A. C. Casimiro, S. Ferreira, *Principal Higgs Bundles and Schottky Representations*. P. Cerejeiras, M. Reissig, I. Sabadini, J. Toft (Eds.). Current Trends in Analysis, its Applications and Computation. Springer International Publishing, Cham, 2022, 89–97. DOI: 10.1007/978-3-030-87502-2_10

[10] A. C. Casimiro, C. Florentino, S. Lawton, A. Oliveira, *Topology of moduli spaces of free group representations in real reductive groups*. Forum Mathematicum **28**(2016), no. 2, 275–294. DOI: 10.1515/forum-2014-0049

[11] K. Corlette, *Flat G-bundles with canonical metrics*. Journal of Differential Geometry **28**(1988), 361–382. DOI: 10.4310/JDG/1214442469

[12] M. Culler, P. B. Shalen, *Varieties of group representations and splittings of 3-manifolds*. Annals of Mathematics **117**(1983), no. 1, 109–146. DOI: 10.2307/2006973

[13] I. Dolgachev, *Lectures on Invariant Theory*. London Mathematical Society Lecture Note Series 296. Cambridge University Press, Cambridge, 2003. DOI: 10.1017/CBO9780511615436

[14] S. K. Donaldson, *Twisted Harmonic Maps and the Self-Duality Equations*. Proceedings of the London Mathematical Society **55**(1987), no. 1, 127–131. DOI: 10.1112/plms/s3-55.1.127

[15] J.-M. Drezet, *Luna's slice theorem and applications*. Algebraic Group Actions and Quotients. Hindawi Publishing Corporation, Cairo, 2004, 39–89.

[16] C. Florentino, P. B. Gothen, A. Nozad, *Homotopy type of moduli spaces of G-Higgs bundles and reducibility of the nilpotent cone*. Bulletin des Sciences Mathématiques **150**(2019), 84–101. DOI: 10.1016/j.bulsci.2018.10.002

[17] C. Florentino, S. Lawton, *The topology of moduli spaces of free group representations*. Mathematische Annalen **345**(2009), no. 2, 453–489. DOI: 10.1007/s00208-009-0362-4

[18] C. Florentino, S. Lawton, *Character varieties and moduli of quiver representations*. The Tradition of Ahlfors-Bers VI. Vol. 590. Contemporary Mathematics. American Mathematical Society, Providence, RI, 2013, 9–38. DOI: 10.1090/conm/590/11720

[19] C. Florentino, S. Lawton, *Topology of character varieties of Abelian groups*. Topology and its Applications **173**(2014), 32–58. DOI: 10.1016/j.topol.2014.05.009

[20] C. Florentino, S. Lawton, *Flawed groups and the topology of character varieties*. Topology and its Applications **341**(2024), 108756. DOI: 10.1016/j.topol.2023.108756

[21] R. Fricke, H. Vogt, *Vorlesungen über die Theorie der automorphen Funktionen. Band I*. Reprinted by Johnson Reprint Corp., 1965. Teubner, Leipzig, 1897.

[22] W. M. Goldman, *The symplectic nature of fundamental groups of surfaces*. Advances in Mathematics **54**(1984), no. 2, 200–225. DOI: 10.1016/0001-8708(84)90040-9

[23] W. M. Goldman, *Geometric structures on manifolds*. Lecture notes, December 2021. 2021.

[24] B. H. Gross, J. Harris, *Real algebraic curves*. Annales Scientifiques de l'École Normale Supérieure **14**(1981), no. 2, 157–182. DOI: 10.24033/asens.1401

[25] A. Hatcher, *Algebraic Topology*. Cambridge University Press, Cambridge, 2002. DOI: 10.1017/S0013091503214620

[26] N. J. Hitchin, *The self-duality equations on a Riemann surface*. Proceedings of the London Mathematical Society **55**(1987), no. 1, 59–126. DOI: 10.1112/plms/s3-55.1.59

[27] J. E. Humphreys, *Linear Algebraic Groups*. Graduate Texts in Mathematics 21. Springer New York, 1975. DOI: 10.1007/978-1-4684-9443-3

[28] D. Johnson, J. J. Millson, *Deformation spaces associated to compact hyperbolic manifolds*. Discrete Groups in Geometry and Analysis (New Haven, Conn., 1984). Vol. 67. Progress in Mathematics. Birkhäuser Boston, Boston, MA, 1987, 48–106. DOI: 10.1007/978-1-4899-6664-3_3

[29] A. Kapustin, E. Witten, *Electric-magnetic duality and the geometric Langlands program*. Communications in Number Theory and Physics **1**(2007), no. 1, 1–236. DOI: 10.4310/CNTP.2007.v1.n1.a1

[30] Y. Karshon, *An algebraic proof for the symplectic structure of moduli space*. Proceedings of the American Mathematical Society **116**(1992), no. 3, 591–605. DOI: 10.1090/S0002-9939-1992-1112494-2

[31] G. Kempf, L. Ness, *The Length of Vectors in Representation Spaces*. Algebraic Geometry, Proc. Summer Meeting, Univ. Copenhagen, Copenhagen, 1978. Vol. 732. Lecture notes in Mathematics. Springer, Berlin, 1979, 233–243. DOI: 10.1007/BFb0066647

[32] G. R. Kempf, *Instability in invariant theory*. Annals of Mathematics **108** (1978), no. 2, 299–316. DOI: 10.2307/1971168

[33] F. Labourie, *Lectures on Representations of Surface Groups*. Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2013. DOI: 10.4171/127

[34] S. Lawton, *Poisson geometry of SL(3,C)-character varieties relative to a surface with boundary*. Transactions of the American Mathematical Society **361**(2009), no. 5, 2397–2429. DOI: 10.1090/S0002-9947-08-04777-6

[35] B. Loustau, *Minimal surfaces and quasi-Fuchsian structures*. 2015.

[36] J. Marché, *Character Varieties*. 2022.

[37] A. Maret, *A note on Character Varieties*. Preliminary version, compiled May 14, 2025. 2025.

[38] B. M. S. Martin, *Restrictions of representations of surface groups to a pair of free subgroups*. Journal of Algebra **225**(2000), no. 1, 231–249. DOI: 10.1006/jabr.1999.8110

[39] J. S. Milne, *Lie Algebras, Algebraic Groups, and Lie Groups*. 2013.

[40] J. S. Milne, *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2017. DOI: 10.1017/9781316711736

[41] G. Mondello, *Topology of representation spaces of surface groups in PSL(2,R) with assigned boundary monodromy and nonzero Euler number*. Pure and Applied Mathematics Quarterly **12**(2016), no. 3, 399–462. DOI: 10.4310/PAMQ.2016.v12.n3.a3

[42] A. Neeman, *The topology of quotient varieties*. Annals of Mathematics (2) **122**(1985), no. 2, 419–459. DOI: 10.2307/1971309

[43] C. Procesi, *The invariant theory of n x n matrices*. Advances in Mathematics **19**(1976), no. 3, 306–381. DOI: 10.1016/0001-8708(76)90027-X

[44] C. Procesi, G. Schwarz, *Inequalities defining orbit spaces*. Inventiones Mathematicae **81**(1985), no. 3, 539–554. DOI: 10.1007/BF01388587

[45] R. W. Richardson, P. J. Slodowy, *Minimum vectors for real reductive algebraic groups*. Journal of the London Mathematical Society **42**(1990), no. 3, 409–429. DOI: 10.1112/jlms/s2-42.3.409

[46] G. W. Schwarz, *The Topology of Algebraic Quotients*. Topological Methods in Algebraic Transformation Groups (New Brunswick, NJ, 1988). Vol. 80. Progress in Mathematics. Birkhäuser Boston, Boston, MA, 1989, 135–151. DOI: 10.1007/978-1-4612-3702-0_9

[47] A. S. Sikora, *Character varieties*. Transactions of the American Mathematical Society **364**(2012), no. 10, 5173–5208. DOI: 10.1090/S0002-9947-2012-05448-1

[48] A. S. Sikora, *Generating sets for coordinate rings of character varieties*. Journal of Pure and Applied Algebra **217**(2013), no. 11, 2076–2087. DOI: 10.1016/j.jpaa.2013.01.005

[49] C. T. Simpson, *Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization*. Journal of the American Mathematical Society **1**(1988), no. 4, 867–918. DOI: 10.1090/S0894-0347-1988-0944577-9

[50] J. H. C. Whitehead, *Combinatorial Homotopy. I*. Bulletin of the American Mathematical Society **55**(1949), 213–245. DOI: 10.1090/S0002-9904-1949-09175-9