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MODULI SPACE OF HIGGS COHERENT SYSTEMS WITH STABLE UNDERLYING VECTOR BUNDLES OVER A CURVE

ESPACIO MÓDULI DE SISTEMAS COHERENTES DE HIGGS CON HAZ SUBYACENTE ESTABLE SOBRE UNA CURVA

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Abstract

Let X be a smooth, connected complex projective curve of genus at least 2. A Higgs coherent system is defined as an augmented bundle (E, V), where E is a holomorphic vector bundle over X, and V is a linear subspace of the space of Higgs fields of E. The purpose of this paper is twofold. First, we introduce the moduli problem of Higgs coherent systems over X. Second, we construct the fine moduli space for Higgs coherent systems with stable underlying vector bundles. The construction is achieved by establishing the representability of the associated moduli functor by a projective scheme together with a universal family.

Keywords: moduli spaces; Higgs bundles; coherent systems; Higgs coherent systems.

Resumen

Sea X una curva proyectiva compleja, suave y conexa, de género al menos 2. Un sistema coherente de Higgs se define como un haz aumentado (E, V), donde E es un haz vectorial holomorfo sobre X y V es un subespacio lineal del espacio de campos de Higgs de E. El propósito de este artículo es doble. En primer lugar, introducimos el problema móduli de los sistemas coherentes de Higgs sobre X. En segundo lugar, construimos el espacio móduli fino para los sistemas coherentes de Higgs cuyo haz vectorial subyacente es estable. La construcción consiste en establecer que el funtor móduli asociado es representado por un esquema proyectivo junto con una familia universal.

Palabras clave: espacios móduli; haces de Higgs; sistemas coherentes; sistemas coherentes de Higgs.

Mathematics Subject Classification: Primary 14H60; Secondary 14D20, 14D22

1. INTRODUCTION

Let X be a smooth, connected complex projective curve of genus $g \geq 2$, and denote by ω_X its canonical bundle. The moduli spaces of Higgs bundles and coherent systems over X have been extensively studied due to their relevance in connecting geometry, topology, and physics.

A Higgs bundle over X is a pair (E, φ), where E is a vector bundle over X and φ is a Higgs field of E; that is, φ is a section in $H^0(X, \mathrm{End}(E) \otimes \omega_X)$. Higgs bundles were first studied by N. J. Hitchin as the solutions of the Yang-Mills equations on a Riemann surface [13], establishing a connection between differential equations and algebraic geometry. Since then, they have become central objects intersecting geometry, topology, and physics ([5], [26], [23]). Diverse perspectives include foundational work by M. F. Atiyah and R. Bott [1], the generalization to higher dimensions by C. Simpson [25], the study of its moduli space by T. Hausel [12], and its important role in N. Bao Châu's proof of the Fundamental Lemma [2]. On the other hand, a coherent system is a pair (E, V), where E is a vector bundle over X and V is a linear subspace of $H^0(X, E)$. Coherent systems have contributed significantly to the understanding of the geometry of the moduli space of vector bundles [3], [4], [20].

In this paper, we study augmented bundles that are related to Higgs bundles and coherent systems, called **Higgs coherent systems**. These augmented bundles can be understood as a collection of Higgs bundles approached from the perspective of coherent systems, where we consider a vector bundle together with a linear subspace of Higgs fields rather than a single Higgs field. Specifically, a Higgs coherent system of type (n, d, k) is a pair (E, V) where E is a vector bundle over X of rank n and degree d, and V is a k-dimensional subspace of $H^0(X, \mathrm{End}(E)\otimes\omega_X)$. The aim of this paper is to study the moduli problem of Higgs coherent systems whose underlying vector bundles are stable and to construct the moduli space of such augmented bundles.

Let n, d, k be three integers such that $(n, d) = 1$ and $0 < k \leq n^2(g-1)+1$. Denote by $SH(n, d, k)$ the set of Higgs coherent systems (E, V) of type (n, d, k) such that E is a stable vector bundle. The stability assumption allows us to describe the isomorphism classes of these Higgs coherent systems and construct a parameter scheme for them. This parameter scheme inherits some of the geometric properties of the moduli space of stable bundles, as stated in the following result.

Theorem I. *(Theorem 3.4) The space of isomorphism classes of Higgs coherent systems in $SH(n, d, k)$ is a smooth irreducible projective scheme of dimension $(k+1)(n^2(g-1)+1)-k^2$.*

We construct a family of Higgs coherent systems in $SH(n, d, k)$ with the universal property parameterized by the above parameter scheme; this allows us to prove the existence of a fine moduli space. The result is presented as follows.

Theorem II. *(Theorem 3.7) There exists a fine moduli space for Higgs coherent systems in $SH(n, d, k)$ in the category of smooth irreducible projective schemes.*

As a corollary of the above result, we obtain the moduli space of Higgs coherent systems of type $(1, d, k)$, previously studied in [8], as established in Corollary 3.9.

The paper is organized as follows. In Section 2, we introduce Higgs coherent systems of type (n, d, k) over X and their basic properties. Then, we introduce the concept of families parameterized by a scheme and set out the problem we are interested in; the moduli problem of Higgs coherent systems of type (n, d, k) over X. Our main results are stated and proved in Section 3.

Notation: We work over the field of complex numbers, denoted by $\mathbb{C}$. Unless stated otherwise, by scheme we always mean a noetherian projective scheme of finite type over $\mathbb{C}$; by a point, we always mean a closed point; and by a vector bundle, we always mean a holomorphic vector bundle. Let X and Y be two schemes; we denote by $X \times Y$ the fiber product $X \times_{\mathrm{Spec}(\mathbb{C})} Y$. For a vector bundle E over X, we denote by $\deg(E)$ its degree, by $\mathrm{rk}(E)$ its rank, and by $h^i(E)$ the dimension of the cohomology group $H^i(X,E)$. We will write $H^i(E)$ instead of $H^i(X,E)$ when no confusion arises. Let n and d be two integers with $n \geq 1$. For a vector bundle of type (n,d), we mean a vector bundle of rank n and degree d. Let $\alpha\colon F \to E$ be a morphism of vector bundles; we denote by $H^i(\alpha)\colon H^i(F) \to H^i(E)$ the induced morphism in the cohomology groups. If x is a point in X, we denote by E_x the fiber of E at x. Similarly, if $\mathcal{E}$ is a vector bundle over a product $X \times Y$ and y is a point in Y, we denote by $\mathcal{E}_y$ the vector bundle over X obtained by the restriction of $\mathcal{E}$ to $X \times \{y\}$. Let $f\colon X \to Y$ be a morphism of schemes; for $i \geq 0$, we denote by $R^i f_*(-)$ the i-th right derived functor of the direct image functor $f_*(-)$.

2. Moduli problem of Higgs coherent systems

Let X be a smooth, connected complex projective curve of genus $g \geq 2$, and let ω_X denote its canonical bundle. In this section, we introduce Higgs coherent systems over X and study their main properties. Subsequently, we introduce the concept of families parameterized by a scheme and set out the problem of Higgs coherent systems over X.

2.1. Higgs coherent systems.

Let n, d, k be three integers, where $n, k \geq 1$. We begin with the definition of a Higgs coherent system of type (n,d,k) over X.

Definition 2.1. *A* **Higgs coherent system of type** (n,d,k) *on* X *is a pair* (E,V) *consisting of a vector bundle* E *over* X *of type* (n,d)*, and a linear subspace* V *of* $H^0(X, \mathrm{End}(E) \otimes \omega_X)$ *of dimension* k*. We will denote by* $H(n,d,k)$ *the set of all Higgs coherent systems of type* (n,d,k) *over* X*.*

We refer to the space $H^0(X, \mathrm{End}(E) \otimes \omega_X)$ as the twisted endomorphisms of E or as the space of Higgs fields of E. The pair $(0,0)$ is referred to as the trivial Higgs coherent system and is sometimes denoted by $0 := (0,0)$.

Let E be a vector bundle of type (n,d). By Serre's duality and Riemann-Roch's theorem, we have that

$$h^0(\mathrm{End}(E) \otimes \omega_X) = n^2(g-1) + h^0(\mathrm{End}(E)). \tag{2.1}$$

Consequently, the dimension of the twisted endomorphisms of E depends on the dimension of the space of automorphisms of E. In particular, if L is a line bundle, it follows that $\mathrm{End}(L) \otimes \omega_X = \omega_X$. Then, the space of twisted endomorphisms of a line bundle is the space of holomorphic 1-forms $H^0(X, \omega_X)$. In particular, $H(1,d,k) \neq \emptyset$ if and only if $k \leq g$.

From now on, we assume that $n \geq 2$. The next result is related to the non-emptiness of $H(n,d,k)$.

Proposition 2.2. *For any type (n,d,k), the set of Higgs coherent systems $H(n,d,k)$ is nonempty. Moreover, let (E,V) be a Higgs coherent system in $H(n,d,k)$:*

i. if $k > n^2(g-2)+1$, then E is not simple;

ii. if $k > n^2(g-\frac{1}{2}) - \frac{n}{2} + 1$, then E is decomposable or unstable.

Proof. Let E be a vector bundle of type (n,d). By (2.1), we have that

$$h^0(\mathrm{End}(E)\otimes\omega_X) \geq n^2(g-1)+1.$$

If $k \leq n^2(g-1)+1$, there always exists a linear subspace of dimension k of the space $H^0(X,\mathrm{End}(E)\otimes\omega_X)$, and thus $H(n,d,k)\neq\emptyset$. Now, if $k > n^2(g-1)+1$, we can always produce a decomposable vector bundle E such that $h^0(\mathrm{End}(E))$, and hence $h^0(\mathrm{End}(E)\otimes\omega_X)$, is big enough. We proceed as follows: Let r be an integer, and $x\in X$ a point; we consider the vector bundle of type (n,d)

$$E_r := \mathcal{O}_X((-r)x)\oplus\mathcal{O}_X((d+r)x)\oplus\mathcal{O}_X^{\oplus(n-2)}.$$

We have that

$$\begin{aligned}\mathrm{End}(E_r) =& \mathcal{O}_X((d+2r)x)\oplus\mathcal{O}_X^{\oplus n-2}((d+r)x)\oplus\mathcal{O}_X^{\oplus n-2}((r)x)\oplus\mathcal{O}_X^{\oplus(n-2)^2+2}\oplus\\ &\mathcal{O}_X((-d-2r)x)\oplus\mathcal{O}_X^{\oplus n-2}((-d-r)x)\oplus\mathcal{O}_X^{\oplus n-2}((-r)x).\end{aligned}$$

Take $r > \max\{0,-d\}$. It follows that

$$\begin{aligned}h^0(\mathrm{End}(E_r)) &= (n-2)^2+2+h^0(\mathcal{O}_X((d+2r)x)+(n-2)[h^0(\mathcal{O}_X((d+r)x)+h^0(\mathcal{O}_X((r)x))]\\ &\geq (n-2)^2+2+h^0(\mathcal{O}_X((d+2r)x)\\ &\geq (n-2)^2+d+2r+3-g.\end{aligned}$$

Therefore,

$$\begin{aligned}h^0(\mathrm{End}(E_r)\otimes\omega_X) &= n^2(g-1)+h^0(\mathrm{End}(E_r))\\ &\geq n^2(g-1)+(n-2)^2+d+2r+3-g\\ &\geq n^2(g-1)+d-g+2r.\end{aligned}$$

Take $r > \max\{0,-d,\frac{1-d+g}{2},\frac{k-n^2(g-1)-d+g}{2}\}$. Then, $h^0(\mathrm{End}(E_r)\otimes\omega_X)\geq k$, and hence there is a subspace V_r of dimension k of $H^0(X,\mathrm{End}(E_r)\otimes\omega_X)$, which implies that (E_r,V_r) lies in $H(n,d,k)$. For the last part, if E is a simple vector bundle, then $k\leq h^0(\mathrm{End}(E)\otimes\omega_X) = n^2(g-1)+1$. If E is an indecomposable and semistable vector bundle, then $h^0(\mathrm{End}(E))\leq 1+\frac{n(n-1)}{2}$ ([6, Proposition 1]). Therefore, $k\leq h^0(\mathrm{End}(E)\otimes\omega_X)\leq n^2(g-\frac{1}{2})-\frac{n}{2}+1$, which completes the proof. ◻

2.2. Morphisms of Higgs coherent systems.

Now we define a morphism between Higgs coherent systems. For this, first note that a morphism of vector bundles, $\alpha\colon F \to E$, induces the following morphisms

$$\begin{aligned} &\alpha \otimes id_{F^\vee\otimes\omega_X}\colon F \otimes F^\vee \otimes \omega_X \to E \otimes F^\vee \otimes \omega_X \text{ and}\\ &\alpha^\vee \otimes id_{E\otimes\omega_X}\colon E^\vee \otimes E \otimes \omega_X \to F^\vee \otimes E\otimes\omega_X, \end{aligned}$$

and thus, we have the induced homomorphisms on the cohomology groups

$$\begin{tikzcd} & H^0(X,\mathrm{End}(E)\otimes\omega_X) \arrow[d, "\alpha_2"] \\ H^0(X,\mathrm{End}(F)\otimes\omega_X) \arrow[r, "\alpha_1"] & H^0(X, F^\vee\otimes E\otimes\omega_X) \end{tikzcd}, \tag{2.2}$$

where

$$\begin{aligned} &\alpha_1(\psi) = H^0(\alpha\otimes id_{F^\vee\otimes\omega_X})(\psi) = (\alpha\otimes id_{F^\vee\otimes\omega_X})\circ\psi, \text{ for all } \psi\in H^0(X,\mathrm{End}(F)\otimes\omega_X) \text{ and}\\ &\alpha_2(\varphi) = H^0(\alpha^\vee\otimes id_{E\otimes\omega_X})(\varphi) = (\alpha^\vee\otimes id_{E\otimes\omega_X})\circ\varphi, \text{ for all } \varphi\in H^0(X,\mathrm{End}(E)\otimes\omega_X). \end{aligned}$$

The above two homomorphisms share a common codomain, which allows us to relate subspaces of $H^0(X,\mathrm{End}(F)\otimes\omega_X)$ to subspaces of $H^0(X,\mathrm{End}(E)\otimes\omega_X)$. We use this fact to define a morphism between Higgs coherent systems.

Definition 2.3. *A* **morphism** *of Higgs coherent systems, $h_\alpha\colon (F,W)\to(E,V)$, is a morphism $\alpha\colon F\to E$ such that $\alpha_1(W)\subseteq\alpha_2(V)$.*

Remark 2.4. *Let (F,W) and (E,V) be two Higgs coherent systems.*

i. A morphism from the trivial Higgs coherent system $(0,0)$ to a Higgs coherent system (E,V), by convention, consists of the inclusion in each component: $0\subset E$ and $0\subset V$. The morphism $(E,V)\to(0,0)$ means that E maps to 0 and V maps to 0

ii. Let $h_\alpha\colon (F,W)\to(E,V)$ be a morphism of Higgs coherent systems. For any $\psi\in H^0(X,\mathrm{End}(F)\otimes\omega_X)$, there exists $\varphi\in H^0(X,\mathrm{End}(E)\otimes\omega_X)$ such that $\alpha_1(\psi)=\alpha_2(\varphi)$. Since $H^0(X,\mathrm{End}(E)\otimes\omega_X)=\mathrm{Hom}(E,E\otimes\omega_X)$, any section φ of $\mathrm{End}(E)\otimes\omega_X$ corresponds to a morphism $\varphi\colon E\to E\otimes\omega_X$. Therefore, $\alpha_1(\psi)=\alpha_2(\varphi)$ if and only if the following diagram commutes:

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & E \arrow[d, "\varphi"] \\ F\otimes\omega_X \arrow[r, "\alpha\otimes id_{\omega_X}"'] & E\otimes\omega_X \end{tikzcd}. \tag{2.3}$$

iii. A morphism $h_\alpha\colon (F,W)\to(E,V)$ is determined by the morphisms α, α_1, and α_2. In particular, if $\alpha\colon F\to E$ is injective, then α_1 is injective. Analogously, if $\alpha\colon F\to E$ is surjective, then α_2 is injective

iv. Let $h_\alpha\colon (F,W)\to(E,V)$ be a morphism of Higgs coherent systems such that α is surjective. The injectivity of α_2 restricted to V, together with the condition $\alpha_1(W)\subseteq\alpha_2(V)$, implies that the map

$$\begin{aligned}\hat\alpha\colon H^0(X,\mathrm{End}(F)\otimes\omega_X) &\dashrightarrow H^0(X,\mathrm{End}(E)\otimes\omega_X)\\ \psi &\mapsto \alpha_2^{-1}(\alpha_1(\psi))\end{aligned}\tag{2.4}$$

is well defined on W and $\hat\alpha(W)\subseteq V$. In this case, h_α is written as $(\alpha,\alpha_1,\alpha_2,\hat\alpha)$, or simply $(\alpha,\hat\alpha)$

Let $h_\alpha\colon (F,W)\to(E,V)$ be a morphism of Higgs coherent systems. If $\alpha\colon F\to E$ is an isomorphism, then α_1 and α_2 are injective, and hence the homomorphism in (2.4) is well-defined for all Higgs fields of F. We use these facts to define when the Higgs coherent systems (F,W) and (E,V) are isomorphic.

Definition 2.5. Two Higgs coherent systems, (F,W) and (E,V), are **isomorphic systems** if there exists a morphism $h_\alpha\colon (F,W)\to(E,V)$ with α being an isomorphism such that $\hat\alpha(W)=V$. In this case, we say that h_α is an **isomorphism** and we write $(F,W)\sim(E,V)$.

Remark 2.6. Let $h_\alpha=(\alpha,\hat\alpha)\colon (F,W)\to(E,V)$ be an isomorphism. From its definition in (2.4), it follows that $\hat\alpha$ is an isomorphism and

$$\begin{aligned}\hat\alpha = H^0(\alpha\otimes\alpha^{-\vee}\otimes id_{\omega_X})\colon H^0(X,\mathrm{End}(F)\otimes\omega_X) &\to H^0(X,\mathrm{End}(E)\otimes\omega_X)\\ \psi &\mapsto (\alpha\otimes\alpha^{-\vee}\otimes id_{\omega_X})\circ\psi.\end{aligned}\tag{2.5}$$

In particular, (F,W) and (E,V) have the same type. Moreover, if (F,ψ) and (E,φ) are two Higgs bundles, they are isomorphic if and only if there exists an isomorphism $\alpha\colon F\to E$ such that $\hat\alpha(\psi)=\varphi$. In this sense, the definition of isomorphic Higgs coherent systems naturally extends that of isomorphic Higgs bundles to linear subspaces.

2.3. Families of Higgs coherent systems.

In what follows, we introduce the notion of how Higgs coherent systems deform. That is, we define the concept of families with which we will work with. Roughly speaking, a family of Higgs coherent systems parameterized by a scheme S is a pair $(\mathcal{E},\mathcal{V})$ where $\mathcal{E}$ is a family of vector bundles parameterized by S and $\mathcal{V}$ is a locally free sheaf of rank k over S, whose fiber at each s in S is precisely $H^0(X,\mathrm{End}(\mathcal{E}_s)\otimes\omega_X)$.

Let $\mathcal{E}$ be a family of vector bundles of type (n,d) parameterized by a scheme S. For any point s in S, we are interested in the subspaces of $H^0(X,\mathrm{End}(\mathcal{E}_s)\otimes\omega_X)$, so it is natural to consider the direct image $R^0\pi_{2^*}(\mathrm{End}(\mathcal{E})\otimes\pi_X^*(\omega_X))$ with respect to the projection to the second factor $\pi_2\colon X\times S\to S$. However, in general, the fibers of this sheaf are not equal to $H^0(X,\mathrm{End}(\mathcal{E}_s)\otimes\omega_X)$ (see [16, Corollary 4.2.9]). Instead of $R^0\pi_{2^*}(\mathrm{End}(\mathcal{E})\otimes\pi_X^*(\omega_X))$, we consider the first direct image $R^1\pi_{2^*}(\mathrm{End}(\mathcal{E}))$. That is,

$$\begin{tikzcd} \mathrm{End}(\mathcal{E}) \arrow[d] & R^1\pi_{2^*}(\mathrm{End}(\mathcal{E})) \arrow[d] \\ X\times S \arrow[r, "\pi_2"] & S \end{tikzcd}.$$

We define $\mathcal{R}^1_{\mathcal{E}} := R^1\pi_{2^*}(\mathrm{End}(\mathcal{E}))$. The following result proves that a locally free sheaf whose dual is a quotient of $\mathcal{R}^1_{\mathcal{E}}$ can be regarded as a family of subspaces of twisted endomorphisms of the fibers of $\mathcal{R}^1_{\mathcal{E}}$

Lemma 2.7. *Let $\mathcal{V}$ be a locally free sheaf of rank k such that $\mathcal{V}^\vee$ is a quotient of $\mathcal{R}^1_{\mathcal{E}}$. For any s in S, it follows that $\mathcal{V}_s$ is a k-dimensional subspace of $H^0(X,\mathrm{End}(\mathcal{E}_s)\otimes\omega_X)$.*

Proof. Let $q\colon \mathcal{R}^1_{\mathcal{E}}\to\mathcal{V}^\vee$ be the quotient over S. For any point s in S, we have a surjective morphism

$$q_s\colon (\mathcal{R}^1_{\mathcal{E}})_s\to(\mathcal{V}^\vee)_s.$$

It follows that $R^2\pi_{2^*}(\mathrm{End}(\mathcal{E}))=0$ because X is a curve. Since the projection to the second factor π_2 is proper, by [10, Theorem 12.11] or [18, Corollary 3], we have that

$$(R^1\pi_{2^*}(\mathrm{End}(\mathcal{E})))_s = H^1(X,\mathrm{End}(\mathcal{E}_s)).$$

On the other hand, since $\mathcal{V}$ is locally free, we have that $(\mathcal{V}^\vee)_s=\mathcal{V}_s^\vee$. Therefore, taking the dual of q_s and using Serre's duality, $\mathcal{V}_s$ is a k-dimensional subspace of $H^0(X,\mathrm{End}(\mathcal{E}_s)\otimes\omega_X)$, as claimed. □

Let $\mathcal{E}$ and $\mathcal{V}$ be as in the above result. For all s in S, the pair $(\mathcal{E}_s,\mathcal{V}_s)$ is a Higgs coherent system of type (n,d,k). This motivates the following definition.

Definition 2.8. *A* **family of Higgs coherent systems** *of type (n,d,k) parameterized by a scheme S is a pair $(\mathcal{E},\mathcal{V})$, where $\mathcal{E}$ is a family of vector bundles of type (n,d) parameterized by S and $\mathcal{V}$ is a locally free sheaf of rank k over S such that $\mathcal{V}^\vee$ is a quotient of $\mathcal{R}^1_{\mathcal{E}}$. Two families parameterized by S, $(\mathcal{E},\mathcal{V})$, and $(\mathcal{E}',\mathcal{V}')$ are said to be* **equivalent** *if, for all points s in S, the Higgs coherent systems $(\mathcal{E}_s,\mathcal{V}_s)$ and $(\mathcal{E}'_s,\mathcal{V}'_s)$ are isomorphic. In this case, we write $(\mathcal{E},\mathcal{V})\approx(\mathcal{E}',\mathcal{V}')$.*

To define the notion of an induced family by a morphism of schemes, we will use the following result.

Lemma 2.9. *Let $(\mathcal{E}, \mathcal{V})$ be a family of Higgs coherent systems of type (n, d, k) parameterized by S. Let $\phi\colon S' \to S$ be a morphism. The pair $(\phi^*(\mathcal{E}), \phi^*(\mathcal{V}))$ is a family of Higgs coherent systems of type (n, d, k) parameterized by S'.*

Proof. Let $q\colon \mathcal{R}^1_{\mathcal{E}} \to \mathcal{V}^\vee$ be the quotient associated with the family $(\mathcal{E}, \mathcal{V})$. By definition, $\phi^*(\mathcal{E}) = (id_X \times \phi)^*(\mathcal{E})$ is a family of vector bundles of type (n, d) parameterized by S'. It is enough to prove that $\phi^*(\mathcal{V})^\vee$ is a quotient of $\mathcal{R}^1_{\phi^*(\mathcal{E})}$. Since the pull-back commutes with the tensor product, the projection formula ([17, Proposition 2.32]) yields

$$\begin{aligned}\mathcal{R}^1_{\phi^*(\mathcal{E})} &= R^1\pi_{2^*}(\mathrm{End}(\phi^*(\mathcal{E}))\\ &= R^1\pi_{2^*}(\phi^*(\mathrm{End}(\mathcal{E})))\\ &= \phi^*(R^1\pi_{2^*}(\mathrm{End}(\mathcal{E})))\\ &= \phi^*(\mathcal{R}^1_{\mathcal{E}}).\end{aligned}$$

Moreover, $\mathcal{V}$ is locally free, hence $\phi^*(\mathcal{V})^\vee = \phi^*(\mathcal{V}^\vee)$. Finally, since pullback is a right exact functor, the morphism $\phi^*(q)\colon \mathcal{R}^1_{\phi^*(\mathcal{E})} \to \phi^*(\mathcal{V})^\vee$ is surjective. Therefore, $(\phi^*(\mathcal{E}), \phi^*(\mathcal{V}))$ is a family of Higgs coherent systems, which completes the proof. □

Let $\phi\colon S' \to S$ be a morphism, and let $(\mathcal{E}, \mathcal{V})$ be a family of Higgs coherent systems of type (n, d, k) parameterized by S. We define **the induced family by** ϕ, denoted as $\phi^*(\mathcal{E}, \mathcal{V})$, to be $((\phi)^*(\mathcal{E}), \phi^*(\mathcal{V}))$.

The notion of the induced family satisfies the following functorial properties (cf. [19, Conditions 1.4 (c)]), which allows us to define the moduli problem for Higgs coherent systems.

Remark 2.10. *Let $\phi'\colon S'' \to S'$ and $\phi\colon S' \to S$ be two morphisms of schemes and $(\mathcal{E}, \mathcal{V})$ and $(\mathcal{E}', \mathcal{V}')$ be two families parameterized by S. Then*

i. $(\phi \circ \phi')^*((\mathcal{E}, \mathcal{V})) = \phi'^* \circ \phi^*((\mathcal{E}, \mathcal{V}))$

ii. $id_S^*((\mathcal{E}, \mathcal{V})) = (\mathcal{E}, \mathcal{V})$

iii. $(\mathcal{E}, \mathcal{V}) \approx (\mathcal{E}', \mathcal{V}') \Rightarrow \phi^*(\mathcal{E}, \mathcal{V}) \approx \phi^*(\mathcal{E}', \mathcal{V}')$

We define the moduli problem of Higgs coherent systems of type (n, d, k) over X as follows:

$$\boxed{(H(n, d, k), \sim, (\mathcal{E}, \mathcal{V}), \approx).}$$

That is, this data defines the following functor: Denote by $\mathcal{F}(S)$ the set of equivalence classes of families of Higgs coherent systems of type (n, d, k) parameterized by S. We define the **moduli functor for Higgs coherent systems of type** (n, d, k)

as follows:

$$\begin{aligned} Fam_{(n,d,k)} \colon Sch &\longrightarrow Sets \\ S &\mapsto \mathcal{F}(S) \\ (\phi \colon S' \to S) &\mapsto \phi^* \colon \mathcal{F}(S) \to \mathcal{F}(S') \\ &\qquad\qquad (\mathcal{E}, \mathcal{V}) \mapsto \phi^*(\mathcal{E}, \mathcal{V}). \end{aligned} \tag{2.6}$$

To construct the moduli space for the above problem, it is necessary to endow $H(n,d,k)/\sim$ with a geometric-algebraic structure that reflects how Higgs coherent systems deform in families. Equivalently, we are interested in determining if the moduli functor $Fam_{(n,d,k)}$ is representable. In Section 3, we construct a projective scheme which is a fine moduli space for the moduli problem of Higgs coherent systems whose underlying vector bundles are stable.

Remark 2.11. In [24], the author introduces coherent ρ-systems, where ρ is a representation of $\mathrm{GL}_n(\mathbb{C})$ on a finite dimensional vector space (see [24, Section 1.1]). The existence of a moduli space is established there in the case where ρ is a tensor power of the standard representation, yielding a projective moduli space constructed via GIT ([24, Main Theorem]). When ρ is the representation of $\mathrm{GL}_n(\mathbb{C})$ on the space of $(n \times n)$-matrices by conjugation, a coherent ρ-system corresponds to a Higgs coherent system. This representation does not arise from a tensor power of the standard representation, and therefore the existence of a moduli space for Higgs coherent systems does not follow from the main result of [24]. In Section 3, we construct the moduli space for this case by a different approach, without relying a priori on GIT or on representation-theoretic methods, and assuming stability of the underlying vector bundles.

3. Moduli of Higgs coherent systems with stable underlying vector bundles

In this section, we study Higgs coherent systems with stable underlying vector bundles and construct the fine moduli space for the corresponding moduli problem.

From now on, we fix a possible type (n,d,k) with $(n,d) = 1$ and $k \leq n^2(g-1)+1$. Let (E,V) be a Higgs coherent system of type (n,d,k). We define

$$H_E(V) = \{V' \in \mathrm{Grass}(k, H^0(X, \mathrm{End}(E) \otimes \omega_X) \mid (E,V) \sim (E,V')\}.$$

The next result is related to the isomorphism classes of Higgs coherent systems (E,V), where E is a simple vector bundle.

Proposition 3.1. *Let (E,V) be a Higgs coherent system of type (n,d,k) such that E is a simple vector bundle. Then, $H_E(V) = \{V\}$.*

Proof. Let V' be a subspace of $H^0(X, \mathrm{End}(E) \otimes \omega_X)$, and suppose that $(E,V) \sim (E,V')$. There exists an isomorphism $\alpha \colon E \to E$ such that $\hat{\alpha}(V) = V'$. Since E is simple, there exists $c \in \mathbb{C}^*$ such that $\alpha = c \cdot id_E$. It follows that $\hat{\alpha}(V) = (\alpha \otimes \alpha^{-\vee} \otimes id_{\omega_X})(V) = (c \cdot id_E \otimes c^{-1} \cdot id_{E^\vee} \otimes id_{\omega_X})(V) = V$. Therefore, $V = V'$, which completes the proof. □

Remark 3.2. Let (E, V) and (E', V') be two Higgs coherent systems of type (n, d, k) such that E and E' are simple vector bundles. From Proposition 3.1 and the fact that if $E \cong E'$, then $H^0(X, \mathrm{Hom}(E, E')) = \mathbb{C}$, it follows that $(E, V) \sim (E, V')$ if and only if $V = V'$. The equality $V = V'$ means that V and V' are equal as subspaces of $H^0(X, E^\vee \otimes E' \otimes \omega_X)$.

Definition 3.3. A Higgs coherent system (E, V) of type (n, d, k) is **strongly stable** if E is stable. We will denote by $SH(n, d, k)$ the subset of $H(n, d, k)$ consisting of strongly stable Higgs coherent systems of type (n, d, k).

Let $\mathcal{E}$ be a family of stable vector bundles parameterized by S; that is, for all s in S, the vector bundle $\mathcal{E}_s$ is stable. It follows that $\mathcal{R}^1_{\mathcal{E}}$ is a locally free sheaf of rank $n^2(g-1)+1$, and thus there is a bijection between locally free sheaves that are quotients of $\mathcal{R}^1_{\mathcal{E}}$ and locally free sub-sheaves of $\mathcal{R}^{1^\vee}_{\mathcal{E}}$. Under this bijection, by a family of strongly stable Higgs coherent systems of type (n, d, k) parameterized by S, we mean a pair $(\mathcal{E}, \mathcal{V})$ where $\mathcal{E}$ is a family of stable vector bundles parameterized by S and $\mathcal{V}$ is a locally free subsheaf of rank k of $\mathcal{R}^{1^\vee}_{\mathcal{E}}$. Denote by $\mathcal{F}^s(S)$ the set of equivalence classes of families of strongly stable Higgs coherent systems of type (n, d, k) parameterized by S. We define the **moduli functor for strongly stable Higgs coherent systems of type** (n, d, k) as in (2.6), replacing $\mathcal{F}(S)$ with $\mathcal{F}^s(S)$.

Now, we recall some properties of the moduli space of stable vector bundles. Let $M^s(n, d)$ denote the moduli space of stable vector bundles of type (n, d) over X. It is a smooth projective variety of dimension $n^2(g-1)+1$ ([19, Remark 5.9]), and for any vector bundle E in $M^s(n, d)$, there is a canonical isomorphism $T_E M^s(n, d) = H^1(X, \mathrm{End}(E))$ ([11, Theorem 2.7 and Corollary 2.8]). We have the following result.

Theorem 3.4. *The k-th Grassmannian bundle* $\mathrm{Grass}(k, \Omega^1 M^s(n, d))$ *parametrizes the isomorphism classes of strongly stable Higgs coherent systems. More precisely, there is a set-theoretic bijection between* $\mathrm{Grass}(k, \Omega^1 M^s(n, d))$ *and the space* $SH(n, d, k)/\sim$. *In particular,* $SH(n, d, k)/\sim$ *admits the structure of a smooth irreducible projective scheme of dimension* $(k+1)(n^2(g-1)+1) - k^2$.

Proof. The bijection is a consequence of the above properties of $M^s(n, d)$ and Proposition 3.1. We endow $SH(n, d, k)/\sim$ with the scheme structure induced by this bijection. Since $M^s(n, d)$ is smooth, irreducible, and projective, it follows that $\mathrm{Grass}(k, \Omega^1 M^s(n, d))$ is a smooth, irreducible projective scheme. Moreover, its dimension is equal to $\dim(M^s(n, d)) + \dim(\mathrm{Grass}(k, \Omega^1_E M^s(n, d)))$, where E is a stable vector bundle. Therefore,

$$\dim(\mathrm{Grass}(k, \Omega^1 M^s(n, d))) = (k+1)(n^2(g-1)+1) - k^2,$$

which completes the proof. □

Before proving that $\mathrm{Grass}(k, \Omega^1 M^s(n,d))$ is the fine moduli space, we first construct a family of strongly stable Higgs coherent systems parameterized by $\mathrm{Grass}(k, \Omega^1 M^s(n,d))$. To do this, since $(n,d) = 1$, there is a universal vector bundle $\mathcal{E}$ over $X \times M^s(n,d)$ ([19, Theorem 5.12]). We consider the pair $(\hat{\mathcal{E}}, \mathcal{V})$, defined as follows:

- $\hat{\mathcal{E}}$ denotes the pull-back $(id_X \times \pi)^*(\mathcal{E})$:

$$\begin{tikzcd}
\hat{\mathcal{E}} \arrow[d] & \mathcal{E} \arrow[d] \\
X \times \mathrm{Grass}(k, \Omega^1 M^s(n,d)) \arrow[r, "id_X \times \pi"] & X \times M^s(n,d)
\end{tikzcd} .$$

- $\mathcal{V}$ denotes the tautological subbundle of rank k of $\pi^*(\Omega^1 M^s(n,d))$. The tautological subbundle fits into the universal exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{V} \arrow[r] & \pi^*(\Omega^1 M^s(n,d)) \arrow[r] \arrow[d] & Q \arrow[r] & 0 \\
& & \mathrm{Grass}(k, \Omega^1 M^s(n,d)) & &
\end{tikzcd} , \tag{3.1}$$

where Q denotes the tautological quotient.

Lemma 3.5. *The pair $(\hat{\mathcal{E}}, \mathcal{V})$ is a family of strongly stable Higgs coherent systems parameterized by* $\mathrm{Grass}(k, \Omega^1 M^s(n,d))$.

Proof. Since $\mathcal{E}$ is a family of stable vector bundles of type (n,d), we have that $\hat{\mathcal{E}}$ is also a family of stable vector bundles of type (n,d). From the commutative diagram

$$\begin{tikzcd}
X \times \mathrm{Grass}(k, \Omega^1 M^s(n,d)) \arrow[r, "id_X \times \pi"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\circlearrowleft"] & X \times M^s(n,d) \arrow[d, "\pi_2"] \\
\mathrm{Grass}(k, \Omega^1 M^s(n,d)) \arrow[r, "\pi"'] & M^s(n,d)
\end{tikzcd} ,$$

and the projection formula, we have that

$$\begin{aligned}
R^1 \pi_{2^*}(\mathrm{End}(\hat{\mathcal{E}})) &= R^1 \pi_{2^*}((id_X \times \pi)^*(\mathrm{End}(\mathcal{E}))) \\
&= \pi^*(R^1 \pi_{2^*}(\mathrm{End}(\mathcal{E}))).
\end{aligned}$$

Then,

$$\begin{aligned}\mathcal{R}_{\hat{\mathcal{E}}}^{1^\vee} = R^1\pi_{2^*}(\mathrm{End}(\hat{\mathcal{E}}))^\vee &= \pi^*(R^1\pi_{2^*}(\mathrm{End}(\mathcal{E})))^\vee \\ &= \pi^*(R^1\pi_{2^*}(\mathrm{End}(\mathcal{E}))^\vee) \\ &= \pi^*(\mathcal{R}_{\mathcal{E}}^{1^\vee}).\end{aligned}$$

From the universal property of $\mathcal{E}$, it follows that $\mathcal{R}^1_{\mathcal{E}} = TM^s(n,d)$ ([14, Appendix 2.C.II and Theorem 10.2.1]). Therefore, $\mathcal{V} \subset \pi^*(\Omega^1 M^s(n,d)) = \mathcal{R}_{\hat{\mathcal{E}}}^{1^\vee}$, and thus $(\hat{\mathcal{E}}, \mathcal{V})$ is a family of strongly stable Higgs coherent systems, as claimed. □

Remark 3.6. Note that the dual sequence of (3.1) yields a quotient $\mathcal{V}^\vee$ of $\mathcal{R}^1_{\hat{\mathcal{E}}}$. Hence, by the above result, the pair $(\hat{\mathcal{E}}, \mathcal{V})$ is also a family of Higgs coherent systems parameterized by $\mathrm{Grass}(k, \Omega^1 M^s(n,d))$ in the sense of Definition 2.8.

In the next result, we will prove that the above family satisfies the universal property, and thus there exists a fine moduli space for the moduli problem of strongly stable Higgs coherent systems.

Theorem 3.7. *The k-th Grassmannian bundle,* $\mathrm{Grass}(k, \Omega^1 M^s(n,d))$*, together with the family* $(\hat{\mathcal{E}}, \mathcal{V})$ *is a fine moduli space for the moduli problem of strongly stable Higgs coherent systems of type* (n,d,k)*. Moreover, it is a smooth irreducible projective variety of dimension* $(k+1)(n^2(g-1)+1) - k^2$*.*

Proof. The Grassmannian bundle $\mathrm{Grass}(k, \Omega^1 M^s(n,d))$ parametrizes the isomorphism classes of strongly stable Higgs coherent systems (Theorem 3.4), and the pair $(\hat{\mathcal{E}}, \mathcal{V})$ is a family parameterized by $\mathrm{Grass}(k, \Omega^1 M^s(n,d))$ (Lemma 3.5). It is enough to prove that $(\hat{\mathcal{E}}, \mathcal{V})$ satisfies the universal property. To do this, note that from the universal property of $\mathcal{E}$ and the definition of $\mathcal{V}$, the restriction of $(\hat{\mathcal{E}}, \mathcal{V})$ to $X \times \{(E,V)\}$ is isomorphic to (E,V). We claim that the universal properties of $\mathcal{E}$ and $\mathcal{V}$ imply that $(\hat{\mathcal{E}}, \mathcal{V})$ satisfies the universal property. Indeed, let $(\mathcal{E}', \mathcal{V}')$ be a family of strongly stable Higgs coherent systems parameterized by T. By the universal property of $\mathcal{E}$, there exists a unique morphism $\phi\colon T \to M^s(n,d)$ such that $\mathcal{E}'$ is equivalent to $\phi^*(\mathcal{E})$. Moreover, by the universal property of $\mathcal{V}$, there exists a unique morphism $\hat{\phi}\colon T \to \mathrm{Grass}(k, \Omega^1 M^s(n,d))$ such that $\hat{\phi}^*(\mathcal{V}) = \mathcal{V}'$ ([9, Proposition 8.17(1)]). We have that

$$\hat{\phi}^*(\hat{\mathcal{E}}) = (id_X \times \hat{\phi})^*((id_X \times \pi)^*(\mathcal{E})) = (id_X \times (\pi \circ \hat{\phi}))^*(\mathcal{E}) = (\pi \circ \hat{\phi})^*(\mathcal{E}).$$

Note that, for all $t \in T$, $\mathcal{V}'_t = \hat{\phi}^*(\mathcal{V})_t = \mathcal{V}_{\hat{\phi}(t)} = \hat{\phi}(t)$, and $\hat{\phi}^*(\hat{\mathcal{E}})_t = (\pi \circ \hat{\phi})^*(\mathcal{E})_t = \mathcal{E}_{\pi(\hat{\phi}(t))} = \mathcal{E}_{\pi(\mathcal{V}'_t)} = \mathcal{E}'_t$. Therefore, $\hat{\phi}^*(\hat{\mathcal{E}}, \mathcal{V})$ is equivalent to $(\mathcal{E}', \mathcal{V}')$, as claimed. □

Remark 3.8. It is known that if $(n,d) \neq 1$, the universal family of stable vector bundles $\mathcal{E}$ does not exist, that is a fine moduli space for stable vector bundles does not exist ([22, Theorem 2]). For this case, such a universal family exists locally in the étale topology ([25, Theorem 1.21. (4)]). Since sheaves in the Zariski topology can be consider as sheaves in the étale topology [15, Proposition 4.15], for any type (n, d, k) with $k < n^2(g-1)+1$, we can proceed locally with the above arguments to obtain a local version of Theorem 3.7.

In [8], the authors study Higgs coherent systems of type $(1, d, k)$. They consider notions of families and the equivalence between them that differ from the corresponding notions addressed in this work. In particular, for the concept of a family, they consider a zero direct image instead of the first direct image, as in our case (see [8, Definición 3.3.]). Nevertheless, their definition has a solvable issue: in general, for a family $(\mathcal{E}, \mathcal{V})$ parameterized by T and a point t in T, it is not necessarily true that $(\mathcal{E}, \mathcal{V})_t$ is a Higgs coherent system (cf. [16, Corollary 4.2.9]), as claimed in [8, Proposición 3.1.1 (a)]. This issue can be solved, for example, by requiring the extra condition of injectivity, as in coherent systems (cf. [21, Definition 1.11]), on the natural morphism between the fiber of the zero direct image and the space of twisted endomorphisms.

We now apply our approach to construct the moduli space of Higgs coherent systems of type $(1, d, k)$. Let $\mathrm{Pic}^d(X)$ be the Picard variety of X and $\mathcal{L}^d$ its Poincaré bundle. It is known that the tangent space at a line bundle L in $\mathrm{Pic}^d(X)$ is $H^1(X, O_X)$. Since $\mathrm{Pic}^d(X)$ is a smooth variety, its tangent sheaf is locally free and has rank g. Moreover, since $\mathrm{Pic}^d(X)$ is isomorphic to $\mathrm{Pic}^0(X)$, which is an abelian variety, its tangent bundle is trivial ([18, Page 42 (iii)]). By the universal property of $\mathcal{L}^d$, it follows that $\mathcal{R}^{1^\vee}_{\mathcal{L}^d} = \Omega^1 \mathrm{Pic}^d(X) = H^0(X, \omega_X) \otimes O_{\mathrm{Pic}^d(X)}$ ([14, Theorem 10.2.1]), and hence the k-th Grassmannian bundle $\mathrm{Grass}(k, \Omega^1 \mathrm{Pic}^d(X))$ is the product $\mathrm{Pic}^d(X) \times \mathrm{Grass}(k, H^0(X, \omega_X))$. In particular, from Theorem 3.7, for Higgs coherent systems of type $(1, d, k)$, with $k \leq g$, we obtain the following result (cf. [8, Teorema 3.2]).

Corollary 3.9. *The moduli fine space of Higgs coherent systems of type* $(1, d, k)$ *is* $\mathrm{Pic}^d(X) \times \mathrm{Grass}(k, H^0(X, \omega_X))$ *together with the family* $(\hat{\mathcal{L}}^d, \mathcal{V})$*. Moreover, it is a smooth, irreducible projective variety of dimension* $g(k+1) - k^2$*.*

The study of the moduli problem of Higgs coherent systems, without additional conditions on the underlying vector bundles, can be treated using geometric invariant theory (GIT). More specifically, the moduli problem (2.6) admits a GIT-problem; that is, there exists a parameter scheme $\mathcal{S}(n, d, k)$ for Higgs coherent systems of type (n, d, k), together with a linearizable action of a reductive group G, such that two Higgs coherent systems are isomorphic if and only if their corresponding points in the parameter scheme lie in the same orbit. Therefore, endowing $H(n, d, k)/\sim$ with an algebraic structure is equivalent to giving an algebraic structure to $\mathcal{S}(n, d, k)/G$. For a treatment of this equivalence, we refer the reader to [7] and forthcoming articles by the author.

In general, GIT constructions ensure the existence of coarse moduli spaces. In our setting, the construction of the fine moduli space in Theorem 3.7 is carried out without initially relying on GIT and fundamentally depends on the geometric properties of the moduli space of stable bundles. This emphasizes the importance of the stable setting within the general theory and clarifies the role it plays in the study of Higgs coherent systems. We hope that the development of the theory of Higgs coherent systems will contribute to a broader understanding of Higgs bundles and coherent systems, as well as their moduli spaces.

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