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AN OVERVIEW ON CURVE SEMISTABLE AND NUMERICALLY FLAT HIGGS BUNDLES

UNA VISIÓN GENERAL DE FIBRADOS DE HIGGS SEMIESTABLES DE CURVA Y NUMÉRICAMENTE PLANOS

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Abstract

After recalling the basic notions concerning Higgs-Grassmann schemes, we review how these can be used to define generalisations of the notion of positivity conditions, such as numerical flatness, which "feel" the Higgs field. Then we prove several properties of Higgs bundles over smooth projective varieties defined over an algebraically closed field of characteristic 0, satisfying these conditions. Finally, we discuss how one can relate them to semistability of the so-called "curve semistable" Higgs bundles.

Keywords: Higgs bundles; curve semistability; numerical flatness; positivity conditions.

Resumen

Después de recordar las nociones básicas sobre los esquemas de Higgs-Grassmann, examinamos cómo se puede utilizar estas para definir generalizaciones de nociones de positividad, como la planitud numérica, que "detectan" el campo de Higgs. Luego demostramos varias propiedades de fibrados de Higgs sobre variedad proyectivas suaves definidas sobre campos algebraicamentes cerrados de característica 0, que satisfacen estas condiciones. Por último, analizamos cómo se pueden conectar estos con los llamados fibrados de Higgs semiestables en las curvas.

Palabras clave: fibrados de Higgs; semiestabilidad de curvas; planitud numérica; condiciones de positividad.

Mathematics Subject Classification: Primary 14A15, 14F06, 14H60, 14J60.

Introduction

In order to construct the *moduli space of vector bundles over an irreducible smooth projective curve*, Mumford has introduced the so-called (*slope semi*)*stability condition*. Later, Takemoto has extended this notion to any irreducible smooth projective variety (see Definition 0.2). In [21], Miyaoka has related this condition to the *positivity* of a numerical class.

More precisely, let X be an irreducible projective variety over an algebraically closed field. A line bundle L over X is *numerically eventually free* (*nef*, for short, see also [20, Remark 1.4.2]) if for any irreducible curve C on X the following inequality $\int_C c_1(L) \geq 0$ holds. A vector bundle E over X is *nef* if the line bundle $\mathcal{O}_{\mathrm{Gr}_1(E)}(1)$ over $\mathrm{Gr}_1(E)$ (the projective bundle of rank 1 locally free quotients of E, see [20]) is nef; and E is *numerically flat* (*nflat*, for short) if E and $E^\vee$ are both nef. Thus E is semistable if and only if the *normalized hyperplane class* $\lambda_1(E) \in \mathrm{N}^1(\mathrm{Gr}_1(E))$ is nef (see Equation (3.1) and [21, Theorem 3.1]); here $\mathrm{N}^1(\mathrm{Gr}_1(E))$ is the *real vector space of real* 1*-cocycles on* $\mathrm{Gr}_1(E)$ *modulo numerical equivalence.*

Bruzzo and Hernández Ruipérez have generalised in [8] *Miyaoka's criterion* introducing other numerical classes $\lambda_s(E) \in \mathrm{N}^1(\mathrm{Gr}_1(Q_{s,E}))$ and $\theta_s(E) \in \mathrm{N}^1(\mathrm{Gr}_1(E))$

where r is the rank of E, $s \in \{1, \dots, r-1\}$ and $Q_{s,E}$ is the *universal rank s quotient bundle over* $\mathrm{Gr}_s(E)$ (see Equations (3.1) and (3.2), respectively). They have proved that the semistability of E implies the nefness of all these numerical classes; and, conversely, if at least one of these numerical classes is nef then E is semistable ([8, Theorem 1.1]). Moreover, they have given a generalization of this criterion to semistable vector bundles over smooth complex projective varieties.

Here we extend this criterion to irreducible smooth projective varieties defined over an algebraically closed field of characteristic 0 (Theorem 4.1).We emphasize that the proofs given here are purely algebraic, *i.e.*, no analytic method has been used. As a consequence, a vector bundle E over X whose classes $\lambda_s(E)$'s or $\theta_s(E)$'s are nef, respectively, satisfies the following property: the pull-backs of E to any irreducible smooth projective curve are semistable (*curve semistable*, for simplicity; see also Definition 3.1). Moreover, such a vector bundle is semistable with respect to all polarizations of X and $c_2(\mathrm{End}(E)) = 0 \in \mathrm{A}^2(X)$ (the *Abelian group of* 2-*cocycles on X modulo rational equivalence*). Furthermore, this suggests a generalisation to the *Higgs bundles* setting.

Loosely speaking, positivity conditions and semistability for vector bundles over irreducible smooth projective varieties are not "disjoint" properties. More generally, an analogous phenomenon happens in the Higgs bundles framework. In order to generalize all these results, we need to recall main definitions about *Higgs sheaves on projective schemes.*

Let X be a projective scheme over an algebraically closed field of characteristic 0, let Ω^1_X be the cotangent sheaf of X and let $\Omega^p_X = \bigwedge^p \Omega^1_X$ for $p \in \mathbb{N}_{\geq 1}$.

Definition 0.1. A *Higgs sheaf* $\mathfrak{E}$ on X is a pair $(\mathcal{E}, \varphi)$ where $\mathcal{E}$ is an $\mathcal{O}_X$-coherent sheaf equipped with a morphism $\varphi\colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ called *Higgs field* such that the composition

$$\varphi \wedge \varphi\colon \mathcal{E} \xrightarrow{\varphi} \mathcal{E} \otimes \Omega^1_X \xrightarrow{\varphi \otimes \mathrm{Id}} \mathcal{E} \otimes \Omega^1_X \otimes \Omega^1_X \to \mathcal{E} \otimes \Omega^2_X$$

vanishes. A Higgs subsheaf of $\mathfrak{E}$ is a *φ-invariant subsheaf* $\mathcal{F}$ of $\mathcal{E}$, *i.e.*, $\varphi(\mathcal{F}) \subseteq \mathcal{F} \otimes \Omega^1_X$. A *quotient Higgs* of $\mathfrak{E}$ is a quotient sheaf of $\mathcal{E}$ such that the corresponding kernel is φ-invariant. A *Higgs bundle* is a Higgs sheaf on X whose underlying coherent sheaf is locally free.

Assuming X to be also an irreducible smooth variety, given a Higgs bundle $\mathfrak{E} = (E, \varphi)$ over X, Bruzzo and Hernández Ruipérez in [8] have introduced closed subschemes $\mathfrak{Gr}_t(\mathfrak{E})$ of $\mathrm{Gr}_t(E)$, called the *t-th Higgs-Grassmann schemes of* $\mathfrak{E}$, which parametrizes the rank t quotient Higgs bundles of $\mathfrak{E}$. These schemes enjoy a universal property similar to that of the Grassmann bundles. Let $\mathfrak{Q}_{t,\mathfrak{E}}$ be the restriction of the universal rank t quotient bundle of E to $\mathfrak{Gr}_t(\mathfrak{E})$; this is a rank t quotient Higgs bundles of the pullback of $\mathfrak{E}$ over $\mathfrak{Gr}_t(\mathfrak{E})$. They have defined the numerical classes $\lambda_t(\mathfrak{E}) \in \mathrm{N}^1\left(\mathrm{Gr}_1\left(\mathfrak{Q}_{t,\mathfrak{E}}\right)\right)$ and $\theta_t(\mathfrak{E}) \in \mathrm{N}^1\left(\mathfrak{Gr}_1(\mathfrak{E})\right)$ where $t \in \{1, \dots, r-1\}$ (see

Equations (3.1) and (3.2), respectively). So one has an equivalence between the nefness of these numerical classes and the curve semistability of $\mathfrak{E}$ (see Theorems 3.2 and 3.3); even in this setting, $\mathfrak{E}$ is semistable with respect to all polarizations of X.

To be precise, let H be a polarization of X and let $\mathfrak{E} = (\mathcal{E}, \varphi)$ be a torsion-free Higgs sheaf on X, if not otherwise indicated. One defines the *slope* of $\mathfrak{E}$ as $\mu(\mathfrak{E}) = \dfrac{\deg(\mathfrak{E})}{\operatorname{rank}(\mathfrak{E})} \in \mathbb{Q}$, where $\deg(\mathfrak{E}) = \displaystyle\int_X c_1(\mathcal{E}) \cdot H^{n-1} \in \mathbb{Z}$. In a similar way as for vector bundles there is a notion of stability for torsion-free Higgs sheaves.

Definition 0.2. $\mathfrak{E}$ is *semistable* (respectively, *stable*) if $\mu(\mathfrak{E}) \underset{(<)}{\leq} \mu(\mathfrak{Q})$ for every torsion-free quotient Higgs sheaf $\mathfrak{Q}$ with $0 < \operatorname{rank}(\mathfrak{Q}) < \operatorname{rank}(\mathfrak{E})$. In the other eventuality, $\mathfrak{E}$ is *unstable*.

On the other hand, these Higgs-Grassmann schemes of a Higgs bundle $\mathfrak{E} = (E, \varphi)$ are an ingredient of the positivity conditions for Higgs bundles. The idea is the following: $\det(E)$ has to be nef. Of course this is not enough if $r \geq 2$, so one requires also the "*Higgs nefness*" of all *universal quotient Higgs bundles* $\mathfrak{Q}_{t,\mathfrak{E}}$ recursively (see Definition 1.1). Such a Higgs bundle $\mathfrak{E}$ is called *Higgs nef* (*H-nef*, for short); and it follows to define *Higgs numerically flat* (*H-nflat*, for short) a H-nef Higgs bundle $\mathfrak{E}$ whose dual Higgs bundle $\mathfrak{E}^\vee$ is H-nef as well. This idea has been developed by Bruzzo, Hernández Ruipérez and Graña Otero in [5, 8].

The interplay between semistability and positivity conditions for Higgs bundles has been studied by the author as well as Biswas, Bruzzo, Graña Otero, Gurjar, Hernández Ruipérez, Lanza, Lo Giudice and Peragine in [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 19]. In particular, whether a curve semistable Higgs bundle $\mathfrak{E} = (E, \varphi)$ has $c_2(\operatorname{End}(E)) = 0 \in \mathrm{A}^2(X)$ remains open (Theorem 4.4). Equivalently, the vanishing of Chern classes of H-nflat Higgs bundles is not yet known ([1, Corollary 3.2]). For more on these, please see Remark 4.6.

An equivalent way to define H-nflat Higgs bundles is the following one: a Higgs bundle $\mathfrak{E} = (E, \varphi)$ over an irreducible smooth projective variety is H-nflat if it is curve semistable and $c_1(E) \equiv_{num} 0$ (see Lemma 2.3.a)). So this combined with [18, Corollary] simplifies the problem to prove the vanishing of $c_2(E) \in \mathrm{A}^2(X)$. Other simplifications of this problem are recalled and proved at the end of this paper.

Notations and conventions. $\mathbb{K}$ is an algebraically closed field of characteristic 0, unless otherwise indicated. By a projective variety X we mean a projective integral scheme over $\mathbb{K}$ of dimension $n \geq 1$ and of finite type. If $n \in \{1, 2\}$ we say projective curve or projective surface, respectively. Whenever we consider a morphism $f : C \to X$, we understand that C is an irreducible smooth projective curve. We denote by $\mathrm{N}^1(X)$ the real vector space of real 1-cocycles on X modulo numerical equivalence, and by $\mathrm{A}^2(X)$ the Abelian group of 2-cocycles on X modulo rational equivalence.

1. Higgs-Grassmann schemes and H-ample, H-nef and H-nflat Higgs bundles

Let $\mathfrak{E} = (E, \varphi)$ be a rank $r \geq 2$ Higgs bundle over a smooth projective variety X, and let $s \in \{1, \ldots, r-1\}$ be an integer number. Let $p_s \colon \mathrm{Gr}_s(E) \to X$ be the *Grassmann bundle* parametrizing rank s locally free quotients of E (see [20]). Consider the short exact sequence of vector bundles over $\mathrm{Gr}_s(E)$

$$0 \longrightarrow S_{r-s,E} \xrightarrow{\eta} p_s^* E \xrightarrow{\epsilon} Q_{s,E} \longrightarrow 0\,,$$

where $S_{r-s,E}$ is the *universal rank $r-s$ subbundle* and $Q_{s,E}$ is the *universal rank s quotient bundle of p_s^*E*, respectively. One defines the closed subschemes $\mathfrak{Gr}_s(\mathfrak{E}) \subseteq \mathrm{Gr}_s(E)$ (the *s-th Higgs-Grassmann schemes of* $\mathfrak{E}$) as the zero loci of the composite morphisms

$$(\epsilon \otimes \mathrm{Id}) \circ p_s^*\varphi \circ \eta \colon S_{r-s,E} \to Q_{s,E} \otimes p_s^*\Omega^1_X.$$

The restriction of the previous sequence to $\mathfrak{Gr}_s(\mathfrak{E})$ yields a universal short exact sequence

$$0 \longrightarrow \mathfrak{S}_{r-s,\mathfrak{E}} \xrightarrow{\psi} \rho_s^*\mathfrak{E} \xrightarrow{\eta} \mathfrak{Q}_{s,\mathfrak{E}} \longrightarrow 0,\,,$$

where $\mathfrak{Q}_{s,\mathfrak{E}} = Q_{s,E|\mathfrak{Gr}_s(\mathfrak{E})}$ is equipped with the quotient Higgs field induced by $\rho_s^*\varphi$, and $\rho_s = p_{s|\mathfrak{Gr}_s(\mathfrak{E})}$. The scheme $\mathfrak{Gr}_s(\mathfrak{E})$ enjoys the usual universal property: for a morphism of varieties $f \colon Y \to X$, the morphism $g \colon Y \to \mathrm{Gr}_s(E)$ given by a rank s quotient Q of f^*E factors through $\mathfrak{Gr}_s(\mathfrak{E})$ if and only if φ induces a Higgs field on Q.

Definition 1.1 (see [6, Definition A.2])**.** A Higgs bundle $\mathfrak{E} = (E, \varphi)$ of rank one is said to be *Higgs-ample/Higgs-numerically effective* (*H-ample/H-nef*, for short) if E is *ample/numerically effective* in the usual sense. If $\mathrm{rank}(\mathfrak{E}) \geq 2$, we inductively define H-ampleness/H-nefness by requiring that

a) the determinant line bundle $\det(E)$ is ample/nef, and

b) all Higgs bundles $\mathfrak{Q}_{s,\mathfrak{E}}$ are H-ample/H-nef for all s.

A Higgs bundle $\mathfrak{E}$ is *Higgs-numerically* flat (*H-nflat*, for short) if $\mathfrak{E}$ and $\mathfrak{E}^\vee$ are both H-nef.

Remark 1.2.

a) Note that if E nef/nflat in the usual sense, then $\mathfrak{E}$ is H-nef/H-nflat. If $\varphi = 0$, the Higgs bundle $\mathfrak{E} = (E, 0)$ is H-nef/H-nflat if E is nef/nflat in the usual sense.

b) The recursive condition in this definition can be recasted as follows. Let $1 \leq s_1 < s_2 < \ldots < s_k < r$ and let $\mathfrak{Q}_{(s_1\cdots,s_k),\mathfrak{E}}$ be the rank s_1 universal quotient Higgs bundle over $\mathfrak{Gr}_{s_1}(\mathfrak{Q}_{(s_2,\ldots,s_k),\mathfrak{E}})$, obtained by taking the successive universal quotient Higgs bundles of $\mathfrak{E}$ of rank s_k, then s_{k-1}, all the way to rank s_1. The H-ampleness/H-nefness condition for $\mathfrak{E}$ is equivalent to requiring that all line bundles $\det(\mathfrak{E})$ and $\det(\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}})$ are ample/nef.

c) The first Chern class of an H-nflat Higgs bundle is numerically zero, because the corresponding determinant bundle is nflat. ◇

We refer the reader to [4, Remark 4.2], [3, Example 3.3], examples 2.5 and 3.7 for furthermore considerations on the previous definitions.

2. Properties of H-nef and H-nflat Higgs bundles

Higgs bundles over X which are H-nef satisfy properties analogous to those of nef vector bundles. These properties have been proved in [1, 3, 4, 5]; here we list some of them for completeness.

Lemma 2.1. *Let $\mathfrak{E} = (E, \varphi)$ be an H-nef Higgs bundle over X. The following statements hold.*

a) Let $f\colon Y \to X$ be a morphism of smooth projective varieties. Then $f^*\mathfrak{E}$ is H-nef ([5, Proposition 2.6.(ii)]). If f is also surjective and $f^*\mathfrak{E}$ is H-nef then $\mathfrak{E}$ is H-nef ([1, Lemma 3.4]).

b) Every quotient Higgs bundle of $\mathfrak{E}$ is H-nef ([1, Lemma 3.5]).

c) Tensor products of H-nef Higgs bundles are H-nef ([1, Theorem 3.6]).

d) Exterior and symmetric powers of H-nef Higgs bundles are H-nef (see [4, Propositions 3.5, 4.4 and Lemma 4.5]).

e) $\mathfrak{E}$ is H-nef if and only if the Higgs bundle $\mathfrak{E} \otimes \mathcal{O}_X(D) = (E \otimes \mathcal{O}_X(D), \varphi \otimes Id)$ is H-ample for every ample Cartier $\mathbb{Q}$-divisor D in X ([5, Proposition 2.6.(i)]).

f) $\mathfrak{E}$ is H-nef if and only if $\det(E)$ is nef and for every finite morphism $f\colon C \to X$, any every Higgs quotient bundle $f^*\mathfrak{E} \to \mathfrak{Q} \to 0$, the inequality $\deg \mathfrak{Q} \geq 0$ holds ([3, Corollary 3.7]).

g) The extensions of H-nef Higgs bundles are H-nef (see [3, Theorem 3.9]).

Remark 2.2. In [5], the first part of Lemmata 2.1.a) and 2.1.e) are proved assuming that $\mathbb{K} = \mathbb{C}$, however these proofs work in general by [20, Propositions 6.1.2.(ii) and 6.1.8.(iv)]. ◇

Category of H-nflat Higgs bundles satisfies other properties which have been proved in [1, 2, 6, 19].

Lemma 2.3.

a) *$\mathfrak{E}$ is H-nflat if and only if the pullback of $\mathfrak{E}$ via any $f\colon C \to X$ is semistable and $\int_C f^* c_1(E) = 0$ (see [6, Lemma A.7]).*

b) *Any H-nflat Higgs bundle is semistable ([6, Proposition A.8]).*

c) *Extensions of H-nflat Higgs bundles are H-nflat ([1, Proposition 3.1.(iii)]).*

d) *Tensor products of H-nflat Higgs bundles are H-nflat ([1, Proposition 3.1.(iv)]).*

e) *Kernels and cokernels of morphisms of H-nflat Higgs bundles are H-nflat Higgs bundles ([1, Propositions 3.7 and 3.8]).*

f) *Let $\mathfrak{E}$ be a Higgs bundle over X. $\mathfrak{E}$ is H-nflat if and only if it is* pseudostable, i.e., *it has a filtration*

$$0 = \mathfrak{F}_{m+1} \subsetneq \mathfrak{F}_m \subsetneq \ldots \subsetneq \mathfrak{F}_0 = \mathfrak{E}, \tag{2.1}$$

whose quotients $\mathfrak{F}_i/\mathfrak{F}_{i+1}$ are locally free and stable, and these quotients are also H-nflat ([2, Theorem 3.2]).

Remark 2.4. The "if part" of Lemma 2.3.f) follows by Lemma 2.3.c). Indeed, let $\mathfrak{E}$ be a Higgs bundle and let

$$0 = \mathfrak{F}_0 \subsetneq \mathfrak{F} \subsetneq \mathfrak{F}_1 \subsetneq \ldots \subsetneq \mathfrak{F}_m \subsetneq \mathfrak{F}_{m+1} = \mathfrak{E}$$

be a filtration of $\mathfrak{E}$ in Higgs subbundle whose quotients $\mathfrak{F}, \mathfrak{Q}_1, \ldots, \mathfrak{Q}_{m+1}$ are locally free, stable and H-nflat. As explained in the proof of [2, Theorem 3.2], the subbundle $\mathfrak{F}_1$ is H-nflat. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{F}_1 \longrightarrow \mathfrak{F}_2 \longrightarrow \mathfrak{Q}_2 \longrightarrow 0\,,$$

since $\mathfrak{F}_1$ and $\mathfrak{Q}_2$ are H-nflat, by Lemma 2.3.c) $\mathfrak{F}_2$ is H-nflat. Iterating this reasoning, one proves that $\mathfrak{E}$ is H-nflat. ◇

Even if [6, Lemma A.7 and Proposition A.8] and [2, Theorem 3.2] have been proved where $\mathbb{K} = \mathbb{C}$, we note that that this hypothesis is unnecessary, in the sense these statements work on algebraically closed fields of characteristic 0. Thus, also the Lemmata 2.3.c) and 2.3.d) hold on any algebraically closed field of characteristic 0. The original proofs of the remaining lemmata continue to be valid in this extending setting, therefore we do not repeat them here.

Example 2.5. Let C be a smooth projective curve of genus $g \geq 2$.

a) (see [13, Example 3.2.9]) Let $\mathfrak{E} = (E = L_1 \oplus L_2, \varphi)$ be a rank 2 Higgs bundle over C such that $\varphi(L_1) \subseteq L_2 \otimes \Omega^1_X$ and $\varphi(L_2) = 0$. By [b) 13, Proposition 3.2.2], $\mathfrak{E}$ has only a quotient Higgs bundle which is $(L_1, 0)$. If one takes $g = 2$, $\deg(L_1) = 1$ and $\deg(L_2) = 0$, then $\mathfrak{E}$ is a H-ample Higgs bundle whose underlying vector bundle is not ample.

b) (see [10, Example 2.9]) Let $\mathfrak{E} = \left(E = K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, \varphi\right)$, where K is the canonical bundle of C, $K^{\frac{1}{2}}$ is a line bundle over C whose square is K and

$$\varphi = \begin{pmatrix} 0 & \omega \\ 1 & 0 \end{pmatrix},\ 1 \in \operatorname{Hom}\left(K^{\frac{1}{2}}, K^{-\frac{1}{2}} \otimes K\right),\ \omega \in \mathrm{H}^0(C, K^2).$$

The Higgs bundle $\mathfrak{E}$ is stable, because there are no subbundles of positive degree preserved by φ. Indeed, let L be a line subbundle of E then L is either $K^{-\frac{1}{2}}$ or $K^{\frac{1}{2}}$. In this case,

$$\deg\left(K^{\frac{1}{2}}\right) = g - 1 > 0,\ \deg\left(K^{-\frac{1}{2}}\right) = 1 - g < 0.$$

The line bundle L destabilizes E if and only if $L = K^{\frac{1}{2}}$. However $K^{\frac{1}{2}}$ does not destabilize $\mathfrak{E}$, because this is not φ-invariant. Furthermore, since $\deg(\mathfrak{E}) = 0$ then $\mathfrak{E}$ is H-nflat by Lemma 2.3.a); but the underlying vector bundle is unstable hence it is not nflat (see lemma 2.3.b)). △

The category of H-ample Higgs bundles over X has been studied in [3], hence we refer the reader to there for any further detail.

3. Curve semistable Higgs bundles

From now on, let $\mathfrak{E} = (E, \varphi)$ be a rank r Higgs bundle over a smooth projective polarized variety (X, H), and let $\mathcal{E}$ be the sheaf of sections of E, if not otherwise indicated.

Lemma 2.3.a) justifies the next definition which specializes Definition 0.2.

Definition 3.1. $\mathfrak{E}$ is *curve semistable* if for every morphism $f\colon C \to X$ the pullback Higgs bundle $f^*\mathfrak{E}$ is semistable.

The study of this class of Higgs bundle has started in [8] as a generalisation to Higgs bundles framework of Miyaoka's work [21, Section 3] on semistable vector bundles over smooth projective curves. There Bruzzo and Hernández Ruipérez have introduced the following numerical classes

$$\lambda_s(\mathfrak{E}) = \left[c_1\left(\mathcal{O}_{\mathrm{Gr}_1(Q_{s,\mathfrak{E}})}(1)\right)\right] - \frac{1}{r}\varpi_s^*(c_1(E)) \in \mathrm{N}^1\left(\mathrm{Gr}_1\left(Q_{s,\mathfrak{E}}\right)\right), \tag{3.1}$$

$$\theta_s(\mathfrak{E}) = \left[c_1\left(Q_{s,\mathfrak{E}}\right)\right] - \frac{s}{r}\rho_s^*(c_1(E)) \in \mathrm{N}^1\left(\mathfrak{Gr}_s(\mathfrak{E})\right), \tag{3.2}$$

where $s \in \{1,\dots,r-1\}$ and $\varpi_s\colon \mathrm{Gr}_1\left(Q_{s,\mathfrak{E}}\right) \to \mathfrak{Gr}_s(\mathfrak{E}) \xrightarrow{\rho_s} X$.

We recall that a class $\gamma \in \mathrm{N}^1(X)$ is *numerically effective* (*nef*, for short) if for any irreducible projective curve $C \subseteq X$ the inequality $\gamma \cdot [C] \geq 0$ holds. And one calls γ *strictly nef* class if for any irreducible projective curve $C \subseteq X$ the inequality $\gamma \cdot [C] > 0$ holds. Bruzzo, Graña Otero and Hernández Ruiperéz have proved in [5, 6, 8] the following theorems in the complex setting, here we extend these results to smooth projective varieties defined over an algebraically closed field of characteristic 0, without using any analytic method.

Theorem 3.2 (see [8, Theorem 1.2]). *$\mathfrak{E}$ is curve semistable if and only if each class $\theta_s(\mathfrak{E})$ is* nef*, where $s \in \{1,\dots,r-1\}$.*

Theorem 3.3 (see [8, Theorem 1.2]). *$\mathfrak{E}$ is curve semistable if and only if each class $\lambda_s(\mathfrak{E})$ is* nef*, where $s \in \{1,\dots,r-1\}$.*

Remark 3.4. By definition $\lambda_1(\mathfrak{E}) = \theta_1(\mathfrak{E})$. Thus if $r = 2$ the previous theorems are the same. ◇

Before proving these theorems, we recall the following lemma.

Lemma 3.5 (see [8, Lemma 3.3]). *Let $f\colon Y \to X$ be a finite surjective morphism of smooth projective curves. Then $\mathfrak{E}$ is semistable if and only if $f^*\mathfrak{E}$ is semistable.*

Proof. If $\mathfrak{E}$ is unstable then there exists a torsion-free Higgs subsheaf $\mathfrak{F}$ of $\mathfrak{E}$ such that $\mu(\mathfrak{F}) > \mu(\mathfrak{E})$, hence $\mu\left(f^*\mathfrak{F}\right) > \mu\left(f^*\mathfrak{E}\right)$ *i.e.*, $f^*\mathfrak{E}$ is unstable. Let us assume $f^*\mathfrak{E}$ unstable, without loss of generality one can assume that f is a Galois covering[1] with Galois group G, *i.e.*, the field extension $f^\#\colon \mathbb{K}(X) \to \mathbb{K}(Y)$ is normal (and separable) and the relevant Galois group is G. Let $\mathfrak{F} = (F, f^*\varphi_{|F})$ be the maximal destabilizing Higgs subsheaf of $f^*\mathfrak{E}$. For any $g \in G$, $g^*\mathfrak{F}$ is a destabilizing Higgs subsheaf of $f^*\mathfrak{E}$ of maximal rank; however, by the unicity of the HN-filtration of $f^*\mathfrak{E}$, it has to be $g^*\mathfrak{F} = \mathfrak{F}$. From all this, it follows that $F = f^*E_0$ for some destabilizing subbundle E_0 of E. By [17, Exercise III.9.3.a] f is a flat morphism, and since it is also surjective then f is (by definition) faithfully flat. Thus the composition $E_0 \otimes \Omega^1_X \to E \otimes \Omega^1_X \to (E/E_0) \otimes \Omega^1_X$ vanishes if and only if the composition

[1] In general, if $char(\mathbb{K}) > 0$ then one needs also that f is separable as stated in [8, 21].

$F\otimes f^*\Omega^1_X \to f^*E\otimes f^*\Omega^1_X \to (f^*E/F)\otimes f^*\Omega^1_X$ vanishes. Consider the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & F \arrow[r] \arrow[dr, "f^*\varphi"'] & f^*E\otimes f^*\Omega^1_X \arrow[r] \arrow[d] & (f^*E/F)\otimes f^*\Omega^1_X \arrow[d] & \\
0 \arrow[r] & F\otimes\Omega^1_Y \arrow[r] & f^*E\otimes\Omega^1_Y \arrow[r] & (f^*E/F)\otimes\Omega^1_Y \arrow[r] & 0,
\end{tikzcd} ,$$

since $f^*\varphi_{|F}$ takes values in $F\otimes\Omega^1_Y$ one has the claim, *i.e.*, $\mathfrak{E}$ is unstable. Q.E.D.

Proof of Theorem 3.2. Let $C\subseteq X$ be an irreducible projective curve and assume that the restriction $\theta_s(\mathfrak{E})_{|C}$ of $\theta_s(\mathfrak{E})$ to $\mathfrak{Gr}_s(\mathfrak{E}_{|C})$ is nef. If $\mathfrak{F}=(F,\psi)$ is a rank s torsion-free Higgs quotient sheaf of $\mathfrak{E}_{|C}$ then by [23, Corollary at page 75] F is locally free. By Universal Property of Higgs-Grassmann schemes, there exists a unique section $\sigma\colon C\to\mathfrak{Gr}_s(\mathfrak{E}_{|C})$ such that $\mathfrak{F}=\sigma^*\mathfrak{Q}_{s,\mathfrak{E}|C}$, where $\mathfrak{Q}_{s,\mathfrak{E}|C}$ is the restriction of $\mathfrak{Q}_{s,\mathfrak{E}}$ to $\mathfrak{Gr}_s(\mathfrak{E}_{|C})$. Then

$$0\le\theta_s(\mathfrak{E})_{|C}\cdot[\sigma(C)]=\deg(F)-\frac{s}{r}\deg(E_{|C})=s\left(\mu(F)-\mu(E_{|C})\right);$$

thus if each $\theta_s(\mathfrak{E})$ is nef then $\mathfrak{E}_{|C}$ is semistable, *i.e.*, $\mathfrak{E}$ is curve semistable. *Conversely*, let $\mathfrak{E}$ be curve semistable and let us assume that $\theta_s(\mathfrak{E})$ is not nef for some $s\in\{1,\dots,r-1\}$ *i.e.*, there exists an irreducible projective curve $C'\subseteq\mathfrak{Gr}_s(\mathfrak{E})$ such that $\theta_s(\mathfrak{E})\cdot[C']<0$. Under this hypothesis, C' is not contained in a fibre of ρ_s, hence it surjects onto a projective curve $C\subseteq X$. One may choose a projective curve C'' and a morphism $h\colon C''\to C$ such that $\widetilde{C}=C''\times_C C'$ is a union of projective curves C_j isomorphic to C (see the proof of [21, Theorem 3.1]); $\theta_s(h^*\mathfrak{E})\cdot[C_j]<0$ for any index j evidently. For clarity, one has the following Cartesian diagram

$$\begin{tikzcd}
\mathfrak{Gr}_s(h^*\mathfrak{E}) \arrow[r, "h_s"'] \arrow[d] & \mathfrak{Gr}_s(\mathfrak{E}_{|C}) \arrow[r, hook] \arrow[d] & \mathfrak{Gr}_s(\mathfrak{E}) \arrow[d, "\rho_s"] & C' \arrow[l, hook'] \\
C'' \arrow[r, "h"'] & C \arrow[r, hook, "i"'] & X, &
\end{tikzcd} ,$$

by the universal property of fibre products $\widetilde{C}\hookrightarrow\mathfrak{Gr}_s(h^*\mathfrak{E})$. Let $\mathfrak{E}_j=h_s^*\left(\rho_s^*\mathfrak{E}_{|C}\right)_{|C_j}$ and let $\mathfrak{Q}_j=\mathfrak{Q}_{s,h^*\mathfrak{E}|C_j}$. By Lemma 3.5, $\mathfrak{E}_j$ is a semistable Higgs bundle, therefore $\mu(\mathfrak{Q}_j)\ge\mu(\mathfrak{E}_j)$. On the other hand

$$0>\theta_s(h^*\mathfrak{E})\cdot[C_j]=\left(c_1(Q_j)-\frac{s}{r}\rho_s^*c_1(E)\right)\cdot[C_j]=s\left(\mu(Q_j)-\mu(E_j)\right),$$

where E_j and Q_j are the underlying vector bundles to $\mathfrak{E}_j$ and $\mathfrak{Q}_j$, respectively. But this is a contradiction with the assumptions, hence all classes $\theta_s(\mathfrak{E})$ have to be nef. Q.E.D.

Proof of Theorem 3.3. The class $\lambda_s(\mathfrak{E})$ may be regarded as the numerical class of the hyperplane bundle of the Higgs $\mathbb{Q}$-bundle $\mathfrak{F}_s = \mathfrak{Q}_{s,\mathfrak{E}} \otimes \rho_s^*\left(\det(E)^{-1/r}\right)$ over $\mathfrak{Gr}_s(\mathfrak{E})$. As a consequence

$$c_1(\mathfrak{F}_s) = c_1\left(Q_{s,\mathfrak{E}}\right) - \frac{s}{r}\rho_s^* c_1(E) \Rightarrow \theta_s(\mathfrak{E}) = [c_1(\mathfrak{F}_s)] \in \mathrm{N}^1\left(\mathfrak{Gr}_s(\mathfrak{E})\right).$$

From all this, the classes $\lambda_s(\mathfrak{E})$ are nef if and only if the classes $\theta_s(\mathfrak{E})$ are nef, *i.e.*, if and only if $\mathfrak{E}$ is curve semistable (Theorem 3.2). Q.E.D.

Remark 3.6. Mimicking Definition 3.1, $\mathfrak{E}$ is *curve stable* if for every morphism $f: C \to X$ the pullback Higgs bundle $f^*\mathfrak{E}$ is stable.

By the previous proofs, the curve stability of $\mathfrak{E}$ implies the strictly nefness of $\theta_s(\mathfrak{E})$ and $\lambda_s(\mathfrak{E})$ for any $s \in \{1, \ldots, r-1\}$. ◇

Proof of Lemma 2.3.a). Let us assume $\mathfrak{E}$ satisfies the hypotheses, the classes $c_1\left(\mathcal{O}_{\mathrm{Gr}_1(Q_{s,\mathfrak{E}})}(1)\right)$ are nef for any $s \in \{1, \ldots, r-1\}$ (Theorem 3.3), hence their pullbacks to $\mathfrak{Gr}_s\left(f^*\mathfrak{Q}_{s,\mathfrak{E}}\right)$ are nef. In other words, using the notations introduced in Remark 1.2.b), $\mathfrak{Q}_{(1,s),\mathfrak{E}}$ is nef; and it remains only to prove that $\det\left(\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}\right)$ are nef for all strings of integer numbers $1 \leq s_1 < s_2 < \ldots < s_k < r$. By construction, $\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}$ is a Higgs bundle over $\mathfrak{Gr}_{s_1}\left(\mathfrak{Q}_{(s_2,\cdots,s_k),\mathfrak{E}}\right)$ and there is a morphism $\rho_{(s_1,\cdots,s_k)}\colon \mathfrak{Gr}_{s_1}\left(\mathfrak{Q}_{(s_2,\cdots,s_k),\mathfrak{E}}\right) \to X$ such that $\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}$ is a rank s_1 quotient Higgs bundle of $\rho^*_{(s_1,\cdots,s_k)}\mathfrak{E}$. Thus there exists a unique morphism $g_{(s_1,\cdots,s_k)}\colon \mathfrak{Gr}_{s_1}\left(\mathfrak{Q}_{(s_2,\cdots,s_k),\mathfrak{E}}\right) \to \mathfrak{Gr}_{s_1}(\mathfrak{E})$ such that $\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}} = g^*_{(s_1,\cdots,s_k)}\mathfrak{Q}_{s_1,\mathfrak{E}}$. Therefore

$$\left[c_1\left(\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}\right)\right] = g^*_{(s_1,\cdots,s_k)}\left[c_1\left(\mathfrak{Q}_{s_1,\mathfrak{E}}\right)\right] = g^*_{(s_1,\cdots,s_k)}\theta_{s_1}(\mathfrak{E}),$$

because $\int_C f^*c_1(E) = 0$, hence $\det\left(\mathfrak{Q}_{(s_1,\cdots,s_k),\mathfrak{E}}\right)$ is nef because $\theta_{s_1}(\mathfrak{E})$ is nef by Theorem 3.2. In other words $\mathfrak{E}$ is H-nef. Repeating all this reasoning, considering that also $\mathfrak{E}^\vee$ is curve semistable, $c_1\left(E^\vee\right) = -c_1(E)$ hence $\int_C f^*c_1\left(E^\vee\right) = 0$, one proves in the same way the H-nefness of $\mathfrak{E}^\vee$, *i.e.*, $\mathfrak{E}$ is H-nflat.

The converse also holds, since the proof of [6, Proposition A.8] works on smooth projective varieties defined over an algebraically closed field of characteristic 0. Q.E.D.

Example 3.7. Let X be a minimal smooth surface such that the canonical bundle K_X is nef as well as being big, *i.e.*, X is also a surface of general type. The Chern classes of X satisfy the *Bogomolov-Miyaoka-Yau inequality* ([21, Proposition 7.1])

$$BMY(X) \stackrel{def.}{=} \int_X 3c_2(X) - c_1(X)^2 \geq 0,$$

where $c_k(X) \overset{def.}{=} (-1)^k c_k\left(\Omega^1_X\right)$ for any k. One considers the so-called *Simpson system*, *i.e.*, the Higgs bundle $\mathfrak{S} = (S, \varphi)$, where $S = \Omega^1_X \oplus \mathcal{O}_X$ and

$$\varphi = \begin{pmatrix} 0 & 0 \\ \mathrm{Id} & 0 \end{pmatrix}, \ \mathrm{Id} \in \mathrm{Hom}\left(\Omega^1_X, \Omega^1_X\right).$$

If $BMY(X) = 0$ then the Higgs bundle $\mathfrak{S}(-\beta K_X)$ is H-nef for every rational number[2] $0 < \beta \leq \dfrac{1}{3}$, but $S\,(-\beta K_X)$ is not nef ([3, Theorem 4.2]). Moreover, $\mathrm{End}(\mathfrak{S})$ is H-nflat, by [3, Proposition 4.1] and Lemma 2.3.a), but $\mathrm{End}(S)$ is not nflat since S is not semistable (see Lemma 2.3.b)). △

Remark 3.8. In [14], last example has been extended to n-dimensional varieties X whose canonical bundle is ample and $2(n+1)c_2(X) = nc_1(X)^2 \in \mathrm{A}^2(X)$ (see [14, Corollary 3.4.b]). ◇

4. Curve semistable Higgs bundles whose discriminant class vanishes

We have recalled the definition of curve semistable Higgs bundles, because we want to extend the following theorem to Higgs bundles setting.

Theorem 4.1 (see [22, Theorem 2] and [8, Theorem 1.4])**.** *For a vector bundle E over a smooth projective variety X the following statements are equivalent:*

a) $\theta_1(E)$ *is nef;*

b) E *is curve semistable;*

c) E *is semistable with respect to some polarization* H *and* $c_2\,(\mathrm{End}(E)) = 0 \in \mathrm{A}^2(X)$*;*

d) E *is semistable with respect to some polarization* H *and* $\displaystyle\int_X c_2\,(\mathrm{End}(E)) \cdot H^{n-2} = 0$.

Proof. **(a) is equivalent to (b).** This is [21, Theorem 3.1] when $\dim X = 1$. Let $\dim X \geq 2$. If E is curve semistable then $\theta_1(E)$ is nef by Theorem 3.2. *Conversely*, if $\theta_1(E)$ is nef, let $f\colon C \to X$ be a morphism. Consider the following Cartesian diagram

$$\begin{tikzcd} \mathrm{Gr}_1\,(f^*E) \arrow[r, "\overline{f}"] \arrow[d, "p_{1|C} \equiv f^*(p_1)"'] & \mathrm{Gr}_1(E) \arrow[d, "p_1"] \\ C \arrow[r, "f"'] & X, \end{tikzcd} \quad ,$$

[2]As it is customary, we formally consider twistings by rational divisors, which make sense after pulling back to a (possibly ramified) finite covering of X; on the other hand, the properties of being semistable, H-nef are invariant under such coverings (see Lemmata 2.1.a) and 3.5).

one has

$$\begin{aligned}\overline{f}^{*}\lambda_1(E) &= \overline{f}^{*}\left(\left[c_1\left(Q_{1,E}\otimes p_1^{*}\left(\det(E)^{-1/r}\right)\right)\right]\right)\\ &= \left[c_1\left(Q_{1,f^{*}E}\otimes p_{1|C}^{*}\left(\det(E)^{-1/r}\right)\right)\right] = \lambda_1\left(f^{*}E\right).\end{aligned}$$

Since $Q_{1,E}\otimes p_1^{*}\left(\det(E)^{-1/r}\right)$ is nef then its pullback $Q_{1,f^{*}E}\otimes p_{1|C}^{*}\left(\det(E)^{-1/r}\right)$ via f is nef ([5, Proposition 2.6.(ii)]). Thus $\lambda_1\left(f^{*}E\right)$ is nef as well, and by [21, Theorem 3.1] $f^{*}E$ is semistable. In nother words, one has the claim.

(b) implies (c). By hypothesis, for any morphism $f\colon C\to X$, $f^{*}E$ is semistable, hence $f^{*}\operatorname{End}(E) = f^{*}\left(E\otimes E^{\vee}\right) = f^{*}E\otimes f^{*}E^{\vee}$ is semistable *i.e.*, $\operatorname{End}(E)$ is curve semistable. Since $c_1(\operatorname{End}(E)) = 0$ then $\operatorname{End}(E)$ is nflat (Lemma 2.3.a)), therefore it is semistable (Lemma 2.3.b)) and this implies the semistability of E. Finally [12, Propositions 1.2.9 and 1.3] proves that $c_2(\operatorname{End}(E)) = 0$.

(c) implies (b). If $\dim X = 1$ there is nothing to prove of course. Let $\dim X = n\geq 2$, repeating the reasoning of [15, Lemma 2.8] one may assume $\mathbb{K}\subseteq\mathbb{C}$. Consider the following Cartesian diagram

$$\begin{tikzcd} X_{\mathbb{C}} \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}(\mathbb{C}) \arrow[r] & \operatorname{Spec}(\mathbb{K}), \end{tikzcd},$$

by hypothesis $\operatorname{End}(E)$ is a degree 0 semistable Higgs bundle with $c_2(\operatorname{End}(E)) = 0$, and by the same lemma $f^{*}\operatorname{End}(E)$ is a semistable vector bundle over $X_{\mathbb{C}}$ such that $c_2\left(f^{*}\operatorname{End}(E)\right) = 0$ (see [15, Theorem 2.7]). If there exists a morphism $g\colon C\to X$ such that $g^{*}E$ is unstable, then by [15, Lemma 2.8] $\overline{g}^{*}E$ is unstable, where $\overline{g}$ is the base change morphism of g over $\overline{C} = C\times_{\mathbb{K}}\mathbb{C}$. This gives rise to a contradiction: $\overline{g}^{*}\operatorname{End}(E)$ is the pullback of $f^{*}\operatorname{End}(E)$ over $\overline{C}$ and it is unstable, but by [8, Theorem 1.4] $\overline{g}^{*}\operatorname{End}(E)$ is semistable. To avoid this absurd, E is curve semistable.

(c) implies (d). This is trivial.

(d) implies (c). $\operatorname{End}(E)$ is semistable with $\displaystyle\int_X c_2(\operatorname{End}(E)) = 0$, while $c_1(\operatorname{End}(E)) = 0$ of course. Let

$$\{0\} = F_0 \subsetneq F_1 \subsetneq \ldots \subsetneq F_{m-1} \subsetneq F_m = \operatorname{End}(E)$$

be a Jordan-Hölder filtration of $\operatorname{End}(E)$, by [18, Corollary 6] this can be chosen in a way that the quotients $Q_i = F_i/F_{i-1}$ have vanishing Chern classes for any $i\in\{1,\ldots,m\}$. In other words, $\operatorname{End}(E)$ is an iterating extension of vector bundles with vanishing Chern classes, thus the same statement holds for the Chern classes of $\operatorname{End}(E)$; in particular $c_2(\operatorname{End}(E)) = 0$. Q.E.D.

Moreover, the previous theorem is equivalent to the following one.

Theorem 4.2 (see [1, Corollary 3.2]). *On a smooth projective variety X, the following statements are equivalent.*

a) Let E be a curve semistable vector bundle over X. Then E is semistable with respect to some polarization H and $c_2(\mathrm{End}(E)) = 0 \in \mathrm{A}^2(X)$.

b) The Chern classes of any nflat vector bundle over X vanish.

Proof. If a) holds, let E be a nflat vector bundle over X. By Lemma 2.3.b), E is semistable. Furthermore, applying also [5, Proposition 2.6.(ii)], E is curve semistable hence $c_2(E) = c_2(\mathrm{End}(E)) = 0$. By [18, Corollary 6], E is extension of vector bundles whose Chern classes vanish, hence the Chern classes of E vanish.

If b) holds, let E be a curve semistable vector bundle over X. For any $f\colon C \to X$ one has $f^*\,\mathrm{End}(E) \cong \mathrm{End}(f^*E)$. As proved in Theorem 4.1, $\mathrm{End}(E)$ is curve semistable. Since $\mathrm{End}(E)$ satisfies the hypotheses of Lemma 2.3.a), it is nflat hence $c_2(\mathrm{End}(E)) = 0$. By Lemmata 2.1.b) and 2.3.b), E is semistable. Q.E.D.

Remark 4.3. Since Theorem 4.1 proves that Theorem 4.2.a) holds, one has another proof of Theorem 4.2.b). This has been proved by [12, Proposition 1.3] originally. ◇

In the Higgs bundles setting, Theorem 4.1 changes as it follows.

Theorem 4.4. *Let $\mathfrak{E} = (E, \varphi)$ a rank r Higgs bundle over a smooth projective variety X. Consider the following statements:*

a) $\theta_1(\mathfrak{E}), \ldots, \theta_{r-1}(\mathfrak{E})$ are nef;

b) $\mathfrak{E}$ is curve semistable;

c) $\mathfrak{E}$ is semistable with respect to some polarization H and $c_2(\mathrm{End}(E)) = 0 \in \mathrm{A}^2(X)$;

d) $\mathfrak{E}$ is semistable with respect to some polarization H and $\int_X c_2(\mathrm{End}(E))\cdot H^{n-2} = 0$.

The following implications hold

$$(a) \Longleftrightarrow (b) \Longleftarrow (c) \Longleftrightarrow (d).$$

It is enough to repeat the proof of Theorem 4.1. However if $\mathfrak{E}$ is curve semistable then $\mathrm{End}(\mathfrak{E}) = (\mathrm{End}(E), \mathrm{End}(\varphi))$ is H-nflat (by Lemmata 2.3.d) and 2.3.a)), hence $\mathfrak{E}$ is semistable, but it is unknown whether $c_2(\mathrm{End}(E)) = 0$.

Remark 4.5. If $\mathfrak{E}$ is semistable and $c_2(\mathrm{End}(E)) = 0$ then $\mathfrak{E}$ is semistable with respect to any polarization of X. This follows from the fact that the semistability of H-nflat Higgs bundles does not depend on the polarization of X (Lemma 2.3.b)). ◇

From all this, it is interesting to study whether the condition 4.4.*b)* implies the condition 4.4.*c)*. To simplify the exposition of the corresponding topics, one introduces the following class

$$\Delta(E) = \frac{1}{2r} c_2(\mathrm{End}(E)) = c_2(E) - \frac{r-1}{2r} c_1(E)^2 \in \mathrm{A}^2(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{A}^2(X)_{\mathbb{Q}},$$

which is called *discriminant class of* E (see [18, Theorem 7]). We recall the following conjecture.

Conjecture 1 (**Bruzzo and Graña Otero Conjecture**)**.** Let $\mathfrak{E}$ be a curve semistable Higgs bundle over X. Then $\mathfrak{E}$ is semistable with respect to some polarization H and $\Delta(E) = 0$.

To be more precise, the Conjecture 1 can be simplified using [24, Lemma 3.7], as remarked in [19], and [15, Theorem 2.7].

Conjecture 2. Let $\mathfrak{E}$ be a curve semistable Higgs bundle over a smooth complex projective surface X. Then $\mathfrak{E}$ is semistable with respect to some polarization and $\Delta(E) = 0$.

In other words, to prove Conjecture 1 it is enough to study curve semistable Higgs bundles over smooth projective surfaces; and without loss of generality, it is enough also to assume $\mathbb{K} = \mathbb{C}$.

Remark 4.6. The best of our knowledge, the previous conjecture has been proved in the following cases:

a) when $r = 2$, by [7, Theorems 4.5, 4.8 and 4.9];

b) when X has nef tangent bundle, by [10, Corollary 3.15];

c) when $\dim X = 2$ and $\kappa(X) \in \{-\infty, 0\}$ (the Kodaira dimension of X). Indeed, the statement for ruled surfaces follows by [10, Proposition 3.11]. Since rational surfaces are rationally connected, the statement follows by [10, Theorem 3.6]. The statement for Abelian surfaces follows by [10, Corollary 3.8], the case of K3 surfaces follows by [9, Theorem 6.4] and thus for Enriques surfaces and hyperelliptic surfaces follow by [10, Proposition 3.12]. All this complete this case;

d) when $\dim X = 2$, $\kappa(X) = 1$ and other technical hypotheses, see [11, Proposition 5.6]; we have generalized this result to all elliptic surfaces ([15, Theorem 4.18]), recently;

e) when X is a simply-connected Calabi-Yau variety, by [2, Theorem 4.1];

f) if X satisfies the Conjecture 1 and Y is a fibred projective variety over X with rationally connected fibres, then Y does the same, by [10, Proposition 3.11];

g) if X satisfies the Conjecture 1 then any finite étale quotient Y of X does the same, by [10, Proposition 3.12];

h) when $\mathfrak{E}$ has a Jordan-Hölder filtration whose quotient are H-nflat and have rank at most 2, by [13, corollaries 4.3.6 and 4.3.7];

i) particular Higgs bundles described in [4]. ◇

Furthermore, the previous conjectures are equivalent to a third one ([1, Corollary 3.2] and Lemma 2.3.f)).

Conjecture 3. Let $\mathfrak{E}$ be a stable H-nflat Higgs bundle over a smooth complex projective surface X. Then its Chern classes vanish.

Indeed, to compute Chern classes of $\mathfrak{E}$, one replaces it with graded module $\bigoplus_{i=0}^{m} \mathfrak{F}_i/\mathfrak{F}_{i+1}$ associated to the filtration (2.1). Thus and by Remark 1.2.c), the second Chern class of $\mathfrak{E}$ is the sum of the second Chern class of the $\mathfrak{F}_i/\mathfrak{F}_{i+1}$'s, which are stable and H-nflat Higgs bundles (Lemma 2.3.f)).

Remark 4.7. When the Higgs field vanishes, [12, Proposition 1.3] proves the vanishing of Chern classes for nflat vector bundles over smooth projective varieties. This last result has been extended to compact Kähler manifolds by [16, Corollary 1.19]. ◇

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