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MIXED HODGE STRUCTURES AND E-POLYNOMIALS

ESTRUCTURAS DE HODGE MIXTAS Y POLINOMIOS E

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Abstract

We provide a self-contained review of mixed Hodge structures (MHS) with a focus on the computation of E-polynomials for complex quasi-projective varieties. We discuss the discrepancy between MHS on diffeomorphic varieties, specifically for the cotangent bundle of an abelian variety. Furthermore, we compute the E-polynomials for $SL_n(\mathbb{C})$ and $PGL_n(\mathbb{C})$ and explore the link between these invariants and point-counting over finite fields.

Keywords: E-polynomials; Hodge polynomials; mixed Hodge structures; moduli spaces.

Resumen

En este trabajo se presenta una revisión autocontenida de las estructuras de Hodge mixtas (MHS), con énfasis en el cálculo de los polinomios E para variedades cuasiproyectivas complejas. Se analiza la discrepancia entre las MHS asociadas a variedades difeomorfas, en particular para el fibrado cotangente de una variedad abeliana. Además, calculamos los polinomios E de los grupos algebraicos $SL_n(\mathbb{C})$ y $PGL_n(\mathbb{C})$, y se explora la relación entre estos invariantes y el conteo de puntos sobre cuerpos finitos.

Palabras clave: polinomios E; polinomios de Hodge; estructuras de Hodge mixtas; espacios móduli.

Mathematics Subject Classification: Primary 14C30, 32S35; Secondary 14D07, 14L10.

Introduction

The theory of mixed Hodge structures (MHS), established by Deligne [1] in the 1970s, provides a robust framework for studying the topology of complex algebraic varieties that are not necessarily smooth or projective. A central numerical invariant in this theory is the *mixed Hodge polynomial* [4], which encodes the dimensions of the graded pieces of the Hodge and weight filtrations on the cohomology of a variety. Despite their theoretical elegance, these polynomials are notoriously difficult to compute in practice; consequently, explicit expressions are known only for a restricted class of varieties.

As a more tractable alternative, one often utilizes the E-polynomial (also referred to as the Hodge-Deligne polynomial). This invariant is obtained by specializing the compactly supported Hodge polynomial at $t = -1$. While the E-polynomial carries less information than the full Hodge polynomial, it possesses a remarkable additive property with respect to stratifications and behaves predictably under the Künneth isomorphism. In recent years, these tools have proven indispensable in the study of moduli spaces of Higgs bundles and character varieties, where novel techniques in arithmetic geometry have linked E-polynomials to point-counting over finite fields [2, 3, 4].

This work provides a concise introduction to the formalism of both Hodge and E-polynomials. We present a series of computations and examples inspired by the problem sets from the course taught by Alfonso Zamora during the "Workshop on Character Varieties and Higgs Bundles," organized by DFG-CONARE in August 2025. In particular, we detail the discrepancy between MHS on diffeomorphic varieties and verify the E-polynomials for the algebraic groups $SL_n(\mathbb{C})$ and $PGL_n(\mathbb{C})$. Finally, we conclude with a brief discussion on certain open problems and directions for future study.

The paper is structured as follows: Section 1 reviews the foundational aspects of pure and mixed Hodge theory, concluding with the formal definition of mixed Hodge structures and E-polynomials. In Section 2, we present the specific computations and examples derived from the aforementioned workshop, including the study of diffeomorphic varieties and algebraic groups. Finally, Section 3 offers concluding remarks and discusses directions for future research.

1. Preliminaries

We begin by recalling the basic notions of Hodge theory that will be used throughout this work.

Definition 1. A *pure Hodge structure* of weight k consists of a finitely generated $\mathbb{Z}$-module $V_{\mathbb{Z}}$ together with a decomposition of its complexification

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{p+q=k} V^{p,q},$$

called the *Hodge decomposition*, such that $\overline{V^{p,q}} = V^{q,p}$, where the overline denotes complex conjugation.

Associated to a pure Hodge structure of weight k, one defines a decreasing filtration of $V_{\mathbb{C}}$, called the *Hodge filtration*, by setting

$$F^p(V_{\mathbb{C}}) := \bigoplus_{i \geq p} V^{i,k-i}.$$

This construction yields a finite decreasing filtration

$$V_{\mathbb{C}} = F^0(V_{\mathbb{C}}) \supseteq F^1(V_{\mathbb{C}}) \supseteq \cdots \supseteq \{0\}.$$

Conversely, a pure Hodge structure of weight k can be equivalently described by a decreasing filtration $F^\bullet(V_{\mathbb{C}})$ satisfying

$$V_{\mathbb{C}} = F^p(V_{\mathbb{C}}) \oplus \overline{F^{k-p+1}(V_{\mathbb{C}})} \quad \text{for all } p.$$

In this case, the Hodge decomposition is recovered as

$$V^{p,q} = F^p(V_{\mathbb{C}}) \cap \overline{F^q(V_{\mathbb{C}})}, \qquad p+q=k.$$

Deligne showed that the De Rham cohomology has a pure Hodge structure for some special varieties.

Theorem 1 (Hodge Decomposition)**.** *Let X be a compact Kähler manifold of complex dimension m. For each $k = 0, 1, \dots, 2m$, the de Rham cohomology group $H^k_{\mathrm{dR}}(X)$ carries a pure Hodge structure of weight k, given by the decomposition*

$$H^k_{\mathrm{dR}}(X) \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q}(X),$$

satisfying the reality condition

$$\overline{H^{p,q}(X)} = H^{q,p}(X).$$

The dimensions

$$h^{p,q} := \dim_{\mathbb{C}} H^{p,q}(X)$$

are called the Hodge numbers *of X.*

When dealing with complex algebraic varieties that are neither smooth nor projective, the Hodge decomposition generally fails to hold. Deligne resolved this issue by introducing the notion of *mixed Hodge structures.*

Definition 2. A *mixed Hodge structure* on a finitely generated $\mathbb{Z}$-module $V_{\mathbb{Z}}$ consists of the following data:

- an increasing filtration $W_\bullet(V_{\mathbb{Q}})$ on $V_{\mathbb{Q}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, called the *weight filtration*,

$$\cdots \subseteq W_{k-1}(V_{\mathbb{Q}}) \subseteq W_k(V_{\mathbb{Q}}) \subseteq W_{k+1}(V_{\mathbb{Q}}) \subseteq \cdots;$$

- a decreasing filtration $F^\bullet(V_{\mathbb{C}})$ on $V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, called the *Hodge filtration*,

$$V_{\mathbb{C}} = F^0(V_{\mathbb{C}}) \supseteq F^1(V_{\mathbb{C}}) \supseteq \cdots \supseteq \{0\}.$$

For each integer k, the k-th *graded piece* of the weight filtration is defined as the quotient

$$\mathrm{Gr}^W_k(V_{\mathbb{Q}}) := W_k(V_{\mathbb{Q}})/W_{k-1}(V_{\mathbb{Q}}).$$

These data are required to satisfy the condition that, for each integer k, the induced filtration $F^\bullet$ on the complexified graded piece

$$\mathrm{Gr}^W_k(V_{\mathbb{C}}) := \mathrm{Gr}^W_k(V_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \mathbb{C}$$

defines a pure Hodge structure of weight k.

Theorem 2 (Deligne)**.** *Let X be a quasi-projective algebraic variety, not necessarily smooth, complete, or irreducible. Then, for each $k \in \mathbb{Z}$, the singular cohomology group $H^k(X, \mathbb{Q})$ and the singular cohomology with compact support $H^k_c(X, \mathbb{Q})$ carry natural mixed Hodge structures.*

For each integer k, we define

$$H^{k;p,q}(X) := \mathrm{Gr}_F^p \, \mathrm{Gr}_{p+q}^W(H^k(X,\mathbb{C})).$$

The *mixed Hodge numbers* of X are then given by

$$h^{k,p,q}(X) := \dim_{\mathbb{C}} H^{k;p,q}(X).$$

These numbers satisfy the symmetry property $h^{k,p,q}(X) = h^{k,q,p}(X)$ and, in contrast with the pure case, it may occur that $p+q \neq k$. We call the *k-weights* of the mixed Hodge structure the pairs (p,q) such that $h^{k,p,q}(X) \neq 0$.

Theorem 3. *Let X and Y be complex quasi-projective varieties. Then:*

1. *For all k, p, q, we have*
$$H^{k,q,p}(X) \;\cong\; H^{k,p,q}(X).$$
2. *The weight and Hodge filtrations are preserved by algebraic maps. Therefore, the mixed Hodge structures are likewise preserved.*
3. *The Hodge and weight filtrations are preserved by the Künneth isomorphism. Therefore, the mixed Hodge structures are likewise preserved.*

Definition 3. Let X be a complex algebraic variety.

The **Hodge polynomial** (or **mixed Hodge polynomial**) of X is defined by

$$\mu(X;t,u,v) = \sum_{k,p,q} h^{k,p,q}(X)\, t^k u^p v^q, \tag{1.1}$$

where $h^{k,p,q}(X) := \dim \mathrm{Gr}_F^p \, \mathrm{Gr}_{p+q}^W H^k(X,\mathbb{C})$ are the mixed Hodge numbers of the singular cohomology of X.

Analogously, the **compactly supported mixed Hodge polynomial** is defined by

$$\mu_c(X;t,u,v) = \sum_{k,p,q} h_c^{k,p,q}(X)\, t^k u^p v^q, \tag{1.2}$$

where $h_c^{k,p,q}(X) = \dim \mathrm{Gr}_F^p \, \mathrm{Gr}_{p+q}^W H_c^k(X,\mathbb{C})$ are the mixed Hodge numbers of the cohomology with compact support.

The ***E*-polynomial** (or **Hodge-Deligne polynomial**) is obtained by specializing the compactly supported Hodge polynomial at $t = -1$:

$$E(X;u,v) = \mu_c(X;-1,u,v) = \sum_{k,p,q} (-1)^k h_c^{k,p,q}(X)\, u^p v^q. \tag{1.3}$$

2. Main results

We start by showing that diffeomorphic complex varieties can have different Hodge structures.

Proposition 4. *Let Σ_g be a compact Riemann surface of genus g, and let $J = \mathrm{Pic}^0(\Sigma_g)$ be its Jacobian variety. Then the cotangent bundle T^*J admits two natural realizations as complex algebraic varieties which are diffeomorphic as smooth manifolds but carry different mixed Hodge structures.*

More precisely, identifying T^*J as a trivial vector bundle, we have a diffeomorphism of smooth manifolds $T^*J \simeq (\mathbb{C}^*)^{2g}$. The mixed Hodge structure induced on $H^k(T^*J)$ by the algebraic structure of T^*J is pure of weight k, whereas the mixed Hodge structure induced by the algebraic variety $(\mathbb{C}^*)^{2g}$ is pure Tate of weight $2k$ and type (k, k).

Proof. We start by computing the mixed Hodge structure of $(\mathbb{C}^*)^{2g}$. We begin with degree 0.

Recall that for any complex algebraic variety X we have

$$H^0(X, \mathbb{C}) = H^{0,0,0}(X),$$

that is, the 0-th cohomology is pure of type $(0, 0)$ and weight 0.

By the Künneth isomorphism we obtain

$$H^{0,0,0}((\mathbb{C}^*)^{2g}) \cong H^0((\mathbb{C}^*)^{2g}) \cong (H^0(\mathbb{C}^*))^{\otimes 2g} \cong \mathbb{C}^{\otimes 2g} \cong \mathbb{C}.$$

Therefore

$$h^{0,0,0}((\mathbb{C}^*)^{2g}) = \dim H^{0,0,0}((\mathbb{C}^*)^{2g}) = 1.$$

For general degree k, by the Künneth formula we have

$$H^k((\mathbb{C}^*)^{2g}) \cong \bigoplus_{i+j=k} H^i((\mathbb{C}^*)^{2g-1}) \otimes H^j(\mathbb{C}^*).$$

Since $H^j(\mathbb{C}^*) = 0$ for $j > 1$, this reduces to

$$H^k((\mathbb{C}^*)^{2g}) \cong [H^k((\mathbb{C}^*)^{2g-1}) \otimes H^0(\mathbb{C}^*)] \oplus [H^{k-1}((\mathbb{C}^*)^{2g-1}) \otimes H^1(\mathbb{C}^*)].$$

Iterating the Künneth decomposition, we obtain the following description for general k. Write $(\mathbb{C}^*)^{2g} = \prod_{r=1}^{2g} \mathbb{C}^*_r$. Since $H^j(\mathbb{C}^*) = 0$ for $j > 1$, every summand in the Künneth decomposition of $H^k((\mathbb{C}^*)^{2g})$ is obtained by choosing, for each factor $\mathbb{C}^*_r$, either $H^0(\mathbb{C}^*_r)$ or $H^1(\mathbb{C}^*_r)$, in such a way that the total degree adds up to k.

Equivalently, one must choose exactly k factors contributing H^1 and the remaining $2g-k$ factors contributing H^0. Thus

$$H^k((\mathbb{C}^*)^n) \cong \bigoplus_{i_1+\cdots+i_n=k} H^{i_1}(\mathbb{C}^*) \otimes \cdots \otimes H^{i_n}(\mathbb{C}^*),$$

where $i_j \in \{0,1\}$ since $H^i(\mathbb{C}^*) = 0$ for $i \geq 2$. Thus each summand corresponds to choosing a subset $I \subset \{1,\ldots,2g\}$ with $|I| = k$.

Since $H^1(\mathbb{C}^*) \cong \mathbb{C}$ is one-dimensional, each tensor factor $\bigotimes_{i\in I} H^1(\mathbb{C}^*)$ is canonically isomorphic to $\mathbb{C}$. Therefore we obtain

$$H^k((\mathbb{C}^*)^{2g}) \cong \bigoplus_{\substack{I\subset\{1,\ldots,2g\}\\ |I|=k}} \mathbb{C}.$$

Since the number of subsets $I \subset \{1,\ldots,2g\}$ with $|I| = k$ is $\binom{2g}{k}$, it follows that

$$\dim_{\mathbb{C}} H^k((\mathbb{C}^*)^{2g}) = \binom{2g}{k}.$$

Recall that on $\mathbb{C}^*$, we have $H^0(\mathbb{C}^*) = H^{0;0,0}(\mathbb{C}^*)$ and $H^1(\mathbb{C}^*) = H^{1;1,1}(\mathbb{C}^*)$, hence each tensor factor in the above sum can be written as

$$H^{i_1;i_1,i_1}(\mathbb{C}^*) \otimes \cdots \otimes H^{i_{2g};i_{2g},i_{2g}}(\mathbb{C}^*),$$

where $i_j \in \{0,1\}$ and $\sum_{j=1}^{2g} i_j = k$.

Since Hodge bidegrees add under tensor products, every summand lies in $H^{k;k,k}((\mathbb{C}^*)^{2g})$. Therefore

$$H^k((\mathbb{C}^*)^{2g}) = H^{k;k,k}((\mathbb{C}^*)^{2g}),$$

and in particular $H^k((\mathbb{C}^*)^{2g})$ is pure of Hodge type (k,k) and weight $2k$.

We now compute the (pure) Hodge structure of $J = \mathrm{Pic}^0(\Sigma_g)$.

It is well known that J is a complex abelian variety of dimension g; in particular, J is smooth and projective. Hence, for each k, the mixed Hodge structure on $H^k(J,Q)$ is *pure* of weight k, and we have a Hodge decomposition

$$H^k(J,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(J), \qquad \overline{H^{p,q}(J)} = H^{q,p}(J).$$

Moreover, for an abelian variety the cotangent bundle is trivial so

$$H^{1,0}(J) = H^0(J,\Omega^1_J) \cong \mathbb{C}^g, \qquad H^{0,1}(J) \cong \overline{H^{1,0}(J)} \cong \mathbb{C}^g.$$

In particular, $\dim_{\mathbb{C}} H^{1,0}(J) = \dim_{\mathbb{C}} H^{0,1}(J) = g$.

A standard fact for complex tori (hence for abelian varieties) is that the cohomology algebra is the exterior algebra on H^1:

$$H^*(J,\mathbb{C}) \;\cong\; \bigwedge\nolimits^{*} H^1(J,\mathbb{C}).$$

Using the decomposition $H^1(J,\mathbb{C}) = H^{1,0}(J)\oplus H^{0,1}(J)$ and that Hodge bidegrees add under wedge products, we obtain for each k:

$$H^k(J,\mathbb{C}) \;\cong\; \bigwedge\nolimits^{k}(H^{1,0}(J)\oplus H^{0,1}(J)) \;\cong\; \bigoplus_{p+q=k} \bigwedge\nolimits^{p} H^{1,0}(J) \;\otimes\; \bigwedge\nolimits^{q} H^{0,1}(J).$$

Therefore,

$$H^{p,q}(J) \;\cong\; \bigwedge\nolimits^{p} H^{1,0}(J)\otimes \bigwedge\nolimits^{q} H^{0,1}(J), \qquad p+q=k,$$

and taking dimensions we conclude that the Hodge numbers are

$$h^{p,q}(J) = \dim_{\mathbb{C}} H^{p,q}(J) = \binom{g}{p}\binom{g}{q}.$$

In particular, $H^k(J)$ is pure of weight k.

$\square$

There is a close relation between the standard and compactly supported Hodge polynomials, and hence between the corresponding E-polynomials, given by Poincaré duality.

Proposition 5. *Let X be a smooth connected complex algebraic variety of complex dimension d. Then the compactly supported mixed Hodge polynomial $\mu_c(X;t,u,v)$ and the mixed Hodge polynomial $\mu(X;t,u,v)$ satisfy the relation*

$$\mu_c(X;t,u,v) = (t^2uv)^d\,\mu\Big(X;\tfrac{1}{t},\tfrac{1}{u},\tfrac{1}{v}\Big).$$

Proof. Let X be a smooth connected complex variety of complex dimension d. Since X is smooth, the mixed Hodge structure on $H^k(X)$ has weights $\geq k$, so that $h^{k,p,q}(X) = 0$ whenever $p+q<k$.

By Poincaré duality for compact support, which is compatible with mixed Hodge structures, there is an isomorphism of mixed Hodge structures

$$H^k_c(X) \;\simeq\; (H^{2d-k}(X))^{\vee}(-d).$$

In terms of Hodge numbers, this implies

$$h_c^{k,p,q}(X) = h^{2d-k,\,d-p,\,d-q}(X). \tag{2.1}$$

Using (2.1) we compute:

$$\mu_c(X;t,u,v) = \sum_{k,p,q} h_c^{k,p,q}(X)\, u^p v^q t^k = \sum_{k,p,q} h^{2d-k,\,d-p,\,d-q}(X)\, u^p v^q t^k.$$

Making the change of variables $k' = 2d-k$, $p' = d-p$, $q' = d-q$, we obtain

$$\begin{aligned}\mu_c(X;t,u,v) &= \sum_{k',p',q'} h^{k,p',q'}(X)\, u^{d-p'} v^{d-q'} t^{2d-k'} \\ &= (t^2uv)^d \sum_{k',p',q'} h^{k',p',q'}(X) \Big(\tfrac{1}{t}\Big)^{k'} \Big(\tfrac{1}{u}\Big)^{p'} \Big(\tfrac{1}{v}\Big)^{q'}.\end{aligned}$$

That is,

$$\mu_c(X;t,u,v) = (t^2uv)^d\, \mu\Big(X;\ \tfrac{1}{t},\ \tfrac{1}{u},\ \tfrac{1}{v}\Big),$$

which is the desired identity. □

Proposition 6. *Let $T \subset GL_n(\mathbb{C})$ be a maximal torus. Then $T \cong (\mathbb{C}^*)^n$ and, for every $0 \le k \le n$, $H^k(T)$ is pure of Hodge type (k,k) and*

$$h^{k;p,q}(T) = \begin{cases} \binom{n}{k}, & (p,q) = (k,k); \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. The maximal torus of $GL_n(\mathbb{C})$ is the group D_n of invertible diagonal matrices of size n. Observe that $D_n \cong (\mathbb{C}^*)^n$. By an argument analogous to the one used in the proof of Proposition 4, we obtain

$$\mu(T;t,u,v) = \sum_{k=0}^{n} \binom{n}{k} t^k u^k v^k.$$

The Poincaré polynomial is obtained by setting $u = v = 1$:

$$P(T;t) = \sum_{k=0}^{n} \binom{n}{k} t^k.$$

The E-polynomial is obtained by setting $t = -1$:

$$E(T;x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k = (x-1)^n.$$

Using proposition 5, we also obtain the corresponding compactly supported versions.

□

Proposition 7. *For all $n \geq 1$, the E-polynomials of the complex algebraic groups $SL_n(\mathbb{C})$ and $PGL_n(\mathbb{C})$ are given by*

$$e(SL_n(\mathbb{C}); q) = q^{\binom{n}{2}} \prod_{i=2}^{n} (q^i - 1), \qquad e(PGL_n(\mathbb{C}); q) = q^{\binom{n}{2}} \prod_{i=2}^{n} (q^i - 1).$$

Proof. Consider the determinant morphism

$$SL_n(\mathbb{C}) \longrightarrow GL_n(\mathbb{C}) \xrightarrow{\det} \mathbb{C}^*,$$

which is a Zariski locally trivial fibration.

It is well known that

$$e(GL_n(\mathbb{C}); q) = \prod_{i=0}^{n-1} (q^n - q^i) = q^{\binom{n}{2}} \prod_{i=1}^{n} (q^i - 1), \qquad e(\mathbb{C}^*; q) = q - 1.$$

By multiplicativity of the E-polynomial for Zariski locally trivial fibrations, we obtain

$$e(GL_n(\mathbb{C}); q) = e(SL_n(\mathbb{C}); q)\, e(\mathbb{C}^*; q),$$

and therefore

$$e(SL_n(\mathbb{C}); q) = \frac{e(GL_n(\mathbb{C}); q)}{e(\mathbb{C}^*; q)} = q^{\binom{n}{2}} \prod_{i=2}^{n} (q^i - 1).$$

For $PGL_n(\mathbb{C})$, consider the quotient morphism

$$\mathbb{C}^* \longrightarrow GL_n(\mathbb{C}) \longrightarrow PGL_n(\mathbb{C}).$$

Since $\mathbb{C}^*$ acts freely on $GL_n(\mathbb{C})$ by scalar multiplication and the action is algebraic, the quotient is a Zariski locally trivial principal $\mathbb{C}^*$-bundle. Hence

$$e(GL_n(\mathbb{C}); q) = e(PGL_n(\mathbb{C}); q)\, e(\mathbb{C}^*; q),$$

and we conclude that

$$e(PGL_n(\mathbb{C}); q) = \frac{e(GL_n(\mathbb{C}); q)}{e(\mathbb{C}^*; q)} = q^{\binom{n}{2}} \prod_{i=2}^{n} (q^i - 1).$$

This coincides with the E-polynomial of $SL_n(\mathbb{C})$, as claimed. □

Proposition 8. *The E-polynomial is a counting polynomial for $\mathbb{C}$, $\mathbb{C}^*$ and $\mathbb{P}^n$. More precisely, for every finite field $\mathbb{F}_q$ one has*

$$E(\mathbb{C}; q) = \#\mathbb{C}(\mathbb{F}_q), \qquad E(\mathbb{C}^*; q) = \#\mathbb{C}^*(\mathbb{F}_q), \qquad E(\mathbb{P}^n; q) = \#\mathbb{P}^n(\mathbb{F}_q).$$

Proof. Let $\mathbb{F}_q$ be a finite field with q elements.

(1) The affine line. The set $\mathbb{C}(\mathbb{F}_q)$ consists of all elements of $\mathbb{F}_q$, hence $\#\mathbb{C}(\mathbb{F}_q) = q$. On the other hand, the E-polynomial of $\mathbb{C}$ is $E(\mathbb{C}; x) = x$, so $E(\mathbb{C}; q) = q$.

(2) The multiplicative group. We have $\mathbb{C}^*(\mathbb{F}_q) = \mathbb{F}_q^\times$, which has $q-1$ elements. Since $E(\mathbb{C}^*; x) = x - 1$, it follows that $E(\mathbb{C}^*; q) = q - 1$.

(3) Projective space. A point of $\mathbb{P}^n(\mathbb{F}_q)$ is a line through the origin in $\mathbb{F}_q^{n+1}$, so

$$\#\mathbb{P}^n(\mathbb{F}_q) = \frac{q^{n+1}-1}{q-1} = 1 + q + q^2 + \cdots + q^n.$$

Equivalently, using the decomposition $\mathbb{P}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0$ and additivity of the E-polynomial, we obtain

$$E(\mathbb{P}^n; x) = x^n + x^{n-1} + \cdots + x + 1,$$

hence $E(\mathbb{P}^n; q) = q^n + q^{n-1} + \cdots + q + 1 = \#\mathbb{P}^n(\mathbb{F}_q)$.

Thus in all three cases the value of the E-polynomial at q coincides with the number of $\mathbb{F}_q$-points. □

Proposition 9. *Let Ψ be the operator on monomials defined by*

$$\Psi(q^i t^n) = \sum_{m\geq 1} \frac{(q^i t^n)^m}{m},$$

and let Ψ^{-1} be the operator defined by

$$\Psi^{-1}(q^i t^n) = \sum_{m\geq 1} \frac{\mu(m)}{m}(q^i t^n)^m,$$

where μ denotes the Möbius function. Then Ψ^{-1} is the inverse of Ψ, i.e.

$$\Psi^{-1} \circ \Psi = \Psi \circ \Psi^{-1} = \mathrm{id}$$

on $\mathbb{Q}[q][[t]]$.

Proof. Let $x = q^i t^n$. By definition,

$$\Psi(x) = \sum_{m\geq 1} \frac{x^m}{m}.$$

Applying Ψ^{-1}, we obtain

$$\Psi^{-1}(\Psi(x)) = \sum_{m\geq 1} \frac{1}{m} \Psi^{-1}(x^m) = \sum_{m\geq 1} \frac{1}{m} \sum_{\ell\geq 1} \frac{\mu(\ell)}{\ell} x^{m\ell}$$
$$= \sum_{m\geq 1} \sum_{\ell\geq 1} \frac{\mu(\ell)}{m\ell} x^{m\ell}.$$

Setting $k = m\ell$, this can be rewritten as

$$\Psi^{-1}(\Psi(x)) = \sum_{k\geq 1} \left(\frac{1}{k} \sum_{\ell | k} \mu(\ell) \right) x^k.$$

Using the classical identity

$$\sum_{d|k} \mu(d) = \begin{cases} 1, & k = 1; \\ 0, & k > 1. \end{cases}$$

we see that all terms vanish except for $k = 1$, whose coefficient is 1. Hence

$$\Psi^{-1}(\Psi(x)) = x.$$

The same computation, with Ψ and Ψ^{-1} interchanged, shows that

$$\Psi(\Psi^{-1}(x)) = x.$$

Therefore Ψ^{-1} is indeed the inverse of Ψ on $\mathbb{Q}[q][[t]]$. □

3. Further research

The results discussed in this work provide a foundation for further exploration of the Hodge theory of moduli spaces. In particular, while the computation of E-polynomials for character varieties associated with the groups $SL_n(\mathbb{C})$ and $GL_n(\mathbb{C})$ has already been addressed by several authors, a natural direction for future research is the extension of these techniques to character varieties associated with the symplectic group $Sp_{2n}(\mathbb{C})$ and the orthogonal groups $O_n(\mathbb{C})$ and $SO_n(\mathbb{C})$. These cases involve different topological invariants and symmetry structures, which are expected to give rise to weight filtrations exhibiting behavior distinct from those observed for the groups studied in this work.

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