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EXACT SOLUTIONS AND STABILITY OF A NONLINEAR SCALAR FIELD IN A PETROV TYPE D COSMOLOGY

SOLUCIONES EXACTAS Y ESTABILIDAD DE UN CAMPO ESCALAR NO LINEAL EN UNA COSMOLOGÍA DE PETROV TIPO D

Andrés Angulo-Sibaja [1] Rodrigo Alvarado Marín [2]

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[1] Universidad de Costa Rica, Escuela de Física, San José, Costa Rica. E-Mail: andres.angulosibaja@ucr.ac.cr
[2] Universidad de Costa Rica, Centro de Investigaciones Espaciales (CINESPA) y Escuela de Física, San José, Costa Rica. E-Mail: rodrigo.alvarado@ucr.ac.cr

Abstract

We present analytical solutions for a universe with a scalar field equivalent to a mixture of three perfect fluids: dark energy, dust and stiff matter. The space-time is an anisotropic and homogenous universe of Petrov Type D that expands isotropically in two spatial axes. We also determine the singularities with the Kretschmann scalar, the Hubble parameter, the deceleration parameter, and the temperature in terms of time. Finally, we study the Jacobi stability of the universe with the scalar field and conclude that the model is stable at all times.

Keywords: cosmology; exact solution; scalar field; Einstein's equations; temperature; Hubble; deceleration parameter; Kretschmann; singularity; Jacobi stability.

Resumen

Se presentan soluciones analíticas para un universo con un campo escalar equivalente a una mezcla de tres fluidos perfectos: energía oscura, polvo y materia rígida. El espacio-tiempo del universo es anisotrópico y homogéneo del tipo Petrov D que se expande isotrópicamente en dos direcciones espaciales. También se determina el parámetro de Hubble, parámetro de desaceleración y la temperatura en términos del tiempo. Por último, se estudia la estabilidad de Jacobi para la evolución dinámica del universo con el campo escalar y se concluye que el modelo es estable para cualquier tiempo.

Palabras clave: cosmología; solución exacta; campo escalar; ecuaciones de Einstein; temperatura; Hubble; parámetro de deceleración; Kretschmann; singularidad; estabilidad de Jacobi.

Mathematics Subject Classification: Primary: 83F05; Secondary: 83C15, 83C20, 35B35, 53B40

1. Introduction

Improvements in cosmological observations by COBE, WMAP, DES, PLANCK, and other missions have positioned relativistic cosmology as a precise science capable of validating theoretical models that aim to explain the composition and dynamical evolution of the universe. Due to these advances, precision cosmology has found evidence that challenges the Cosmological Principle (CP). This principle requires the universe to be homogeneous and isotropic for sufficiently large scales, a cornerstone of the well-established paradigm of ΛCDM.

The upper bound of large-scale structures depends on the model employed. In the concordance model, simulations estimate this limit to be around 370 Mpc [20, 27]. However, there are large structures that surpass this limit. The quasar group U1.27 is part of these huge gravitationally bound formations with a proper size of $\sim$ 500 Mpc [14]. The largest one is the Hercules-Corona Borealis Great Wall with about $2-3$ Gpc of proper size [18]. Other large-scale structures are the "Giant Arc" (GA), a large filamentary crescent-shape structure with a proper size of $\sim$ 1 Gpc, which was discovered at a redshift of $z \sim 0.8$ by [22]; the Giant GRB Ring with $\sim$ 1.72 Gpc [13]; and the "Big Ring" (recently found by the same team that discovered the GA), which has a ring with a diameter of $\sim 200-300$ Mpc at the same distance as the GA [23]. This last work is pending peer review but it is part of the increasing observational evidence of structures larger than the required scale of homogeneity. If this evidence surpasses the improvement of statistical and observational tools, the implications of the CP in the standard model of cosmology must be revised.

Another result challenging the CP is the so-called "axis of evil" of the cosmic microwave background (CMB). The axis of evil includes the dipole anomaly $l = 1$ in which the motion of the solar system is just about 10 degrees apart from the direction of the ecliptic plane, as well as the quadrupole $l = 2$ and octupole $l = 3$, which are mysteriously aligned in a direction perpendicular to this elliptical plane.

There are also variational directions in cosmological parameters such as H_0, coherent peculiar velocities of cosmic objects called bulk flows, alignment of large quasar groups, radio galaxies, and preference in the direction of rotation of galaxies. Further evidence has already been presented in [1], and, more recently, the extensive work in [20] discusses in great detail the current observations that present deviations from the CP.

In response to these challenges, this study focuses on a particular universe with an anisotropic and homogeneous Bianchi type-I spacetime in which the main constraint is that two of the spatial scale factors are equal, so that the solution has a Petrov Type D symmetry [1].

We find solutions of a scalar field and scalar potential equivalent to a mixture of perfect fluids with this Petrov Type D symmetry. The importance of studying cosmological scenarios with different fluids has already been discussed in [1, 12] and more recently in [4]. Studies that explore different fluids and their mixtures in this Petrov type D universe can been seen here [1, 8, 9, 10, 11, 12].

Here, we minimally couple a scalar field to the action and find the field ϕ and the potential $V(\phi)$ in terms of the fluid parameters. The significance of studying scalar fields in cosmology as a source of matter lies in their versatility in seeking viable solutions. For example, a scalar field called the inflaton was introduced to drive a possible expansion in the early universe to solve the problems of flatness and horizon [16]. The quintessence is a scalar field varying in time introduced to explain the late-time cosmic acceleration [28]. A scalar field with a Ratra-Peblees potential is able to track radiation and dust behaviors in the early universe to then transition to a dark energy behavior in the late universe [19].

Regarding our Petrov type D symmetry with a scalar field, the work [5] explored a potential of the form $V(\phi) \sim (1 + \cosh(C(\phi + \phi_0)))^{-2}$ that behaves in early times similarly to dark energy and in late times like a stiff fluid of Zeldovich ($w_\phi = 1$). In [6], the authors found for ϕ and $V(\phi)$ an isobaric fluid $P = -D$ where D is a positive constant, and in the case of [7], the author found the interaction of scalar and spinorial field equivalent to a Chaplygin Gas studied in [3].

Commonly, the potential is defined first to study the equation of state; however, this study is looking for the potential $V(\phi)$ and scalar field ϕ equivalent to dark energy, dust and stiff matter. The properties of these fluids have been well studied and under- stood in the context of the dynamical evolution and composition of the universe.

To further analyze this model, we also calculate the Hubble and deceleration parameters, as well as the temperature. These values are crucial because they act as cosmological observables, validating the model by aligning with the current observations. On the other hand, we also determined the non-removable singularities of this universe: when spacetime breaks down; in this case, we use the Kretschmann invariant.

This study also determines the global stability to exponential deviations to nearby trajectories with the theory of Kosambi-Cartan-Chern (KCC); the KCC theory was applied for the first time to scalar field cosmologies in [15], and used in the context of the Petrov type D universe in [6]. This is an essential tool for discriminating which cosmological models are viable for observations and predictions.

In Section 2, we introduce the symmetry under study and the solutions for the scale factors $K(t)$ and $F(t)$. In Section 3 we calculate the Hubble and deceleration parameters together with the temperature and Kretchmann invariant for the singularities. In Section 4, we present the solutions for the scalar field $\phi(t)$ and potential $V(\phi)$ that generate a source of matter equivalent to a mixture of dark energy, dust and stiff matter. In Section 5, we use the KCC theory to determine the system's stability with this scalar field and potential. Finally, in Section 6 we present our conclusions.

2. Symmetry, Einstein's Equations and the Solutions

The next line-element describes the dynamical evolution of the universe under study [1],

$$ds^2 = Fdt^2 - t^{2/3}K(dx^2 + dy^2) - \frac{t^{2/3}}{K^2}dz^2, \tag{2.1}$$

where F and K are functions of t. This metric defines an anisotropic and homogeneous universe that expands isotropically just in the x and y directions: so this is a symmetry of Petrov Type D.

The functions for the energy density $\mu(t)$ and pressure $P(t)$ were already obtained in [1]. Here again, we supposed that each perfect fluid follows $P = w\mu$, so that the energy density and pressure take the forms

$$\mu(t) = \frac{\alpha_w}{3t^{w+1}}, \quad P(t) = \frac{w\alpha_w}{3t^{w+1}}, \tag{2.2}$$

respectively, where $\alpha_w > 0$. These solutions (2.2) are for $w \neq 1$, if $w = 1$ then $\alpha_w/3 = \beta_w$ for $\beta_w > 0$. Since we supposed a mixture of non-interacting fluids, these thermodynamic variables can be expressed as a linear summation of each variable for the corresponding perfect fluid in the following way:

$$P_T = P_\Lambda + P_D + P_Z, \tag{2.3}$$

$$\mu_T = \mu_\Lambda + \mu_D + \mu_Z, \tag{2.4}$$

where the pressures P_Λ, P_D, and P_Z are for dark energy, dust and stiff matter respectively, while μ_Λ, μ_D, and μ_Z represent their respective energy densities.

Using the solutions (2.2), the total energy density and the total pressure for the mixture of fluids are

$$\mu_T = \Lambda + \frac{D}{t} + \frac{Z}{t^2}, \tag{2.5}$$

$$P_T = -\Lambda + \frac{Z}{t^2}, \tag{2.6}$$

where for dark energy $\Lambda = \alpha_{-1}/3$, dust $D = \alpha_0/3$ and stiff matter $Z = \beta_1$. The scale factor $F(t)$ can be define in terms of the total energy density μ_T as

$$F(t) = \frac{4}{9C_1^2 + 12t^2\mu_T}, \tag{2.7}$$

where C_1 is a constant. The solution for the other scale factor is (see [1])

$$K(t) = K_0 e^{C_I \int \frac{F^{1/2}}{t} dt}, \tag{2.8}$$

where K_0 is a constant; the constant C_1 defines a set of models with different behaviors, depending on its value. If we use the total energy density given by (2.5) and $C_1 = \pm 2/3$, then

$$F(t) = \frac{1}{3\Lambda t^2 + 3Dt + 3Z + 1}. \tag{2.9}$$

We now calculate the integral in the exponent of (2.8)

$$K_\pm(t) = K_0 \left(\frac{2\sqrt{1+3Z}\sqrt{3\Lambda t^2 + 3Dt + 3Z + 1} - 3Dt - 6Z - 2}{2\sqrt{1+3Z}\sqrt{3\Lambda t^2 + 3Dt + 3Z + 1} + 3Dt + 6Z + 2} \right)^{\pm \frac{1}{3\sqrt{1+3Z}}}. \tag{2.10}$$

If stiff matter is decoupled ($Z = 0$), we obtain the expected solution for a universe with a constant negative pressure (dark energy) and dust [6]. If both dust and dark energy are decoupled ($\Lambda = 0$ and $D = 0$), which is the same as setting $t = 0$, the scale factor in the z direction diverges. However, near $t = 0$ and defining

$$K_{0\pm} = \left(\frac{3(-3D^2 + 4\Lambda + 12\Lambda Z)}{16(3Z+1)^2} \right)^{\mp 1/(3\sqrt{3Z+1})}, \tag{2.11}$$

the metric takes the following form:

$$ds^2 \approx d\eta^2 - t^{2/3 \pm 2/(3\sqrt{3Z+1})}(dx^2 + dy^2) - t^{2/3 \mp 4/(3\sqrt{3Z+1})} dz^2, \tag{2.12}$$

which is a universe that for small values of Z, approximates the Kasner behavior: an anisotropic and homogeneous vacuum solution. In the case of the late universe $t \to \infty$ and choosing

$$K_{0\pm} = \left(\frac{2\sqrt{3\Lambda + 9\Lambda Z} - 3D}{2\sqrt{3\Lambda + 9\Lambda Z} + 3D} \right)^{\mp 1/(3\sqrt{1+3Z})}, \tag{2.13}$$

with the temporal variable defined as $t = e^{\sqrt{2\Lambda}\eta}$, we obtain

$$ds^2 = d\eta^2 - e^{2/3\sqrt{3\Lambda}\eta}(dx^2 + dy^2 + dz^2), \tag{2.14}$$

which matches the solution for a universe with only dark energy with this same Petrov type D solution [1]. In fact, (2.14) is a de Sitter spacetime universe expected as an asymptotic solution for this model (2.1), which belongs to the Bianchi I universes.

3. Cosmological parameters and singularities

3.1. The Hubble and the deceleration parameters.

These parameters were previously determined in [2], for a Bianchi I metric

$$ds^2 = d^2dt^2 - a^2dx^2 - b^2dy^2 - c^2dz^2, \tag{3.1}$$

the average Hubble parameter is

$$H = \frac{1}{(abc)^{1/3}d} \frac{d}{dt} (abc)^{1/3}, \tag{3.2}$$

and using (2.1) the result is

$$H(t) = \frac{\sqrt{3\Lambda t^2 + 3Dt + 3Z + 1}}{3t}. \tag{3.3}$$

When $t \to 0$, the early universe approximates $H \to \infty$; in the late universe, as $t \to \infty$, dark energy takes over $H \to \sqrt{\Lambda/3}$.

For the deceleration parameter (see [2])

$$q = -\left(1 + \frac{\dot{H}}{dH^2}\right), \tag{3.4}$$

and using (2.1), we obtain

$$q(t) = -\frac{6\,\Lambda t^2 - 3Dt - 12Z - 4}{6\Lambda t^2 + 6Dt + 6Z + 2}. \tag{3.5}$$

As $t \to 0$, the universe begins decelerating with $q \to 2$; but must transition, at least once, through a final accelerating phase that approximates $q \to -1$ when $t \to \infty$. This transition occurs at a single moment:

$$t_0 = \frac{3D + \sqrt{288\Lambda Z + 9D^2 + 96\Lambda}}{12\Lambda}. \tag{3.6}$$

This behavior arises from dark energy taking over non-relativistic matter at a later time. The accelerating expansion in the late universe has already been identified by Supernovae Type Ia data [24, 25] and by later works, which included kinematic methods of galaxy clusters and the cosmic microwave background [17, 21, 26].

3.2. The universe's temperature.

The equation of state for the universe must satisfy the following thermodynamic relation

$$\frac{dP}{\mu + P} = \frac{dT}{T}, \tag{3.7}$$

as previously established in [4]. In the case of (2.1), the equation of state for dark energy, dust, and stiff matter follows (2.6) and (2.5), which leads to the following temperature

$$T = T_0\left(\frac{Dt + 2Z}{t}\right). \tag{3.8}$$

In the early universe when $t \to 0$, the temperature approaches infinity $T \to \infty$ and is dominated by stiff matter fluid; in the late universe when $t \to \infty$, the temperature is dominated by the dust fluid $T = T_0 D$. The result (3.8) does not depend on dark energy, a characteristic shared with other fluid mixtures [8, 12].

3.3. Singularities.

To determine non-removable singularities, we use the Kretschmann invariant, which is the square of the Riemann curvature tensor

$$\mathcal{K} = R_{\mu\nu\lambda\rho}R^{\mu\nu\lambda\rho}. \tag{3.9}$$

In terms of F we can use (2.8) with $C_1 = \pm 2/3$ so that

$$\mathcal{K} = \frac{\pm 32F^{7/2} + 36F^4 - 24F^3 + 20F^2 + 24F\dot{F}t(1-F) + 9\dot{F}^2t^2}{27F^4t^4}; \tag{3.10}$$

then, with (2.9), we obtain

$$\mathcal{K}_{\pm} = \frac{1}{27t^4}\Big[\pm\sqrt{3At^2+3Bt+3Z+1}+72A^2t^4+36ABt^3 + ((-72Z+48)\Lambda+45B^2)t^2+(144Z+48)Bt+180Z^2+48Z+32\Big]. \tag{3.11}$$

When $t \to 0$ the Kretschmann invariant approximates

$$\mathcal{K}_{\pm} = \frac{4\left(45Z^2+12Z\pm 8\sqrt{3Z+1}+8\right)}{27t^4}. \tag{3.12}$$

There is a singularity proportional to $\sim t^{-4}$, which generally happens when the positive constant $C_1 = +2/3$ is used for different fluid mixtures [1, 6, 12], and is equivalent to the anisotropic vacuum solution of Kasner with exponents $p_1 = 2/3$, $p_2 = 2/3$ and $p_3 = -1/3$. For the negative constant $C_1 = -2/3$, when $Z = 0$, the singularity $\sim t^{-4}$ disappears together with the next one in the series $\sim t^{-3}$, but the $\sim t^{-2}$ remains, so it presents the same behavior as dust and dark energy found in [6].

4. Scalar field solution equivalent to dark energy, dust, and stiff matter

The present study considers the minimal coupling of a scalar field ϕ with the Einstein-Hilbert action

$$S = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa}+\frac{1}{2}\phi_{,\mu}\phi_{,\nu}g^{\mu\nu}-V(\phi)\right). \tag{4.1}$$

Here the lagrangian of the scalar field is $\mathcal{L}_\phi = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$. When the field does not interact with any external source, implying no x^μ dependence, the conserved quantity in the Euler-Lagrange equations is the following stress-energy tensor

$$T^\nu_\mu = \frac{\partial\mathcal{L}}{\partial\phi_{,\nu}}\phi_{,\mu} - \mathcal{L}\delta^\nu_\mu. \tag{4.2}$$

The fluid parameters in terms of the scalar field, using (4.2) and (2.1), are

$$\mu = \frac{1}{2F}\dot{\phi}^2 + V(\phi), \tag{4.3}$$

$$P = \frac{1}{2F}\dot{\phi}^2 - V(\phi). \tag{4.4}$$

If the Einstein field equations are considered along with the scalar field and its potential, then

$$-\frac{F-1}{3Ft^2} = \frac{\dot{\phi}^2+2VF}{2F}, \tag{4.5}$$

$$\frac{F^2-\dot{F}t-F}{3F^2t^2} = \frac{-\dot{\phi}^2+2VF}{2F}. \tag{4.6}$$

By eliminating V from the two last equations, we obtain

$$\dot{\phi}^2 + \frac{2F^2 - 2F - \dot{F}t}{3Ft^2} = 0, \tag{4.7}$$

and using the expression for $F(t)$ from (2.9)

$$\dot{\phi}^2 = \frac{Dt + 2Z}{t^2(3\Lambda t^2 + 3Dt + 3Z + 1)}. \tag{4.8}$$

If we take the positive value of the square root and integrate this last equation, we obtain

$$\begin{aligned}\phi(t) = -\frac{2D}{\sqrt{3\Lambda}} L_1 L_2 \Bigg[& F\left(L_1\sqrt{Dt + 2Z}, L_2\right) - \Pi\left(L_1\sqrt{Dt + 2Z}, \frac{1}{2ZL_1^2}, L_2\right) \\ & + F\left(\sqrt{2Z}L_1, L_2\right) - \Pi\left(\sqrt{2Z}L_1, \frac{1}{2ZL_1^2}, L_2\right)\Bigg].\end{aligned} \tag{4.9}$$

Here, $F(z, k)$ is the incomplete elliptic integral of the first kind, and $\Pi(z, v, k)$ is the incomplete elliptic integral of the third kind, where

$$L_1 \equiv \sqrt{\frac{6\Lambda}{12\Lambda Z - D\sqrt{9D^2 - (36Z + 12)\Lambda} - 3D^2}}, \tag{4.10}$$

$$L_2 \equiv \sqrt{\frac{12\Lambda Z - D\sqrt{9D^2 - (36Z + 12)\Lambda} - 3D^2}{12\Lambda Z + D\sqrt{9D^2 - (36Z + 12)\Lambda} - 3D^2}}. \tag{4.11}$$

The same universe can be modeled without changing the form of the metric (2.1) and scale factors (2.9, 2.10) by establishing a new set of coordinate system $\eta = t/\zeta$ and $x'^i = \zeta x^i$, where $D' = D/\zeta$, $\Lambda' = \Lambda$ and $\zeta = \sqrt{3Z + 1}$. In this case, the scalar field takes the following form:

$$\phi(t) = -\frac{2}{\sqrt{3}}\sqrt{L_1^2 - 1}\left(F\left(\sqrt{L_2 t + 1}, \frac{L_1}{\sqrt{2}}\right) - K\left(\frac{L_1}{\sqrt{2}}\right)\right). \tag{4.12}$$

The scalar potential $V(\phi)$ is obtained from the relation $V = \frac{D}{2t} + \Lambda$ and the time t in terms of ϕ from (4.12), so that

$$V(\phi) = \Lambda - \frac{DL_2}{2}\mathrm{nc}\left(\frac{\sqrt{3}}{2\sqrt{L_1^2 - 1}}(\phi - \phi_0), \frac{L_1}{\sqrt{2}}\right)^2, \tag{4.13}$$

in which $\phi_0 = \frac{2}{\sqrt{3}}\sqrt{L_1^2 - 1}K(L_1/\sqrt{2})$ and $\mathrm{nc}(x, n)$ is a Jacobi Elliptic function.

5. Jacobi Stability

To assess the system's stability, we introduce the Kosambi-Cartan-Chern (KCC) developed in [15] for scalar field cosmologies and applied in [6] to our Petrov Type D symmetry. The general idea is to find a set of second-order ordinary differential equations equivalent to the Euler-Lagrange equations. The Jacobi equation can be obtained by doing a small perturbation, which involves the deviation curvature tensor that determines the system's stability.

The second-order ordinary differential equation is a vector field on the tangent bundle $\mathcal{TM}$ of the manifold $\mathcal{M}$. This field is called semispray and has the following form:

$$S = y^i \frac{\partial}{\partial x^i} - 2G^i(x,y)\frac{\partial}{\partial x^i}, \tag{5.1}$$

where $y^i = \dot{x}^i$ and $G(x,y)$ are some local coefficients. The curve $c(t) = (x^i(t))$ is a geodesic of S if and only if

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x^i, \frac{dx^i}{dt}\right) = 0. \tag{5.2}$$

By infinitesimally perturbing the dynamical system of (5.2) with $\tilde{x}^i(t) = x^i(t) + \epsilon\xi^i(t)$ where ϵ is a small parameter and $\xi^t(t)$ goes along $x(t)$, we have

$$\frac{d^2\xi^i}{dt^2} + 2N^i_j\frac{d\xi^i}{dt} + 2\frac{\partial G^i}{\partial x^j}\xi^j = 0, \tag{5.3}$$

where N^i_j is a nonlinear connection on $\mathcal{M}$ defined as

$$N^i_j = \frac{\partial G^i}{\partial y^j}. \tag{5.4}$$

The KKC-covariant differential on v^i, defined as $\frac{Dv^i}{dt} = \frac{dv^i}{dt} + N^i_j v^j$, now operating on y^i is

$$\frac{Dy^i}{dt} = N^i_j y^j - 2G^i. \tag{5.5}$$

Equation (5.2) can now be written as the Jacobi equation

$$\frac{D^2\xi^i}{dt^2} = P^i_j\xi^j. \tag{5.6}$$

Here P^i_j is the deviation curvature tensor given by

$$P^i_j = -2\frac{\partial G^i}{\partial x^j} - 2G^l G^i_{jl} + y^l\frac{\partial N^i_j}{\partial x^l} + N^i_l N^l_j, \tag{5.7}$$

where the Bernald connections G^i_{jl} are

$$G^i_{jl} \equiv \frac{\partial N^i_j}{\partial y^l}. \tag{5.8}$$

The trajectories in (5.2) are said to be Jacobi stable if and only if all real parts of the eigenvalues of the deviation curvature tensor P^i_j are strictly negative. Otherwise, they are Jacobi unstable if at least one is strictly positive. Such a condition for Jacobi stability requires the following Hurtwats determinants to be positive:

$$H_1 = \left|-(P^1_1 + P^2_2)\right| > 0, \tag{5.9}$$

$$H_2 = \begin{vmatrix} -(P^1_1 + P^2_2) & 0 \\ 1 & (P^1_1 P^2_2 - P^1_2 P^2_1) \end{vmatrix} > 0, \tag{5.10}$$

so that the deviation curvature tensor must obey the following inequalities

$$P^1_1 + P^2_2 < 0, \quad P^1_1 P^2_2 - P^1_2 P^2_1 > 0. \tag{5.11}$$

In summary, to determine the stability we first need to derive the pair of coupled differential equations that completely describe the dynamical evolution with the scalar field. After defining the phase space variables, the next step is to identify the local coefficients $G(x, y)$, obtain the deviation tensor and determine the stability via condition (5.11).

By combining equations (4.5) and (4.6)

$$\frac{\dot{F}}{3tF^2} + 2V = 0, \tag{5.12}$$

the first equation can be derived using the conservation of the stress-energy tensor

$$\ddot{\phi} + \dot{\phi}\left(\frac{1}{t} - \frac{1}{2}\frac{\dot{F}}{F}\right) + F\partial_\phi V = 0, \tag{5.13}$$

and (5.12), so that

$$\ddot{\phi} + F\partial V_\phi - \frac{\dot{\phi}\dot{F}}{2F} - \frac{6F^2 V\dot{\phi}}{\dot{F}} = 0. \tag{5.14}$$

This corresponds to the first equation. The second one arises from (5.12) and its time derivative

$$\ddot{F} - \frac{2\dot{F}^2}{F} + 6VF^2 - \frac{\partial_\phi V\dot{\phi}\dot{F}}{V} = 0. \tag{5.15}$$

Defining the new phase space variables as $x^1 = F, y^1 = \dot{F}, x^2 = \phi$, $y^2 = \dot{\phi}$ and $V' = \partial_{x^2} V$, we identify the local coefficients as

$$G^1 = -\frac{V' y^2 y^1}{2V} + 3(x^1)^2 V - \frac{(y^1)^2}{x^1}, \quad G^2 = \frac{x^1 V'}{2} - \frac{y^2 y^1}{4x^1} - \frac{3(x^1)^2 y^2 V}{y^1}. \tag{5.16}$$

Therefore, the nonlinear connections from (5.4) are

$$N^1_1 = -\frac{V' y^2}{2V} - \frac{2y^1}{x^1}, \quad N^1_2 = -\frac{V' y^1}{2V}, \quad N^2_1 = -\frac{y^2}{4x^1} + \frac{3\left(x^1\right)^2 V y^2}{(y^1)^2}, \tag{5.17}$$

$$N^2_2 = -\frac{y^1}{4x^1} - \frac{3\left(x^1\right)^2 V}{y^1}, \tag{5.18}$$

and the Bernald connections, as defined in (5.8), are

$$G^1_{11} = -\frac{2}{x^1}, \quad G^1_{21} = G^1_{12} = -\frac{V'}{2V}, \quad G^2_{22} = G^1_{22} = 0, \tag{5.19}$$

$$G^2_{11} = -\frac{6(x^1)^2 y^2 V}{(y^1)^3}, \quad G^2_{21} = G^2_{12} = -\frac{1}{4x^1} + \frac{3\left(x^1\right)^2 V}{(y^1)^2}. \tag{5.20}$$

Finally, the components of the deviation curvature tensor from (5.7) are

$$P^1_1 = -\frac{y^1 y^2 V'}{8x^1 V} + \frac{x^1 V'^2}{2V} - \frac{9\left(x^1\right)^2 y^2 V'}{2y^1} - \frac{\left(y^2\right)^2 V''}{2V} + \frac{3\left(y^2\right)^2 V'^2}{4V^2}, \tag{5.21}$$

$$P^1_2 = \frac{y^1 y^2 V''}{2V} - \frac{3y^1 y^2 V'^2}{4V^2} - \frac{3\left(x^1\right)^2 V'}{2} + \frac{\left(y^1\right)^2 V'}{8x^1 V}, \tag{5.22}$$

$$P^2_2 = -x^1 V'' + \frac{9\left(x^1\right)^2 y^2 V'}{2y^1} - \frac{y^1 y^2 V'}{8x^1 V} + 3x^1 V - \frac{9\left(x^1\right)^4 V^2}{(y^1)^2}$$
$$- \frac{3\left(y^1\right)^2}{16\,(x^1)^2}, \tag{5.23}$$

$$P^2_1 = -\frac{3V'}{4} + \frac{3y^1 y^2}{16\,(x^1)^2} - \frac{9\left(x^1 y^2\right)^2 V'}{2\,(y^1)^2} + \frac{45\left(x^1\right)^4 y^2 V^2}{(y^1)^3} \tag{5.24}$$

$$- \frac{3\left(x^1\right)^3 V V'}{(y^1)^2} + \frac{\left(y^2\right)^2 V'}{8x^1 V}. \tag{5.25}$$

The derivative of the potential $\partial_\phi V$ is given by (5.13), whereas the double derivative $\partial_{\phi\phi} V$ is

$$V'' = -\frac{\dot{F}^2}{F^3} + \frac{1}{F^2}\left(\frac{3\ddot{\phi}\dot{F}}{2\dot{\phi}} + \frac{\ddot{F}}{2} + \frac{\dot{F}}{t}\right) + \frac{1}{F}\left(\frac{1}{t^2} - \frac{\dddot{\phi}}{\dot{\phi}} + \frac{\ddot{\phi}}{\dot{\phi} t}\right). \tag{5.26}$$

Now, combining all these equations with (4.7), we express the conditions (5.11) in terms of the fluids parameters as

$$
\begin{aligned}
P^1_1 + P^2_2 = &- \frac{(3Z+1)\left(144\Lambda^4 t^6 + 576\Lambda^3 D t^5\right)}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
&- \frac{12\Lambda^2(3Z+1)\left(69D^2 + 8\Lambda(3Z+1)\right)t^4}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
&- \frac{36\Lambda D(3Z+1)\left(11D^2 + 12\Lambda(3Z+1)\right)t^3}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
&- \frac{(3Z+1)\left(27D^4 + 336\Lambda D^2(3Z+1) + 64\Lambda^2(3Z+1)^2\right)t^2}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
&- \frac{4D(3Z+1)^2\left(9D^2 + 60\Lambda Z + 20\Lambda\right)t}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2} \\
&- \frac{12D^2(3Z+1)^3}{16t^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2 (2\Lambda t + D)^2},
\end{aligned}
\tag{5.27}
$$

and

$$
\begin{aligned}
P^1_1 P^2_2 - P^1_2 P^2_1 = &\frac{3D(3Z+1)^2\left(9D\Lambda^2 t^4 + 18D^2\Lambda t^3\right)}{16t^4(2\Lambda t + D)^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2} \\
&+ \frac{3D(3Z+1)^3\left(18\Lambda D t^2 + 8\Lambda(3Z+1)t\right)}{t^4(2\Lambda t + D)^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2} \\
&+ \frac{3D^2(3Z+1)^4}{16t^4(2\Lambda t + D)^2\left(3\Lambda t^2 + 3Dt + 3Z + 1\right)^2}.
\end{aligned}
\tag{5.28}
$$

According to the criteria established in (5.11), the model remains stable throughout the entire evolution of the universe. Most models explored so far have exhibit at least regions with unstable trajectories. These include the Higgs potential $V(\phi) = V_0 + \frac{1}{2}M^2\phi^2 + \frac{\lambda}{4}\phi^4$, exponential potential $V = V_0 e^{\lambda\phi}$, and the Tachyion field $V = V_0\phi^\alpha$ [15]. For the FLRW universe with scalar field equivalent to dark energy and dust, the model also has unstable regions, but in our Petrov type D symmetry, is stable during the entire evolution of the universe [6].

6. Conclusions

This study derived the solutions for a scalar field minimally coupled to gravity and equivalent to a mixture of three fluids in a Petrov Type D universe: dark energy, dust, and stiff matter. The scalar field ϕ and the potential $V(\phi)$ can be simplified via a suitable change of coordinates—see Equations (4.12) and (4.13)—without altering the scale factors or cosmological observables. Additionally, using the Kosambi-Cartan-Chern (KKC) theory [15], we determined the stability of the model by using components of the deviation curvature tensor and the criteria of stability given in Equation (5.11). We conclude that the system is stable during the entire evolution of the universe.

In the early universe $t \to 0$, the behavior of the spacetime approximates a Kasner universe, but with perturbations predominantly dominated by the stiff matter fluid. In the late universe $t \to \infty$, we have the typical behavior of de Sitter spacetime driven by dark energy. Similarly, when $t \to 0$, the average Hubble parameter approximates $H \to \infty$, and when $t \to \infty$ dark energy takes over $H \to \sqrt{\Lambda/3}$. There is a transition t_0 given by Equation (3.6), which separates the decelerating phase with $q \to 2$ in the early universe from the accelerating phase that approximates $q \to -1$ in the late universe, which has been observed in other theoretical results and observational data [12, 24, 25].

The Kretschmann invariant reveals a non-removable singularity proportional to $\sim t^{-4}$ if the positive sign $C_1 = +2/3$ is taken, and $\sim t^{-2}$ if the negative sign $C_1 = -2/3$. In the limits, T tends to infinity as $t \to 0$, while it approaches a constant value $T \to T_0 D$ when $t \to \infty$. Finally, we observed that the temperature does not depend on Λ, because the energy density and pressure of the dark fluid remain constant throughout the entire evolution of the universe.

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